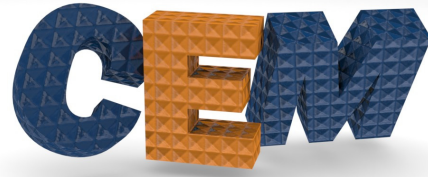


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EE 5337

Computational Electromagnetics (CEM)

Lecture #24

Method of Lines

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Outline



- Formulation of 3D method of lines
- Formulation of 2D method of lines
- Multilayer devices using scattering matrices
- Comparison to RCWA

Definition of Method of Lines



The method of lines was developed by mathematicians to solve partial differential equations (PDEs).

All but one independent variables are discretized.

This leads to a large set of coupled ordinary differential equations (ODEs).

The system of ODEs are solved analytically.

In electromagnetics, the independent variables are usually x , y , and z .

We typically discretize x and y and leave z analytical.

Any method can be used to discretize the independent variables. This includes Fourier transform, finite-differences, finite-elements, etc.

In electromagnetics, the “method of lines” implies that finite-differences are used to discretize x and y .

RCWA is the method of lines, but uses a Fourier transform instead of finite-differences to discretize x and y .

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Sign Convetion



This formulation of the method of lines uses the following sign convention for waves travelling in the $+z$ direction.

$$e^{-jkz}$$

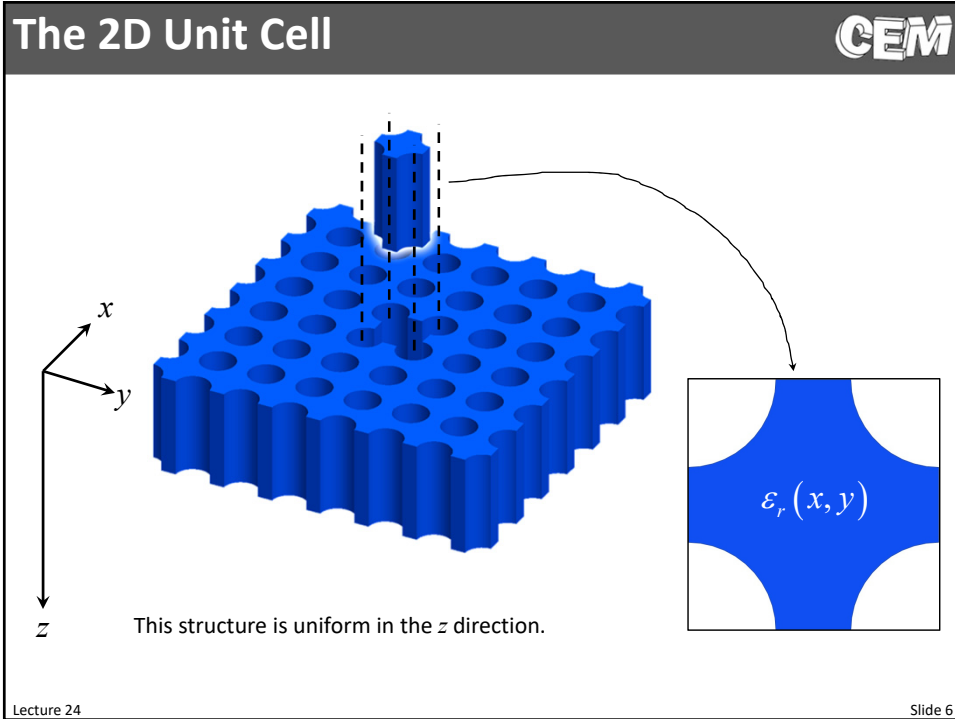
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Formulation of 3D Method of Lines

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Starting Point for MOL



We start with Maxwell's equations in the following form...

$$\frac{\partial E_z}{\partial y'} - \frac{\partial E_y}{\partial z'} = \mu_r \tilde{H}_x$$

$$\frac{\partial \tilde{H}_z}{\partial y'} - \frac{\partial \tilde{H}_y}{\partial z'} = \varepsilon_r E_x$$

$$\frac{\partial E_x}{\partial z'} - \frac{\partial E_z}{\partial x'} = \mu_r \tilde{H}_y$$

$$\frac{\partial \tilde{H}_x}{\partial z'} - \frac{\partial \tilde{H}_z}{\partial x'} = \varepsilon_r E_y$$

$$\frac{\partial E_y}{\partial x'} - \frac{\partial E_x}{\partial y'} = \mu_r \tilde{H}_z$$

$$\frac{\partial \tilde{H}_y}{\partial x'} - \frac{\partial \tilde{H}_x}{\partial y'} = \varepsilon_r E_z$$

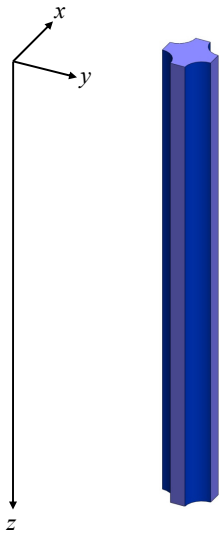
Recall that we normalized the magnetic field and grid according to

$$\tilde{\vec{H}} = -j \sqrt{\frac{\mu_0}{\varepsilon_0}} \vec{H} \quad x' = k_0 x \quad y' = k_0 y \quad z' = k_0 z$$

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z -Uniform Media



We are going to consider Maxwell's equations inside a media that is uniform in the z direction.

The media may be inhomogeneous in the x - y plane, but it must be uniform in the z direction.

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Semi-Analytical Matrix Form of Maxwell's Equations

We can go straight to matrix form using the concept of matrix derivative operators. We keep the z direction analytical.

$$\frac{\partial E_z}{\partial y'} - \frac{\partial E_y}{\partial z'} = \mu_r \tilde{H}_x$$

$$\frac{\partial E_x}{\partial z'} - \frac{\partial E_z}{\partial x'} = \mu_r \tilde{H}_y$$

$$\frac{\partial E_y}{\partial x'} - \frac{\partial E_x}{\partial y'} = \mu_r \tilde{H}_z$$



$$\mathbf{D}_{y'}^e \mathbf{e}_z - \frac{d}{dz'} \mathbf{e}_y = \boldsymbol{\mu}_{xx} \tilde{\mathbf{h}}_x$$

$$\frac{d}{dz'} \mathbf{e}_x - \mathbf{D}_{x'}^e \mathbf{e}_z = \boldsymbol{\mu}_{yy} \tilde{\mathbf{h}}_y$$

$$\mathbf{D}_{x'}^e \mathbf{e}_y - \mathbf{D}_{y'}^e \mathbf{e}_x = \boldsymbol{\mu}_{zz} \tilde{\mathbf{h}}_z$$

$$\frac{\partial \tilde{H}_z}{\partial y'} - \frac{\partial \tilde{H}_y}{\partial z'} = \varepsilon_r E_x$$

$$\frac{\partial \tilde{H}_x}{\partial z'} - \frac{\partial \tilde{H}_z}{\partial x'} = \varepsilon_r E_y$$

$$\frac{\partial \tilde{H}_y}{\partial x'} - \frac{\partial \tilde{H}_x}{\partial y'} = \varepsilon_r E_z$$



$$\mathbf{D}_{y'}^h \tilde{\mathbf{h}}_z - \frac{d}{dz'} \tilde{\mathbf{h}}_y = \boldsymbol{\varepsilon}_{xx} \mathbf{e}_x$$

$$\frac{d}{dz'} \tilde{\mathbf{h}}_x - \mathbf{D}_{x'}^h \tilde{\mathbf{h}}_z = \boldsymbol{\varepsilon}_{yy} \mathbf{e}_y$$

$$\mathbf{D}_{x'}^h \tilde{\mathbf{h}}_y - \mathbf{D}_{y'}^h \tilde{\mathbf{h}}_x = \boldsymbol{\varepsilon}_{zz} \mathbf{e}_z$$

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Eliminate Longitudinal Field Components

We solve the third and sixth equation for \mathbf{e}_z and $\tilde{\mathbf{h}}_z$ and substitute these back into the remaining four equations.

$$\mathbf{D}_{y'}^e \mathbf{e}_z - \frac{d}{dz'} \mathbf{e}_y = \boldsymbol{\mu}_{xx} \tilde{\mathbf{h}}_x \quad \rightarrow \quad \frac{d}{dz'} \mathbf{e}_x = -\mathbf{D}_{x'}^e \boldsymbol{\varepsilon}_{zz}^{-1} \mathbf{D}_{y'}^h \tilde{\mathbf{h}}_x + (\boldsymbol{\mu}_{yy} + \mathbf{D}_{x'}^e \boldsymbol{\varepsilon}_{zz}^{-1} \mathbf{D}_{x'}^h) \tilde{\mathbf{h}}_y$$

$$\frac{d}{dz'} \mathbf{e}_x - \mathbf{D}_{x'}^e \mathbf{e}_z = \boldsymbol{\mu}_{yy} \tilde{\mathbf{h}}_y \quad \rightarrow \quad \frac{d}{dz'} \mathbf{e}_y = -(\boldsymbol{\mu}_{xx} + \mathbf{D}_{y'}^e \boldsymbol{\varepsilon}_{zz}^{-1} \mathbf{D}_{y'}^h) \tilde{\mathbf{h}}_x + \mathbf{D}_{y'}^e \boldsymbol{\varepsilon}_{zz}^{-1} \mathbf{D}_{x'}^h \tilde{\mathbf{h}}_y$$

$$\mathbf{D}_{x'}^e \mathbf{e}_y - \mathbf{D}_{y'}^e \mathbf{e}_x = \boldsymbol{\mu}_{zz} \tilde{\mathbf{h}}_z \quad \Rightarrow \quad \tilde{\mathbf{h}}_z = \boldsymbol{\mu}_{zz}^{-1} (\mathbf{D}_{x'}^e \mathbf{e}_y - \mathbf{D}_{y'}^e \mathbf{e}_x)$$

$$\mathbf{D}_{y'}^h \tilde{\mathbf{h}}_z - \frac{d}{dz'} \tilde{\mathbf{h}}_y = \boldsymbol{\varepsilon}_{xx} \mathbf{e}_x \quad \rightarrow \quad \frac{d}{dz'} \tilde{\mathbf{h}}_x = -\mathbf{D}_{x'}^h \boldsymbol{\mu}_{zz}^{-1} \mathbf{D}_{y'}^e \mathbf{e}_x + (\boldsymbol{\varepsilon}_{yy} + \mathbf{D}_{x'}^h \boldsymbol{\mu}_{zz}^{-1} \mathbf{D}_{x'}^e) \mathbf{e}_y$$

$$\frac{d}{dz'} \tilde{\mathbf{h}}_x - \mathbf{D}_{x'}^h \tilde{\mathbf{h}}_z = \boldsymbol{\varepsilon}_{yy} \mathbf{e}_y \quad \rightarrow \quad \frac{d}{dz'} \tilde{\mathbf{h}}_y = -(\boldsymbol{\varepsilon}_{xx} + \mathbf{D}_{y'}^h \boldsymbol{\mu}_{zz}^{-1} \mathbf{D}_{y'}^e) \mathbf{e}_x + \mathbf{D}_{y'}^h \boldsymbol{\mu}_{zz}^{-1} \mathbf{D}_{x'}^e \mathbf{e}_y$$

$$\mathbf{D}_{x'}^h \tilde{\mathbf{h}}_y - \mathbf{D}_{y'}^h \tilde{\mathbf{h}}_x = \boldsymbol{\varepsilon}_{zz} \mathbf{e}_z \quad \Rightarrow \quad \mathbf{e}_z = \boldsymbol{\varepsilon}_{zz}^{-1} (\mathbf{D}_{x'}^h \tilde{\mathbf{h}}_y - \mathbf{D}_{y'}^h \tilde{\mathbf{h}}_x)$$

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Block Matrix Form



We write the remaining four equations in block matrix form as

$$\begin{aligned} \frac{d}{dz'} \mathbf{e}_y &= -(\boldsymbol{\mu}_{xx} + \mathbf{D}_{y'}^e \boldsymbol{\epsilon}_{zz}^{-1} \mathbf{D}_{y'}^h) \tilde{\mathbf{h}}_x + \mathbf{D}_{y'}^e \boldsymbol{\epsilon}_{zz}^{-1} \mathbf{D}_{x'}^h \tilde{\mathbf{h}}_y \\ \frac{d}{dz'} \mathbf{e}_x &= -\mathbf{D}_{x'}^e \boldsymbol{\epsilon}_{zz}^{-1} \mathbf{D}_{y'}^h \tilde{\mathbf{h}}_x + (\boldsymbol{\mu}_{yy} + \mathbf{D}_{x'}^e \boldsymbol{\epsilon}_{zz}^{-1} \mathbf{D}_{x'}^h) \tilde{\mathbf{h}}_y \end{aligned}$$

$$\frac{d}{dz'} \begin{bmatrix} \mathbf{e}_x \\ \mathbf{e}_y \end{bmatrix} = \begin{bmatrix} -\mathbf{D}_{x'}^e \boldsymbol{\epsilon}_{zz}^{-1} \mathbf{D}_{y'}^h & (\boldsymbol{\mu}_{yy} + \mathbf{D}_{x'}^e \boldsymbol{\epsilon}_{zz}^{-1} \mathbf{D}_{x'}^h) \\ -(\boldsymbol{\mu}_{xx} + \mathbf{D}_{y'}^e \boldsymbol{\epsilon}_{zz}^{-1} \mathbf{D}_{y'}^h) & \mathbf{D}_{y'}^e \boldsymbol{\epsilon}_{zz}^{-1} \mathbf{D}_{x'}^h \end{bmatrix} \begin{bmatrix} \tilde{\mathbf{h}}_x \\ \tilde{\mathbf{h}}_y \end{bmatrix}$$

$$\begin{aligned} \frac{d}{dz'} \tilde{\mathbf{h}}_x &= -\mathbf{D}_{y'}^h \boldsymbol{\mu}_{zz}^{-1} \mathbf{D}_{y'}^e \mathbf{e}_x + (\boldsymbol{\epsilon}_{yy} + \mathbf{D}_{x'}^h \boldsymbol{\mu}_{zz}^{-1} \mathbf{D}_{x'}^e) \mathbf{e}_y \\ \frac{d}{dz'} \tilde{\mathbf{h}}_y &= -(\boldsymbol{\epsilon}_{xx} + \mathbf{D}_{y'}^h \boldsymbol{\mu}_{zz}^{-1} \mathbf{D}_{y'}^e) \mathbf{e}_x + \mathbf{D}_{y'}^h \boldsymbol{\mu}_{zz}^{-1} \mathbf{D}_{x'}^e \mathbf{e}_y \end{aligned}$$

$$\frac{d}{dz'} \begin{bmatrix} \tilde{\mathbf{h}}_x \\ \tilde{\mathbf{h}}_y \end{bmatrix} = \begin{bmatrix} -\mathbf{D}_{y'}^h \boldsymbol{\mu}_{zz}^{-1} \mathbf{D}_{y'}^e & (\boldsymbol{\epsilon}_{yy} + \mathbf{D}_{x'}^h \boldsymbol{\mu}_{zz}^{-1} \mathbf{D}_{x'}^e) \\ -(\boldsymbol{\epsilon}_{xx} + \mathbf{D}_{y'}^h \boldsymbol{\mu}_{zz}^{-1} \mathbf{D}_{y'}^e) & \mathbf{D}_{y'}^h \boldsymbol{\mu}_{zz}^{-1} \mathbf{D}_{x'}^e \end{bmatrix} \begin{bmatrix} \mathbf{e}_x \\ \mathbf{e}_y \end{bmatrix}$$

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Standard PQ Form



We can now write our two equations in the “standard” P and Q form.

$$\frac{d}{dz'} \begin{bmatrix} \mathbf{e}_x \\ \mathbf{e}_y \end{bmatrix} = \mathbf{P} \begin{bmatrix} \tilde{\mathbf{h}}_x \\ \tilde{\mathbf{h}}_y \end{bmatrix} \quad \mathbf{P} = \begin{bmatrix} -\mathbf{D}_{x'}^e \boldsymbol{\epsilon}_{zz}^{-1} \mathbf{D}_{y'}^h & (\boldsymbol{\mu}_{yy} + \mathbf{D}_{x'}^e \boldsymbol{\epsilon}_{zz}^{-1} \mathbf{D}_{x'}^h) \\ -(\boldsymbol{\mu}_{xx} + \mathbf{D}_{y'}^e \boldsymbol{\epsilon}_{zz}^{-1} \mathbf{D}_{y'}^h) & \mathbf{D}_{y'}^e \boldsymbol{\epsilon}_{zz}^{-1} \mathbf{D}_{x'}^h \end{bmatrix}$$

$$\frac{d}{dz'} \begin{bmatrix} \tilde{\mathbf{h}}_x \\ \tilde{\mathbf{h}}_y \end{bmatrix} = \mathbf{Q} \begin{bmatrix} \mathbf{e}_x \\ \mathbf{e}_y \end{bmatrix} \quad \mathbf{Q} = \begin{bmatrix} -\mathbf{D}_{y'}^h \boldsymbol{\mu}_{zz}^{-1} \mathbf{D}_{y'}^e & (\boldsymbol{\epsilon}_{yy} + \mathbf{D}_{x'}^h \boldsymbol{\mu}_{zz}^{-1} \mathbf{D}_{x'}^e) \\ -(\boldsymbol{\epsilon}_{xx} + \mathbf{D}_{y'}^h \boldsymbol{\mu}_{zz}^{-1} \mathbf{D}_{y'}^e) & \mathbf{D}_{y'}^h \boldsymbol{\mu}_{zz}^{-1} \mathbf{D}_{x'}^e \end{bmatrix}$$

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Matrix Wave Equation



We differentiate the “P” equation with respect to z' and substitute the “Q” equation into the result to derive the matrix wave equation.

$$\begin{aligned} \frac{d}{dz'} \begin{bmatrix} \mathbf{e}_x \\ \mathbf{e}_y \end{bmatrix} &= \mathbf{P} \begin{bmatrix} \tilde{\mathbf{h}}_x \\ \tilde{\mathbf{h}}_y \end{bmatrix} & \frac{d}{dz'} \begin{bmatrix} \tilde{\mathbf{h}}_x \\ \tilde{\mathbf{h}}_y \end{bmatrix} &= \mathbf{Q} \begin{bmatrix} \mathbf{e}_x \\ \mathbf{e}_y \end{bmatrix} \\ \frac{d^2}{dz'^2} \begin{bmatrix} \mathbf{e}_x \\ \mathbf{e}_y \end{bmatrix} &= \mathbf{P} \cdot \frac{d}{dz'} \begin{bmatrix} \tilde{\mathbf{h}}_x \\ \tilde{\mathbf{h}}_y \end{bmatrix} \\ \frac{d^2}{dz'^2} \begin{bmatrix} \mathbf{e}_x \\ \mathbf{e}_y \end{bmatrix} &= \mathbf{PQ} \begin{bmatrix} \mathbf{e}_x \\ \mathbf{e}_y \end{bmatrix} \\ \frac{d^2}{dz'^2} \begin{bmatrix} \mathbf{e}_x \\ \mathbf{e}_y \end{bmatrix} - \mathbf{PQ} \begin{bmatrix} \mathbf{e}_x \\ \mathbf{e}_y \end{bmatrix} &= \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \end{bmatrix} \implies \frac{d^2}{dz'^2} \begin{bmatrix} \mathbf{e}_x \\ \mathbf{e}_y \end{bmatrix} - \mathbf{\Omega}^2 \begin{bmatrix} \mathbf{e}_x \\ \mathbf{e}_y \end{bmatrix} = \mathbf{0} \\ & \mathbf{\Omega}^2 = \mathbf{PQ} \end{aligned}$$

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Solution to the Wave Equation



The solution to the wave equation is written the same as in RCWA and TMM.

$$\begin{bmatrix} \mathbf{e}_x \\ \mathbf{e}_y \end{bmatrix} = \mathbf{W} e^{-\lambda z'} \mathbf{c}^+ + \mathbf{W} e^{\lambda z'} \mathbf{c}^-$$

where \mathbf{W} and λ^2 are the eigen-vectors and eigen-values of $\mathbf{\Omega}^2$

The overall solution is then

$$\Psi(\tilde{z}) = \begin{bmatrix} \mathbf{e}_x(\tilde{z}) \\ \mathbf{e}_y(\tilde{z}) \\ \tilde{\mathbf{h}}_x(\tilde{z}) \\ \tilde{\mathbf{h}}_y(\tilde{z}) \end{bmatrix} = \begin{bmatrix} \mathbf{W} & \mathbf{W} \\ -\mathbf{V} & \mathbf{V} \end{bmatrix} \begin{bmatrix} e^{-\lambda \tilde{z}} & \mathbf{0} \\ \mathbf{0} & e^{\lambda \tilde{z}} \end{bmatrix} \begin{bmatrix} \mathbf{c}^+ \\ \mathbf{c}^- \end{bmatrix} \quad \mathbf{V} = \mathbf{QW}\lambda^{-1}$$

Based on previous lectures, you know the rest of this story!

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Interpretation of the Solution



$$\Psi(\tilde{z}) = \mathbf{W} e^{\lambda \tilde{z}} \mathbf{c}$$

$\Psi(z)$ – Overall solution which is the sum of all the modes at plane z' .

\mathbf{c} – Column vector containing the amplitude coefficient of each of the modes. This quantities how much energy is in each mode.

\mathbf{W} – Square matrix who's column vectors describe the “modes” that can exist in the material. These are essentially pictures of the modes which quantify the relative amplitudes of $E_{x'}$, $E_{y'}$, $H_{x'}$, and $H_{y'}$.

$e^{\lambda \tilde{z}}$ – Diagonal matrix describing how the modes propagate. This includes accumulation of phase as well as decaying (loss) or growing (gain) amplitude.

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Solution in a Homogeneous Layer

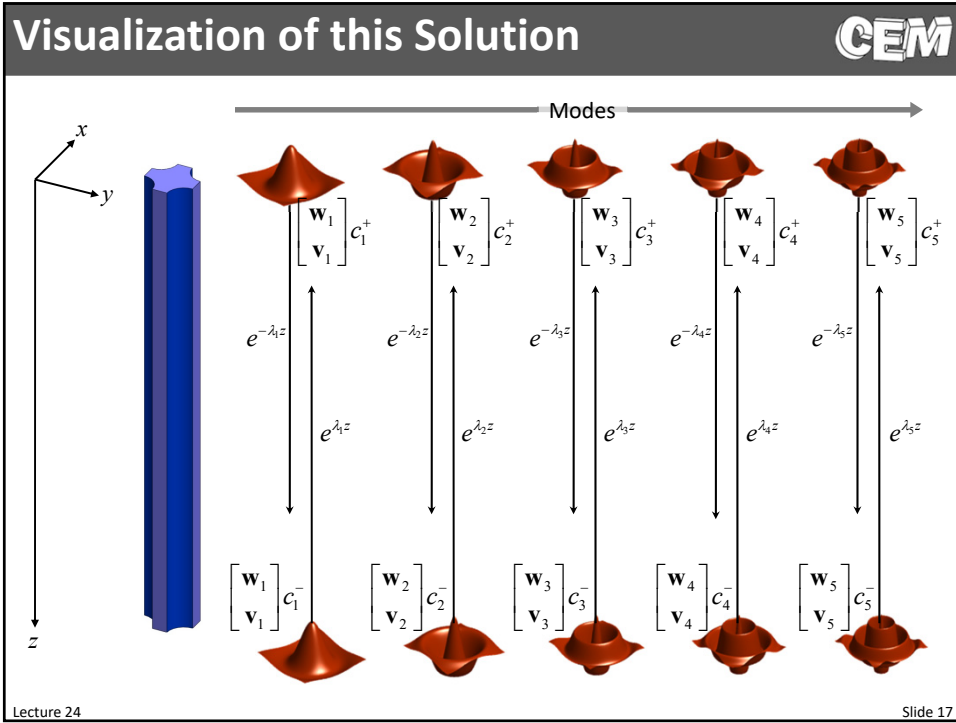


In a homogeneous layer, the \mathbf{P} and \mathbf{Q} matrices are

$$\mathbf{P} = \frac{1}{\epsilon_r} \begin{bmatrix} -\mathbf{D}_{x'}^e \mathbf{D}_{y'}^h & (\mu_r \epsilon_r \mathbf{I} + \mathbf{D}_{x'}^e \mathbf{D}_{x'}^h) \\ -(\mu_r \epsilon_r \mathbf{I} + \mathbf{D}_{y'}^e \mathbf{D}_{y'}^h) & \mathbf{D}_{y'}^e \mathbf{D}_{x'}^h \end{bmatrix} \quad \mathbf{Q} = \frac{1}{\mu_r} \begin{bmatrix} -\mathbf{D}_{x'}^h \mathbf{D}_{y'}^e & (\mu_r \epsilon_r \mathbf{I} + \mathbf{D}_{x'}^h \mathbf{D}_{x'}^e) \\ -(\mu_r \epsilon_r \mathbf{I} + \mathbf{D}_{y'}^h \mathbf{D}_{y'}^e) & \mathbf{D}_{y'}^h \mathbf{D}_{x'}^e \end{bmatrix}$$

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Formulation of 2D Method of Lines

Starting Point



We will start with Maxwell's equations in real-space and in matrix form as derived earlier this lecture.

$$\frac{\partial E_z}{\partial y'} - \frac{\partial E_y}{\partial z'} = \mu_r \tilde{H}_x$$

$$\frac{\partial E_x}{\partial z'} - \frac{\partial E_z}{\partial x'} = \mu_r \tilde{H}_y$$

$$\frac{\partial E_y}{\partial x'} - \frac{\partial E_x}{\partial y'} = \mu_r \tilde{H}_z$$



$$\mathbf{D}_{y'}^e \mathbf{e}_z - \frac{d}{dz'} \mathbf{e}_y = \boldsymbol{\mu}_{xx} \tilde{\mathbf{h}}_x$$

$$\frac{d}{dz'} \mathbf{e}_x - \mathbf{D}_{x'}^e \mathbf{e}_z = \boldsymbol{\mu}_{yy} \tilde{\mathbf{h}}_y$$

$$\mathbf{D}_{x'}^e \mathbf{e}_y - \mathbf{D}_{y'}^e \mathbf{e}_x = \boldsymbol{\mu}_{zz} \tilde{\mathbf{h}}_z$$

$$\frac{\partial \tilde{H}_z}{\partial y'} - \frac{\partial \tilde{H}_y}{\partial z'} = \epsilon_r E_x$$

$$\frac{\partial \tilde{H}_x}{\partial z'} - \frac{\partial \tilde{H}_z}{\partial x'} = \epsilon_r E_y$$

$$\frac{\partial \tilde{H}_y}{\partial x'} - \frac{\partial \tilde{H}_x}{\partial y'} = \epsilon_r E_z$$



$$\mathbf{D}_{y'}^h \tilde{\mathbf{h}}_z - \frac{d}{dz'} \tilde{\mathbf{h}}_y = \boldsymbol{\epsilon}_{xx} \mathbf{e}_x$$

$$\frac{d}{dz'} \tilde{\mathbf{h}}_x - \mathbf{D}_{x'}^h \tilde{\mathbf{h}}_z = \boldsymbol{\epsilon}_{yy} \mathbf{e}_y$$

$$\mathbf{D}_{x'}^h \tilde{\mathbf{h}}_y - \mathbf{D}_{y'}^h \tilde{\mathbf{h}}_x = \boldsymbol{\epsilon}_{zz} \mathbf{e}_z$$

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Reduction to Two Dimensions



For diagonally anisotropic devices that are uniform in the y direction and when there is no wave propagation in the y direction, we have

$$\mathbf{D}_{y'}^e = \mathbf{D}_{y'}^h = \mathbf{0}$$

Maxwell's equations reduce to

$$-\frac{d}{dz'} \mathbf{e}_y = \boldsymbol{\mu}_{xx} \tilde{\mathbf{h}}_x$$

$$\frac{d}{dz'} \mathbf{e}_x - \mathbf{D}_{x'}^e \mathbf{e}_z = \boldsymbol{\mu}_{yy} \tilde{\mathbf{h}}_y$$

$$\mathbf{D}_{x'}^e \mathbf{e}_y = \boldsymbol{\mu}_{zz} \tilde{\mathbf{h}}_z$$

$$-\frac{d}{dz'} \tilde{\mathbf{h}}_y = \boldsymbol{\epsilon}_{xx} \mathbf{e}_x$$

$$\frac{d}{dz'} \tilde{\mathbf{h}}_x - \mathbf{D}_{x'}^h \tilde{\mathbf{h}}_z = \boldsymbol{\epsilon}_{yy} \mathbf{e}_y$$

$$\mathbf{D}_{x'}^h \tilde{\mathbf{h}}_y = \boldsymbol{\epsilon}_{zz} \mathbf{e}_z$$

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Two Independent Modes



Maxwell's equations have decoupled into two independent modes.

$$\begin{aligned} -\frac{d}{dz'} \mathbf{e}_y &= \mu_{xx} \tilde{\mathbf{h}}_x \\ \frac{d}{dz'} \mathbf{e}_x - \mathbf{D}_x^e \mathbf{e}_z &= \mu_{yy} \tilde{\mathbf{h}}_y \\ \mathbf{D}_x^e \mathbf{e}_y &= \mu_{zz} \tilde{\mathbf{h}}_z \end{aligned}$$

$$\begin{aligned} -\frac{d}{dz'} \tilde{\mathbf{h}}_y &= \epsilon_{xx} \mathbf{e}_x \\ \frac{d}{dz'} \tilde{\mathbf{h}}_x - \mathbf{D}_x^h \tilde{\mathbf{h}}_z &= \epsilon_{yy} \mathbf{e}_y \\ \mathbf{D}_x^h \tilde{\mathbf{h}}_y &= \epsilon_{zz} \mathbf{e}_z \end{aligned}$$

E Mode

$$\begin{aligned} \frac{d}{dz'} \tilde{\mathbf{h}}_x - \mathbf{D}_x^h \tilde{\mathbf{h}}_z &= \epsilon_{yy} \mathbf{e}_y \\ -\frac{d}{dz'} \mathbf{e}_y &= \mu_{xx} \tilde{\mathbf{h}}_x \\ \mathbf{D}_x^e \mathbf{e}_y &= \mu_{zz} \tilde{\mathbf{h}}_z \end{aligned}$$

H Mode

$$\begin{aligned} \frac{d}{dz'} \mathbf{e}_x - \mathbf{D}_x^e \mathbf{e}_z &= \mu_{yy} \tilde{\mathbf{h}}_y \\ -\frac{d}{dz'} \tilde{\mathbf{h}}_y &= \epsilon_{xx} \mathbf{e}_x \\ \mathbf{D}_x^h \tilde{\mathbf{h}}_y &= \epsilon_{zz} \mathbf{e}_z \end{aligned}$$

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Apply Dielectric Smoothing



To improve convergence, we can incorporate dielectric smoothing as follows:

E Mode

$$\begin{aligned} \frac{d}{dz'} \tilde{\mathbf{h}}_x - \mathbf{D}_x^h \tilde{\mathbf{h}}_z &= \langle \epsilon_{yy} \rangle \mathbf{e}_y \\ -\frac{d}{dz'} \mathbf{e}_y &= \langle \mu_{xx}^{-1} \rangle^{-1} \tilde{\mathbf{h}}_x \\ \mathbf{D}_x^e \mathbf{e}_y &= \langle \mu_{zz} \rangle \tilde{\mathbf{h}}_z \end{aligned}$$

H Mode

$$\begin{aligned} \frac{d}{dz'} \mathbf{e}_x - \mathbf{D}_x^e \mathbf{e}_z &= \langle \mu_{yy} \rangle \tilde{\mathbf{h}}_y \\ -\frac{d}{dz'} \tilde{\mathbf{h}}_y &= \langle \epsilon_{xx}^{-1} \rangle^{-1} \mathbf{e}_x \\ \mathbf{D}_x^h \tilde{\mathbf{h}}_y &= \langle \epsilon_{zz} \rangle \mathbf{e}_z \end{aligned}$$

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Eliminate Longitudinal Components



We solve for the longitudinal components

$$\tilde{h}_z = \langle \mu_{zz} \rangle^{-1} \mathbf{D}_x^e \mathbf{e}_y$$

$$\mathbf{e}_z = \langle \epsilon_{zz} \rangle^{-1} \mathbf{D}_x^h \tilde{h}_y$$

and substitute these expressions into the remaining equations.

E Mode

$$\frac{d}{dz'} \tilde{h}_x = \langle \epsilon_{yy} \rangle \mathbf{e}_y + \mathbf{D}_{x'}^h \langle \mu_{zz} \rangle^{-1} \mathbf{D}_x^e \mathbf{e}_y$$

$$\frac{d}{dz'} \mathbf{e}_y = -\langle \mu_{xx}^{-1} \rangle^{-1} \tilde{h}_x$$

H Mode

$$\frac{d}{dz'} \mathbf{e}_x = \langle \mu_{yy} \rangle \tilde{h}_y + \mathbf{D}_{x'}^e \langle \epsilon_{zz} \rangle^{-1} \mathbf{D}_x^h \tilde{h}_y$$

$$\frac{d}{dz'} \tilde{h}_y = -\langle \epsilon_{xx}^{-1} \rangle^{-1} \mathbf{e}_x$$

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Standard P and Q Form



We write our matrix equations in the standard **PQ** form.

E Mode

$$\frac{d}{dz'} \mathbf{e}_y = \mathbf{P} \tilde{h}_x$$

$$\frac{d}{dz'} \tilde{h}_x = \mathbf{Q} \mathbf{e}_y$$

$$\mathbf{P} = -\langle \mu_{xx}^{-1} \rangle^{-1}$$

$$\mathbf{Q} = \langle \epsilon_{yy} \rangle + \mathbf{D}_{x'}^h \langle \mu_{zz} \rangle^{-1} \mathbf{D}_x^e$$

H Mode

$$\frac{d}{dz'} \tilde{h}_y = \mathbf{P} \mathbf{e}_x$$

$$\frac{d}{dz'} \mathbf{e}_x = \mathbf{Q} \tilde{h}_y$$

$$\mathbf{P} = -\langle \epsilon_{xx}^{-1} \rangle^{-1}$$

$$\mathbf{Q} = \langle \mu_{yy} \rangle + \mathbf{D}_{x'}^e \langle \epsilon_{zz} \rangle^{-1} \mathbf{D}_x^h$$

...and now you know the rest of the story.

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Multilayer Devices Using Scattering Matrices

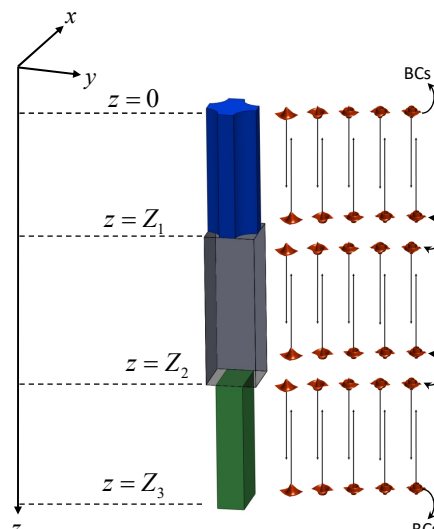
R. C. Rumpf, "Improved formulation of scattering matrices for semi-analytical methods that is consistent with convention," PIERS B, Vol. 35, 241-261, 2011.

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Eigen System in Each Layer





Boundary conditions required that all layers have the same K_x and K_y matrices.

$P_1 = \begin{bmatrix} -D_y^t \epsilon_{z,z}^+ D_y^t & (\mu_{z,yy} + D_y^t \epsilon_{z,z}^+ D_y^t) \\ -(\mu_{z,xx} + D_y^t \epsilon_{z,z}^+ D_y^t) & D_y^t \epsilon_{z,z}^+ D_y^t \end{bmatrix}$ $\Omega_1^z = P_1 Q_1 \rightarrow W_1, \lambda_1 \rightarrow V_1 \quad c_1^+, c_1^-$	$Q_1 = \begin{bmatrix} -D_y^t \mu_{z,z}^+ D_y^t & (\epsilon_{z,yy} + D_y^t \mu_{z,z}^+ D_y^t) \\ -(\epsilon_{z,xx} + D_y^t \mu_{z,z}^+ D_y^t) & D_y^t \mu_{z,z}^+ D_y^t \end{bmatrix}$
$P_2 = \begin{bmatrix} -D_y^t \epsilon_{z,z}^+ D_y^t & (\mu_{z,yy} + D_y^t \epsilon_{z,z}^+ D_y^t) \\ -(\mu_{z,xx} + D_y^t \epsilon_{z,z}^+ D_y^t) & D_y^t \epsilon_{z,z}^+ D_y^t \end{bmatrix}$ $\Omega_2^z = P_2 Q_2 \rightarrow W_2, \lambda_2 \rightarrow V_2 \quad c_2^+, c_2^-$	$Q_2 = \begin{bmatrix} -D_y^t \mu_{z,z}^+ D_y^t & (\epsilon_{z,yy} + D_y^t \mu_{z,z}^+ D_y^t) \\ -(\epsilon_{z,xx} + D_y^t \mu_{z,z}^+ D_y^t) & D_y^t \mu_{z,z}^+ D_y^t \end{bmatrix}$
$P_3 = \begin{bmatrix} -D_y^t \epsilon_{z,z}^+ D_y^t & (\mu_{z,yy} + D_y^t \epsilon_{z,z}^+ D_y^t) \\ -(\mu_{z,xx} + D_y^t \epsilon_{z,z}^+ D_y^t) & D_y^t \epsilon_{z,z}^+ D_y^t \end{bmatrix}$ $\Omega_3^z = P_3 Q_3 \rightarrow W_3, \lambda_3 \rightarrow V_3 \quad c_3^+, c_3^-$	$Q_3 = \begin{bmatrix} -D_y^t \mu_{z,z}^+ D_y^t & (\epsilon_{z,yy} + D_y^t \mu_{z,z}^+ D_y^t) \\ -(\epsilon_{z,xx} + D_y^t \mu_{z,z}^+ D_y^t) & D_y^t \mu_{z,z}^+ D_y^t \end{bmatrix}$

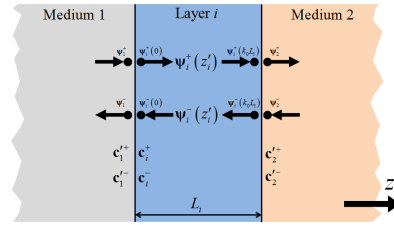
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Field Relations & Boundary Conditions CEM

Field inside the i^{th} layer:

$$\Psi_i(\tilde{z}) = \begin{bmatrix} \mathbf{e}_x(\tilde{z}) \\ \mathbf{e}_y(\tilde{z}) \\ \tilde{\mathbf{h}}_x(\tilde{z}) \\ \tilde{\mathbf{h}}_y(\tilde{z}) \end{bmatrix} = \begin{bmatrix} \mathbf{W}_i & \mathbf{W}_i \\ -\mathbf{V}_i & \mathbf{V}_i \end{bmatrix} \begin{bmatrix} e^{-\lambda_i \tilde{z}} & \mathbf{0} \\ \mathbf{0} & e^{\lambda_i \tilde{z}} \end{bmatrix} \begin{bmatrix} \mathbf{c}_i^+ \\ \mathbf{c}_i^- \end{bmatrix}$$



Boundary conditions at the first interface:

$$\Psi_1 = \Psi_i(0)$$

$$\begin{bmatrix} \mathbf{W}_1 & \mathbf{W}_1 \\ -\mathbf{V}_1 & \mathbf{V}_1 \end{bmatrix} \begin{bmatrix} \mathbf{c}_1^{'+} \\ \mathbf{c}_1'^- \end{bmatrix} = \begin{bmatrix} \mathbf{W}_i & \mathbf{W}_i \\ -\mathbf{V}_i & \mathbf{V}_i \end{bmatrix} \begin{bmatrix} \mathbf{c}_i^+ \\ \mathbf{c}_i^- \end{bmatrix}$$

Boundary conditions at the second interface:

$$\Psi_i(k_0 L_i) = \Psi_2$$

$$\begin{bmatrix} \mathbf{W}_i & \mathbf{W}_i \\ -\mathbf{V}_i & \mathbf{V}_i \end{bmatrix} \begin{bmatrix} e^{-\lambda_i k_0 L_i} & \mathbf{0} \\ \mathbf{0} & e^{\lambda_i k_0 L_i} \end{bmatrix} \begin{bmatrix} \mathbf{c}_i^+ \\ \mathbf{c}_i^- \end{bmatrix} = \begin{bmatrix} \mathbf{W}_2 & \mathbf{W}_2 \\ -\mathbf{V}_2 & \mathbf{V}_2 \end{bmatrix} \begin{bmatrix} \mathbf{c}_2^{'+} \\ \mathbf{c}_2'^- \end{bmatrix}$$

Note: k_0 has been incorporated to normalize L_i .

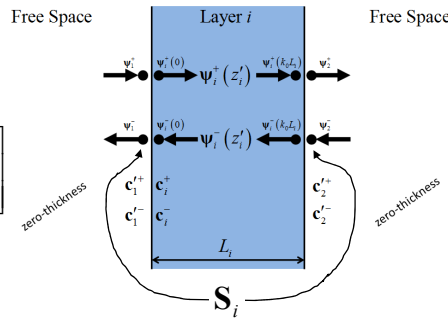
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Adopt the Symmetric S-Matrix Approach CEM

The scattering matrix \mathbf{S}_i of the i^{th} layer is still defined as:

$$\begin{bmatrix} \mathbf{c}_1'^- \\ \mathbf{c}_2'^+ \end{bmatrix} = \mathbf{S}^{(i)} \begin{bmatrix} \mathbf{c}_1'^+ \\ \mathbf{c}_2'^- \end{bmatrix} \quad \mathbf{S}^{(i)} = \begin{bmatrix} \mathbf{S}_{11}^{(i)} & \mathbf{S}_{12}^{(i)} \\ \mathbf{S}_{21}^{(i)} & \mathbf{S}_{22}^{(i)} \end{bmatrix}$$



But the elements are calculated as

$$\mathbf{S}_{11}^{(i)} = (\mathbf{A}_i - \mathbf{X}_i \mathbf{B}_i \mathbf{A}_i^{-1} \mathbf{X}_i \mathbf{B}_i)^{-1} (\mathbf{X}_i \mathbf{B}_i \mathbf{A}_i^{-1} \mathbf{X}_i \mathbf{A}_i - \mathbf{B}_i)$$

$$\mathbf{S}_{12}^{(i)} = (\mathbf{A}_i - \mathbf{X}_i \mathbf{B}_i \mathbf{A}_i^{-1} \mathbf{X}_i \mathbf{B}_i)^{-1} \mathbf{X}_i (\mathbf{A}_i - \mathbf{B}_i \mathbf{A}_i^{-1} \mathbf{B}_i)$$

$$\mathbf{S}_{21}^{(i)} = \mathbf{S}_{12}^{(i)}$$

$$\mathbf{S}_{22}^{(i)} = \mathbf{S}_{11}^{(i)}$$

- Layers are symmetric so the scattering matrix elements have redundancy.
- Scattering matrix equations are simplified.
- Fewer calculations.
- Less memory storage.

$$\mathbf{A}_i = \mathbf{W}_i^{-1} \mathbf{W}_0 + \mathbf{V}_i^{-1} \mathbf{V}_0$$

$$\mathbf{B}_i = \mathbf{W}_i^{-1} \mathbf{W}_0 - \mathbf{V}_i^{-1} \mathbf{V}_0$$

$$\mathbf{X}_i = e^{-\lambda_i k_0 L_i}$$

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Global Scattering Matrix CEM

Scattering matrix for all layers.

$$\mathbf{S}^{(\text{device})} = \mathbf{S}^{(1)} \otimes \mathbf{S}^{(2)} \otimes \mathbf{S}^{(3)}$$

Connection to outside regions

$$\mathbf{S}^{(\text{global})} = \mathbf{S}^{(\text{ref})} \otimes \mathbf{S}^{(\text{device})} \otimes \mathbf{S}^{(\text{tm})}$$

Recall this procedure from Lecture 5.

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Reflection/Transmission Side Scattering Matrices CEM

The reflection-side scattering matrix is

$$\mathbf{S}_{11}^{(\text{ref})} = -\mathbf{A}_{\text{ref}}^{-1} \mathbf{B}_{\text{ref}}$$

$$\mathbf{S}_{12}^{(\text{ref})} = 2\mathbf{A}_{\text{ref}}^{-1}$$

$$\mathbf{S}_{21}^{(\text{ref})} = 0.5(\mathbf{A}_{\text{ref}} - \mathbf{B}_{\text{ref}} \mathbf{A}_{\text{ref}}^{-1} \mathbf{B}_{\text{ref}})$$

$$\mathbf{S}_{22}^{(\text{ref})} = \mathbf{B}_{\text{ref}} \mathbf{A}_{\text{ref}}^{-1}$$

The transmission-side scattering matrix is

$$\mathbf{S}_{11}^{(\text{tm})} = \mathbf{B}_{\text{tm}} \mathbf{A}_{\text{tm}}^{-1}$$

$$\mathbf{S}_{12}^{(\text{tm})} = 0.5(\mathbf{A}_{\text{tm}} - \mathbf{B}_{\text{tm}} \mathbf{A}_{\text{tm}}^{-1} \mathbf{B}_{\text{tm}})$$

$$\mathbf{S}_{21}^{(\text{tm})} = 2\mathbf{A}_{\text{tm}}^{-1}$$

$$\mathbf{S}_{22}^{(\text{tm})} = -\mathbf{A}_{\text{tm}}^{-1} \mathbf{B}_{\text{tm}}$$

$\lim_{L \rightarrow 0}$

$\lim_{L \rightarrow 0}$

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Calculating the Transmitted and Reflected Fields



The electric field source is calculated as

$$\mathbf{c}_{\text{inc}} = \mathbf{W}_{\text{ref}}^{-1} \vec{\mathbf{e}}_{xy}^{\text{inc}} \quad \vec{\mathbf{e}}_{xy}^{\text{inc}} = \begin{bmatrix} p_x e^{-j(k_{x,\text{inc}}x + k_{y,\text{inc}}y)} \\ p_y e^{-j(k_{x,\text{inc}}x + k_{y,\text{inc}}y)} \end{bmatrix}$$

x and y are column vectors containing the coordinates of the points in the cross section.

Recall calculating the polarization components p_x and p_y in TMM.

$$|\vec{p}| = 1$$

Given the global scattering matrix, the coefficients for the reflected and transmitted fields are

$$\mathbf{c}_{\text{ref}} = \mathbf{S}_{11} \mathbf{c}_{\text{inc}} \quad \mathbf{c}_{\text{trn}} = \mathbf{S}_{21} \mathbf{c}_{\text{inc}}$$

The transverse components of the reflected and transmitted fields are then

$$\vec{\mathbf{e}}_{xy}^{\text{ref}} = \mathbf{W}_{\text{ref}} \mathbf{c}_{\text{ref}} = \mathbf{W}_{\text{ref}} \mathbf{S}_{11} \mathbf{c}_{\text{inc}} = \begin{bmatrix} \mathbf{r}_x \\ \mathbf{r}_y \end{bmatrix}$$

$$\vec{\mathbf{e}}_{xy}^{\text{trn}} = \mathbf{W}_{\text{trn}} \mathbf{c}_{\text{trn}} = \mathbf{W}_{\text{trn}} \mathbf{S}_{21} \mathbf{c}_{\text{inc}} = \begin{bmatrix} \mathbf{t}_x \\ \mathbf{t}_y \end{bmatrix}$$

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Calculate the Amplitudes of the Spatial Harmonics



Remove the phase tilt

$$A_{x,\text{ref}}(x, y) = r_x(x, y) \div \exp[-j(k_{x,\text{inc}}x + k_{y,\text{inc}}y)]$$

$$A_{y,\text{ref}}(x, y) = r_y(x, y) \div \exp[-j(k_{x,\text{inc}}x + k_{y,\text{inc}}y)]$$

$$A_{x,\text{trn}}(x, y) = t_x(x, y) \div \exp[-j(k_{x,\text{inc}}x + k_{y,\text{inc}}y)]$$

$$A_{y,\text{trn}}(x, y) = t_y(x, y) \div \exp[-j(k_{x,\text{inc}}x + k_{y,\text{inc}}y)]$$

Calculate amplitudes of the spatial harmonics

$$\mathbf{r}_x = \text{FFT}_{2D} \{A_{x,\text{ref}}(x, y)\}$$

$$\mathbf{r}_y = \text{FFT}_{2D} \{A_{y,\text{ref}}(x, y)\}$$

$$\mathbf{t}_x = \text{FFT}_{2D} \{A_{x,\text{trn}}(x, y)\}$$

$$\mathbf{t}_y = \text{FFT}_{2D} \{A_{y,\text{trn}}(x, y)\}$$

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Calculating the Longitudinal Components



The longitudinal field components are calculated from the transverse components using the divergence equation (see Lecture 4).

$$\mathbf{r}_z = -\mathbf{K}_{z,\text{ref}}^{-1} (\mathbf{K}_x \mathbf{r}_x + \mathbf{K}_y \mathbf{r}_y)$$

$$\mathbf{t}_z = -\mathbf{K}_{z,\text{tm}}^{-1} (\mathbf{K}_x \mathbf{t}_x + \mathbf{K}_y \mathbf{t}_y)$$

Note, the \mathbf{K} matrices are not normalized.

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Calculating the Diffraction Efficiencies



The diffraction efficiencies \mathbf{R} and \mathbf{T} are calculated as

$$|\vec{\mathbf{r}}|^2 = |\mathbf{r}_x|^2 + |\mathbf{r}_y|^2 + |\mathbf{r}_z|^2$$

$$|\vec{\mathbf{t}}|^2 = |\mathbf{t}_x|^2 + |\mathbf{t}_y|^2 + |\mathbf{t}_z|^2$$

$$\mathbf{R} = \frac{\text{Re}[-\mathbf{K}_{z,\text{ref}}/\mu_{r,\text{inc}}]}{\text{Re}[k_z^{\text{inc}}/\mu_{r,\text{inc}}]} |\vec{\mathbf{r}}|^2$$

$$\mathbf{T} = \frac{\text{Re}[\mathbf{K}_{z,\text{tm}}/\mu_{r,\text{tm}}]}{\text{Re}[k_z^{\text{inc}}/\mu_{r,\text{inc}}]} |\vec{\mathbf{t}}|^2$$

Remember that these equations assume the source was given unit amplitude.

Note, the \mathbf{K} matrices are not normalized here.

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Comparison to RCWA

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