Electromagnetics:
Electromagnetic Field Theory

Dispersion Relation

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Lecture Outline

- Dispersion relation
- Index ellipsoids
- Material properties explained by index ellipsoids

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Dispersion Relation

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Derivation in LHI Media (1 of 2)

Start with the wave equation.

$$\nabla^2 \vec{E} + (k_0 n)^2 \vec{E} = 0$$

Solve wave equation

Plane wave solution

$$\vec{E}(\vec{r}) = \vec{P} e^{-j\vec{k}\bullet\vec{r}}$$

Finish Derivation

$$\nabla^2 \left(\vec{P} e^{-j\vec{k}\bullet\vec{r}} \right) + (k_0 n)^2 \left(\vec{P} e^{-j\vec{k}\bullet\vec{r}} \right) = 0$$

Substitute plane wave solution into wave equation.

$$\left(\nabla^2 e^{-j\vec{k}\bullet\vec{r}} \right) + (k_0 n)^2 e^{-j\vec{k}\bullet\vec{r}} = 0$$

Divide both sides by $\vec{P}$.

$$-k^2 e^{-j\vec{k}\bullet\vec{r}} + (k_0 n)^2 e^{-j\vec{k}\bullet\vec{r}} = 0$$

Calculate the Laplacian.

$$-k^2 + (k_0 n)^2 = 0$$

Divide both sides by $e^{-j\vec{k}\cdot\vec{r}}$.

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Derivation in LHI Media (2 of 2)

Starting from the previous slide

$$-k^2 + (k_0 n)^2 = 0$$

$$k^2 = (k_0 n)^2$$

Move $(k_0 n)^2$ to the right-hand side.

Recall that $k^2 = k_x^2 + k_y^2 + k_z^2$ and $k = \omega n / c_0$, the dispersion relation is

$$\boxed{k^2 = k_x^2 + k_y^2 + k_z^2 = (k_0 n)^2 = \left(\frac{\omega n}{c_0}\right)^2}$$

The dispersion relation relates frequency ω to wave number k. For LHI media, it fixes the magnitude of the wave vector to be a constant for all wave directions.

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Index Ellipsoids

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The Index Ellipsoid

From the previous slide, the dispersion relation for a LHI material was:

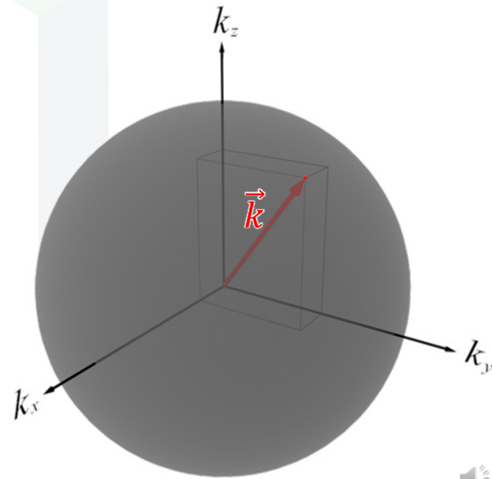
$$k_x^2 + k_y^2 + k_z^2 = (k_0 n)^2$$

This is an equation for a sphere of radius $k_0 n$.

$$x^2 + y^2 + z^2 = r^2$$

Pick a point on the surface.

The vector from the origin to this point is the wave vector $\vec{k}$.



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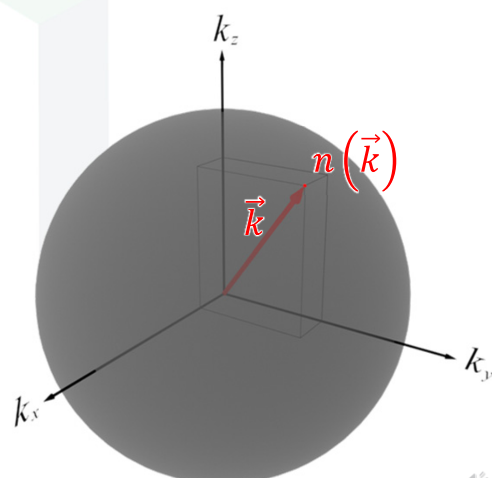
Index Ellipsoids Convey Refractive Index

If frequency k_0 is known, the magnitude of the wave vector $|\vec{k}|$ conveys refractive index n.

$$|\vec{k}| = k_0 n$$

Based on this, the index ellipsoid can be interpreted as a map of the refractive index that a wave experiences as a function of direction of that wave.



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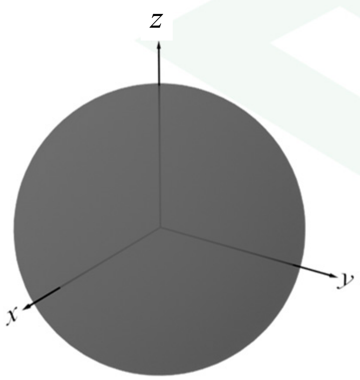
Index Ellipsoid for Isotropic Materials



$$k_x^2 + k_y^2 + k_z^2 = k_0^2 n^2$$

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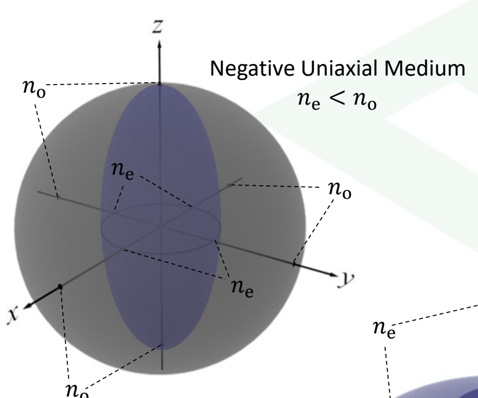
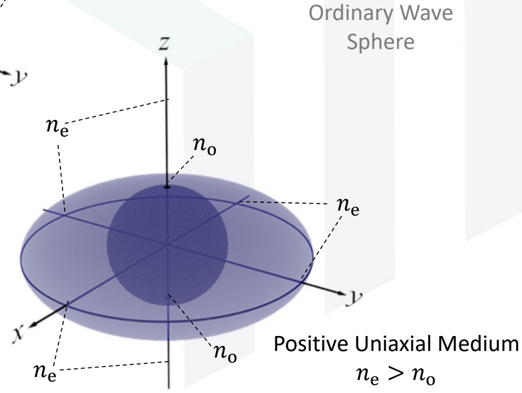
Index Ellipsoid for Uniaxial Materials

$$\left(\frac{k_x^2 + k_y^2 + k_z^2}{n_o^2} - k_0^2\right)\left(\frac{k_z^2 + k_y^2}{n_e^2} + \frac{k_z^2}{n_o^2} - k_0^2\right) = 0$$

Ordinary Wave Sphere

Extraordinary Wave Ellipsoid





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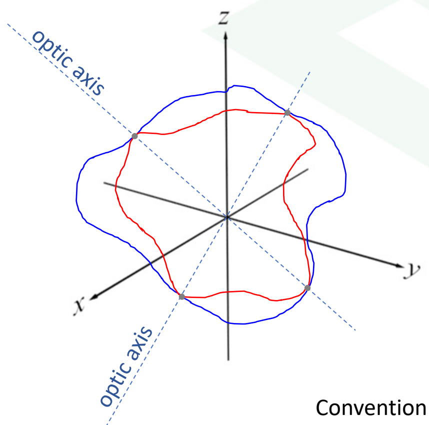
Index Ellipsoid for Biaxial Materials

$$\frac{k_x^2}{\left|\vec{k}\right|^2 - k_0^2 n_x^2} + \frac{k_y^2}{\left|\vec{k}\right|^2 - k_0^2 n_y^2} + \frac{k_z^2}{\left|\vec{k}\right|^2 - k_0^2 n_z^2} = 1$$



Convention
$n_x < n_y < n_z$

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Material Properties Explained by Index Ellipsoids

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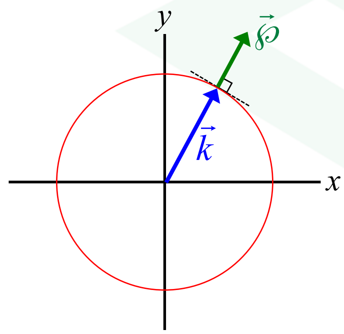
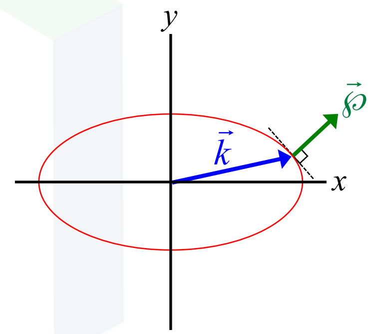
Direction of Phase and Power

Isotropic Materials



Anisotropic Materials



Phase propagates in the direction of $\vec{k}$.

Power propagates in the direction of $\vec{\wp}$ which is normal to the surface of the index ellipsoid.

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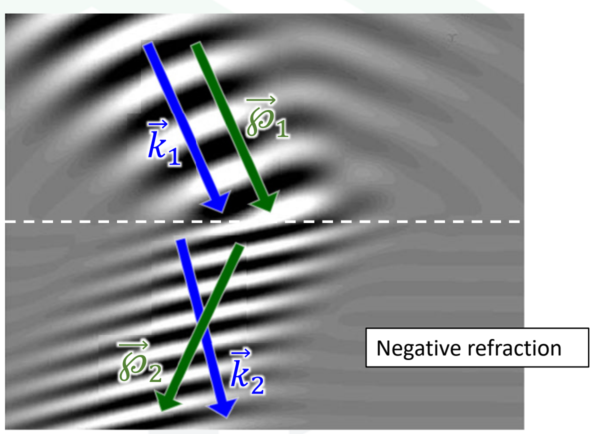
Illustration of $\vec{k}$ Not Same as $\vec{\wp}$



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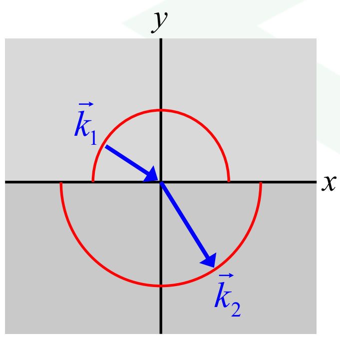
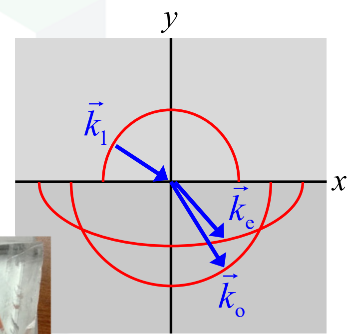
Double Refraction in Anisotropic Materials

<u>Isotropic Materials</u>



<u>Anisotropic Materials</u>



Anisotropic materials have two index ellipsoids – one for each polarization.
Wave power refracts in directions at the same time, producing *double refraction*.

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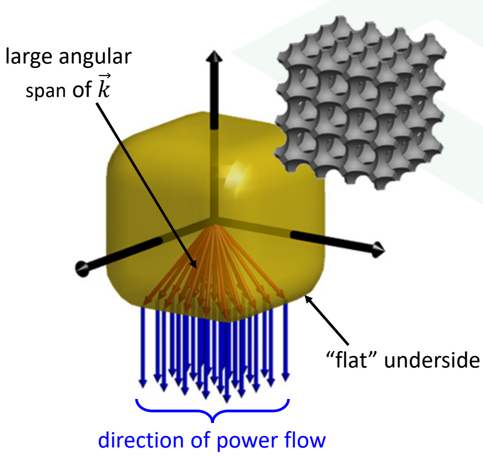
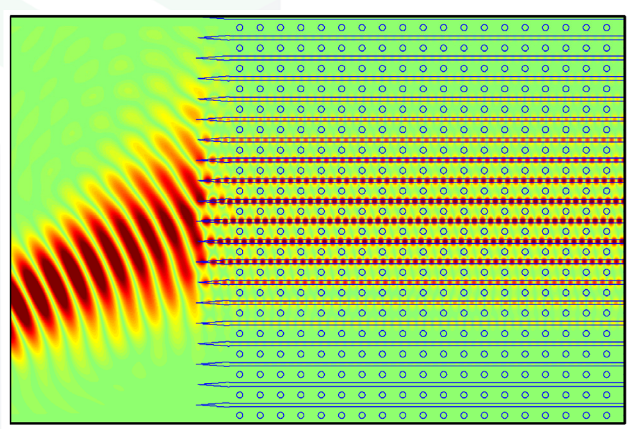
Self-Collimation in Photonic Crystals





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