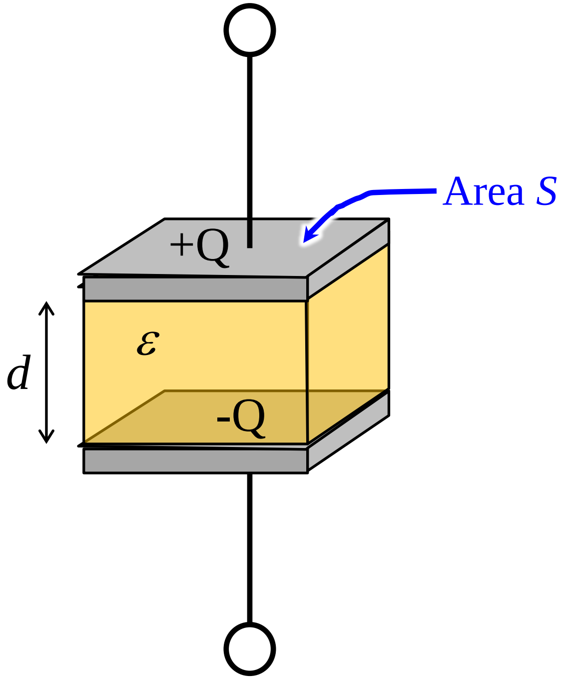
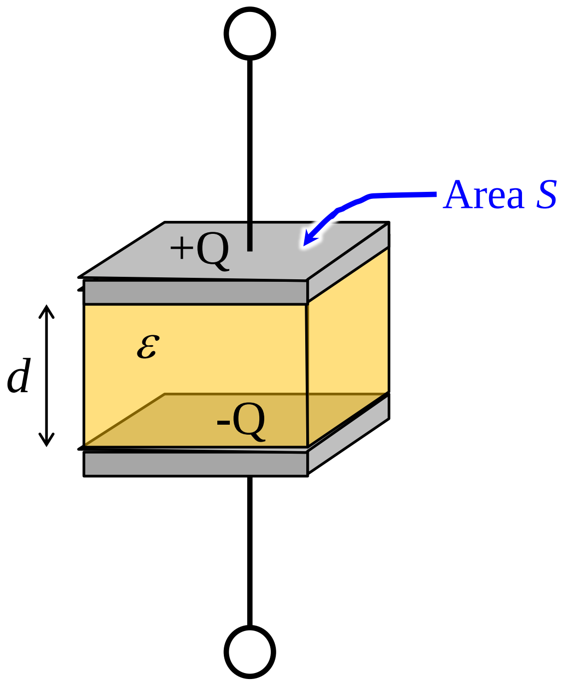


Ignoring fringing fields, the capacitance is

$$C = \varepsilon_0 \varepsilon_r \frac{A}{d}$$



For our example, let the plate dimensions be 1 m x 1m, separated by 1 m of air.

$$C = \left(8.854\times10^{-12}\ \tfrac{\text{F}}{\text{m}}\right)(1.0)\frac{1\ \text{m}^2}{1\ \text{m}} = 8.854\ \text{pF}$$

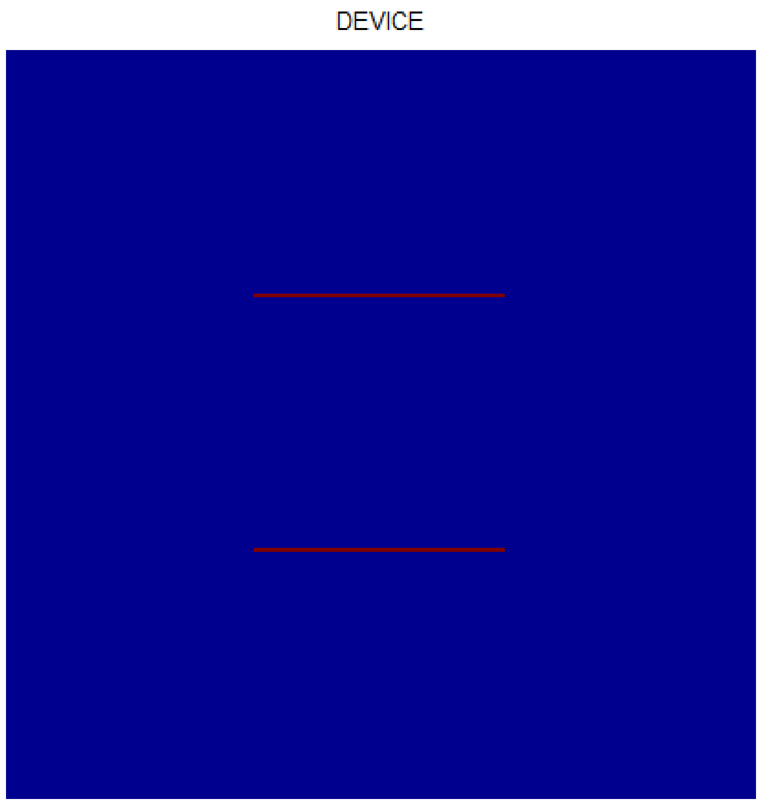
Is this right? To test, the device was analyzed using the finite-difference method.

$$C_{\text{num}} = 18.96\ \text{pF}$$

The model predicts a higher capacitance because there is energy in the fringing fields that was not accounted for previously.

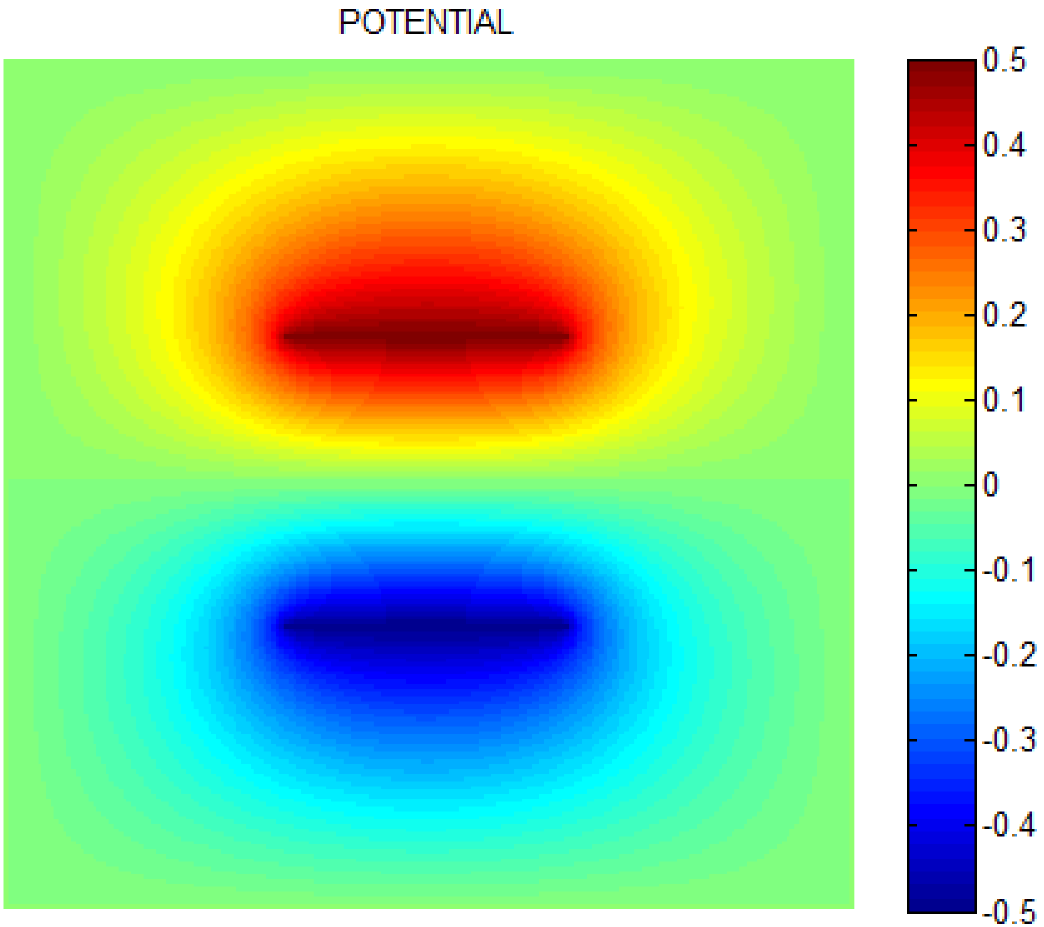
The Model (1 of 4)

Step 1 – We put a device on a 2D grid.



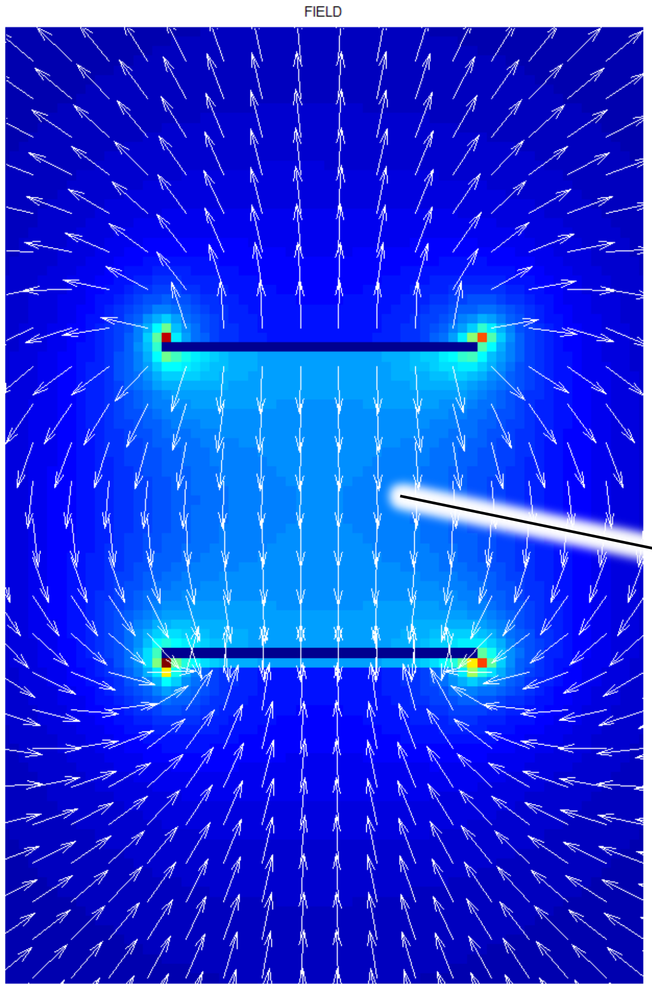
The Model (2 of 4)

Step 2 – Compute the scalar potential by solving $\nabla \bullet \left[\varepsilon_r \left(\nabla V \right) \right] = 0$



The Model (3 of 4)

Step 3 – Calculate the field $\vec{E} = -\nabla V$



We already see that the field is not uniform between the plates due to fringing.

The Model (4 of 4)

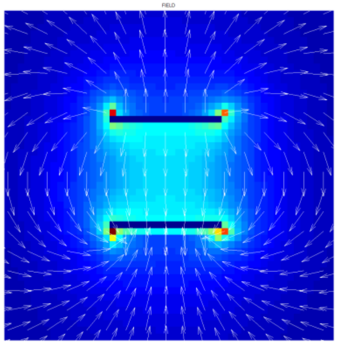
Step 4 – Calculate the electric flux density $\vec{D} = \varepsilon_0 \varepsilon_r \vec{E}$

Step 5 – Calculate the total stored energy $W = \frac{1}{2} \iint_S \left(\vec{D} \bullet \vec{E} \right) ds$

Step 6 – Calculate capacitance $C = \frac{2W}{V_0^2}$

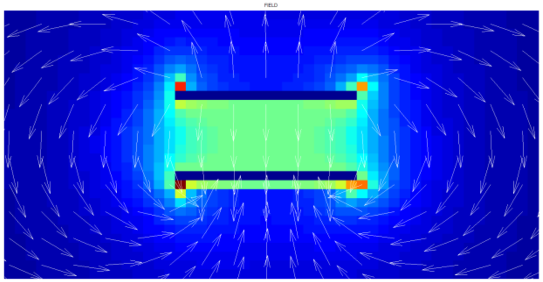
$$C_{\text{num}} = 18.96 \text{ pF}$$

Effect of Separation



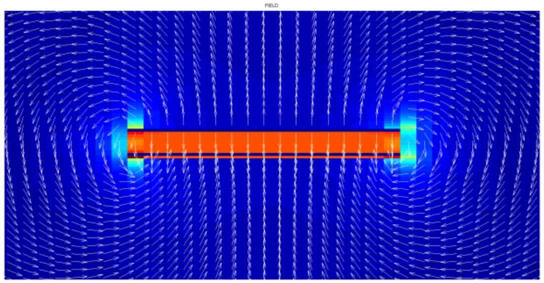
$$\frac{w}{d} = 1$$

53% Error



$$\frac{w}{d} = 2$$

36% Error

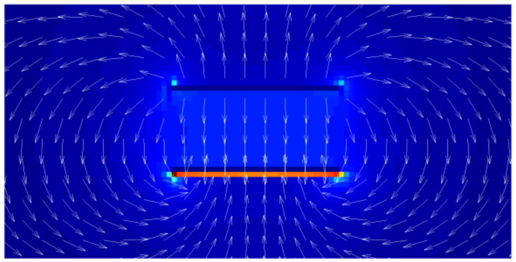


$$\frac{w}{d} = 10$$

6.7% Error

As w becomes much larger than d, the field within the gap is more uniform and less energy resides in the fringing fields. Our simple equation is more accurate.

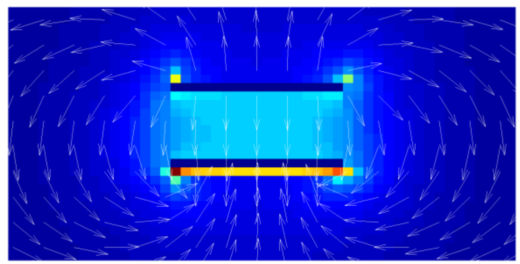
Effect of Dielectric Constant, ε_r



$\varepsilon_r = 1$

$\frac{w}{d} = 2$

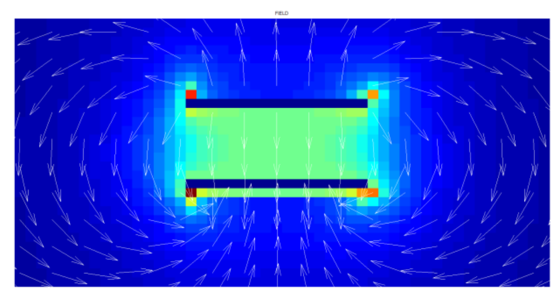
36% Error



$\varepsilon_r = 2$

$\frac{w}{d} = 2$

15% Error



$\varepsilon_r = 5$

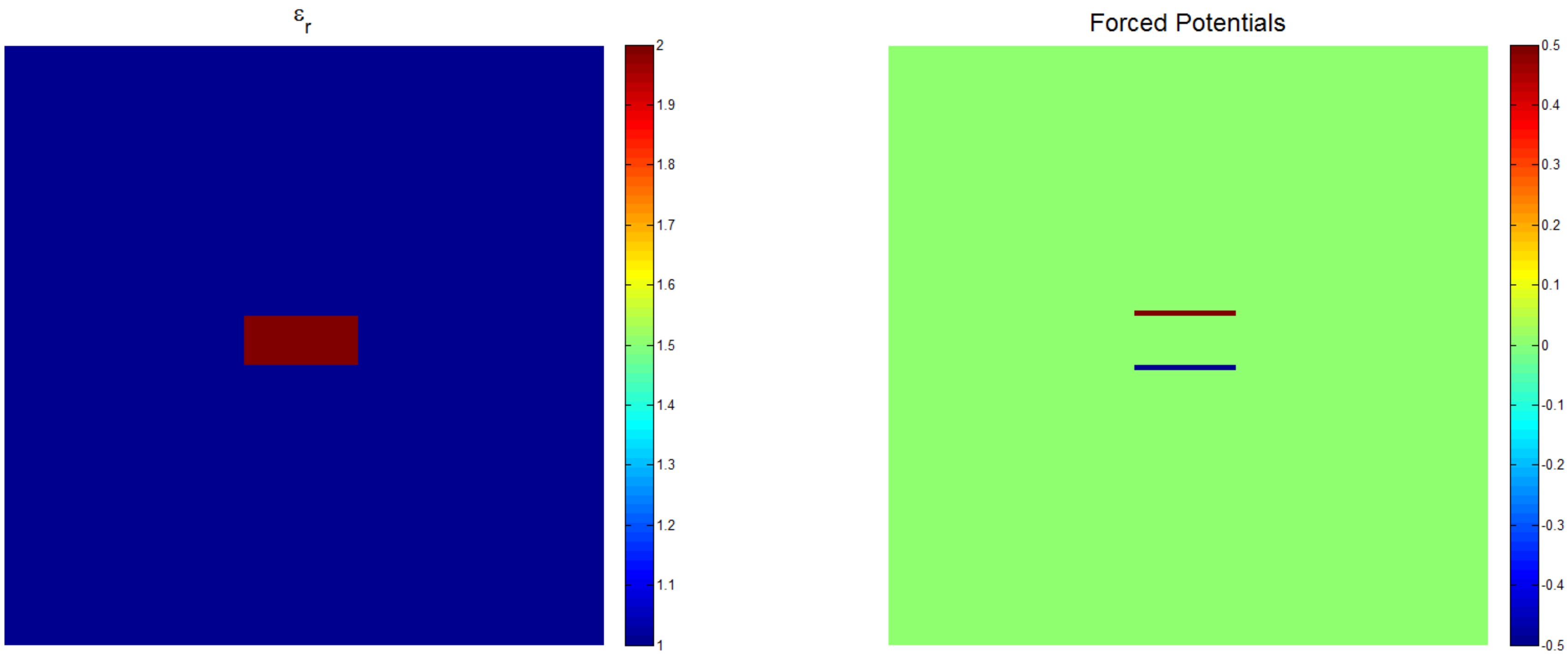
$\frac{w}{d} = 2$

4% Error

As ε_r becomes larger, a greater fraction of energy resides between the plates and the zero-fringing fields approximation becomes more accurate.

How Does the Model Work?

We construct separate grids for the dielectric distribution and the distribution of metals.



How Does the Model Work?

We approximate Laplaces's equation using finite-differences (or finite elements, etc.)

$$\nabla^2 V = 0$$

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0$$

$$\frac{V(i+1,j) - 2V(i,j) + V(i-1,j)}{(\Delta x)^2} + \frac{V(i,j+1) - 2V(i,j) + V(i,j-1)}{(\Delta y)^2} = 0$$

We collect common terms.

$$\left[\frac{2}{(\Delta x)^2} - \frac{2}{(\Delta y)^2}\right] V(i,j) - \frac{1}{(\Delta x)^2} V(i+1,j) - \frac{1}{(\Delta x)^2} V(i-1,j) - \frac{1}{(\Delta y)^2} V(i,j+1) - \frac{1}{(\Delta y)^2} V(i,j-1) = 0$$

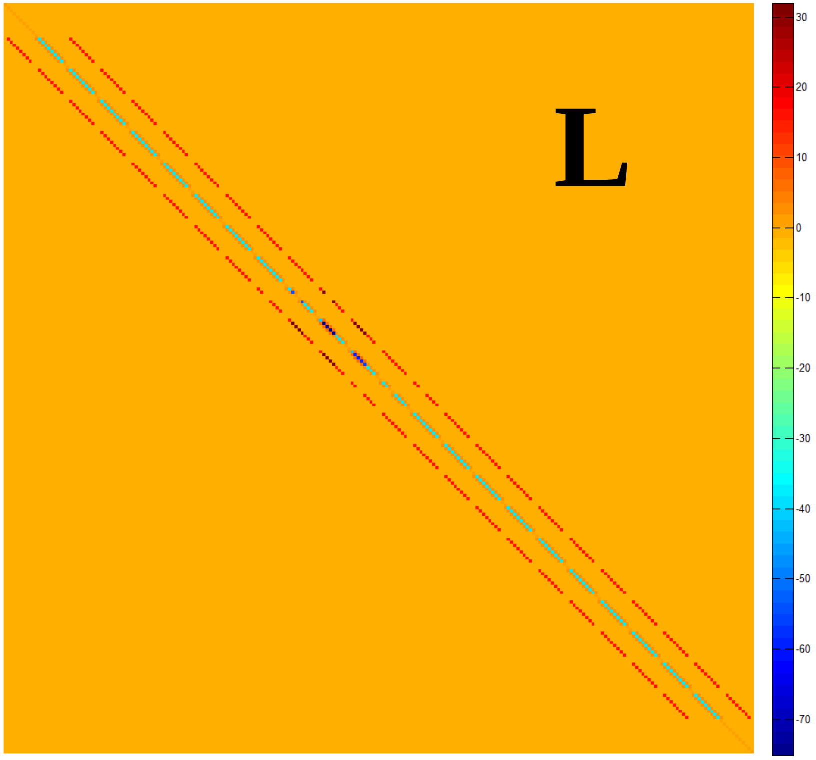
This equation must be satisfied at each point in our grid.

How Does the Model Work?

We write our finite-difference equation at each point on the grid. This large set of equations can be written in matrix form as

$$\mathbf{L}\mathbf{v} = \mathbf{0}$$

$$\mathbf{v} = \begin{bmatrix} V(1,1) \\ V(2,1) \\ V(3,1) \\ \vdots \\ V(N_x, N_y) \end{bmatrix}$$



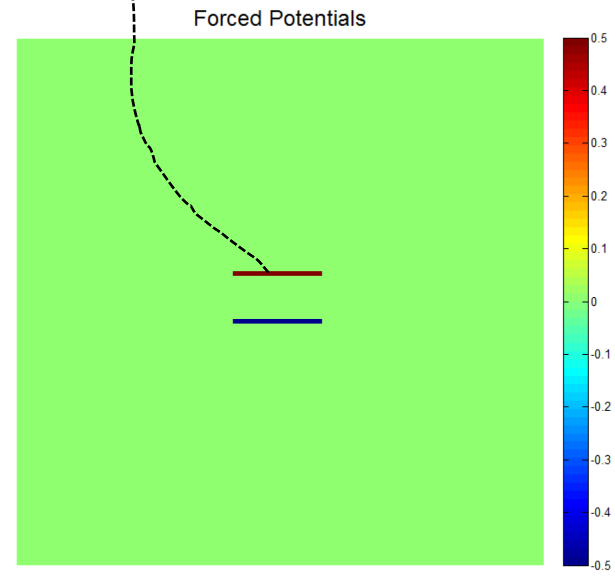
This equation is not yet solvable because

$$\mathbf{v} = \mathbf{L}^{-1}\mathbf{0} = \mathbf{0}$$

How Does the Model Work?

We must incorporate a “source” by enforcing the known potentials.

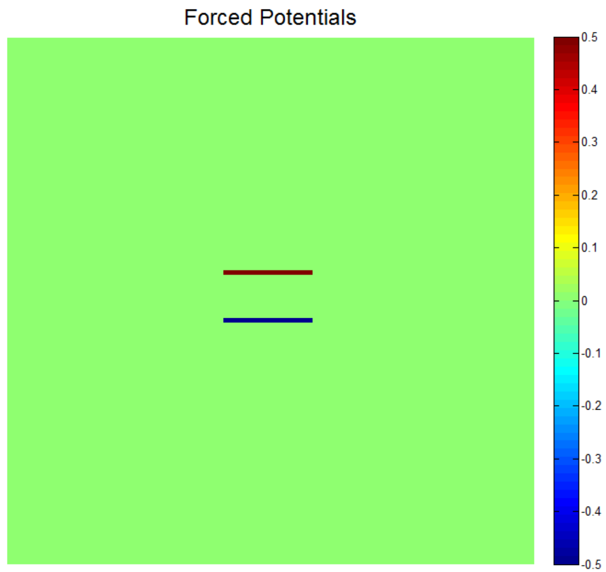
$$
\underbrace{\begin{bmatrix}
(\#) & (\#) & (\#) & \cdots & (\#) & (\#) & (\#) \\
(\#) & (\#) & (\#) & \cdots & (\#) & (\#) & (\#) \\
 & & \ddots & & \iddots & & \\
0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\
 & & \iddots & & \ddots & & \\
(\#) & (\#) & (\#) & \cdots & (\#) & (\#) & (\#) \\
(\#) & (\#) & (\#) & \cdots & (\#) & (\#) & (\#)
\end{bmatrix}}_{\mathbf{L}}
\underbrace{\begin{bmatrix}
V_1 \\ V_2 \\ \vdots \\ V_m^{\text{metal}} \\ \vdots \\ V_{N_x N_y - 1} \\ V_{N_x N_y}
\end{bmatrix}}_{\mathbf{v}}
=
\underbrace{\begin{bmatrix}
0 \\ 0 \\ \vdots \\ V_{\text{applied}} \\ \vdots \\ 0 \\ 0
\end{bmatrix}}_{\mathbf{b}}
$$

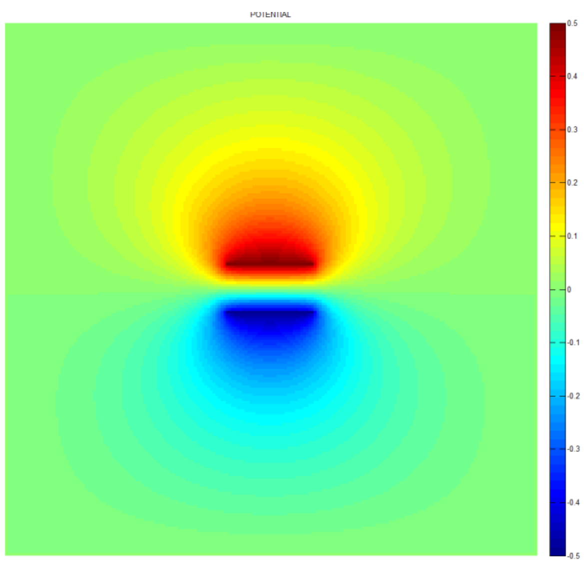


How Does the Model Work?

Calculate the potential

$$\mathbf{v} = \mathbf{L}^{-1}\mathbf{b}$$

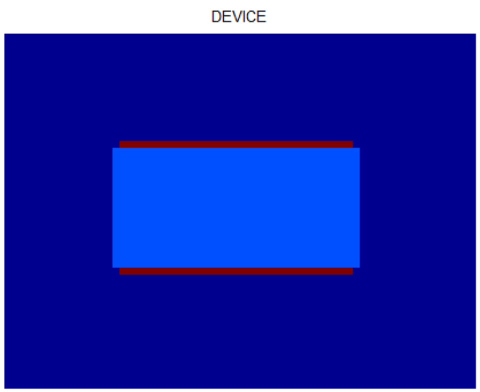


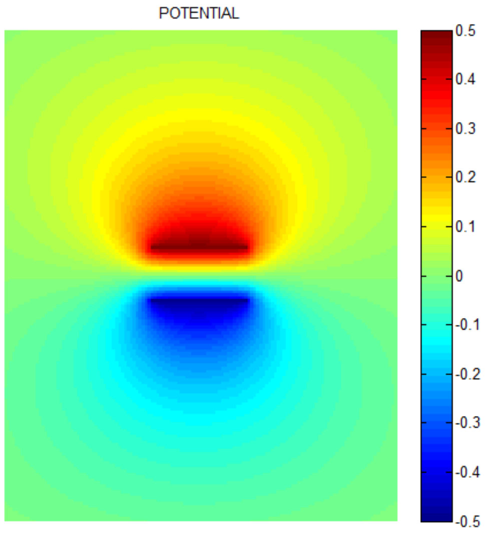


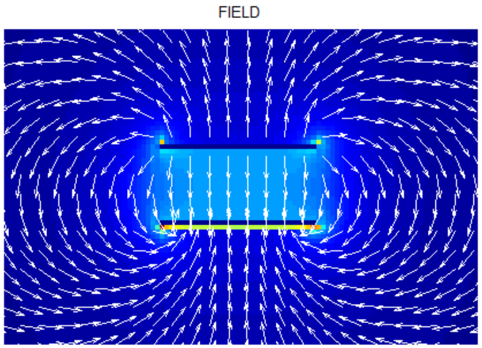
How Does the Model Work?

Calculate the E Field

$$\vec{E} = -\nabla V$$







How Does the Model Work?

Calculate the D Field

$$\vec{D} = \varepsilon_0 \varepsilon_r \vec{E}$$

Calculate total-energy stored

$$W = \frac{1}{2} \iint_{\text{grid}} \left(\vec{D} \bullet \vec{E} \right) ds = \frac{\varepsilon_0}{2} \iint_{\text{grid}} \varepsilon_r \left| \vec{E} \right|^2 ds$$

Calculate capacitance

$$W = \frac{1}{2} C V_0^2 \quad \rightarrow \quad C = \frac{2W}{V_0^2} = \frac{\varepsilon_0}{V_0^2} \iint_{\text{grid}} \varepsilon_r \left| \vec{E} \right|^2 ds$$