EE 4347
Applied Electromagnetics

Topic 3d

Standing Waves &
Multiple Scattering

Lecture Outline

• Standing Waves
• Standing Wave Ratio (SWR)
• Scattering From a Dielectric Slab
• Anti-Reflection Layer
• Bragg GRATINGS
Standing Waves

Two Counter-Propagating Waves (1 of 2)

Suppose we have two counter-propagating waves of equal amplitude travelling in opposite directions.

Observations:
1. Things are boring until the waves overlap.
2. Large fluctuations in amplitude are observed.
3. Locations of the fluctuations are stationary.
4. Total field is zero at some points.
Suppose we have two counter-propagating waves of equal amplitude travelling in opposite directions.

**New Observations:**
1. Fluctuations are smaller.
2. Fluctuations do not go to zero.

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**General Expressions for Forward and Backward Waves**

An incident wave will be reflected from an interface. On the reflection side, there will exist two counterpropagating waves.

- **Incident Wave**
  \[ \vec{E}_i(z) = E_{0,i} e^{-\gamma z} \hat{a}_x \]
  \[ \vec{H}_i(z) = \frac{E_{0,i}}{\eta_i} e^{-\gamma z} \hat{a}_y \]

- **Reflected Wave**
  \[ \vec{E}_r(z) = E_{0,r} e^{+\gamma z} \hat{a}_x \]
  \[ \vec{H}_r(z) = -\frac{E_{0,r}}{\eta_i} e^{+\gamma z} \hat{a}_y \]
Wave Incident on Metal (1 of 2)

To more easily understand what happens on the reflection side, let the wave be incident from a lossless dielectric (i.e. \( \sigma = 0 \)) onto metal (i.e. \( \sigma = \infty \)).

In this case, the material parameters are

\[
\eta_1 = \sqrt{\frac{\mu}{\varepsilon}} \quad \eta_2 = 0
\]

The reflection and transmission coefficients are

\[
r = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} = \frac{0 - \sqrt{\mu/\varepsilon}}{0 + \sqrt{\mu/\varepsilon}} = -1
\]
\[
t = \frac{2\eta_2}{\eta_2 + \eta_1} = \frac{2 \cdot 0}{0 + \sqrt{\mu/\varepsilon}} = 0
\]

In this case, we get zero transmission and 100% reflection with a 180° phase shift.

Wave Incident on Metal (2 of 2)

The propagation constant \( \gamma_1 \) is

\[
\begin{align*}
\alpha_i &= \omega \sqrt{\frac{\mu_i \varepsilon_i}{2}} \left[ 1 + \left( \frac{\sigma_i}{\omega \varepsilon_i} \right)^2 \right]^{-1} = \omega \sqrt{\frac{\mu_i \varepsilon_i}{2}} \left[ 1 + \left( \frac{0}{\omega \varepsilon_i} \right)^2 \right]^{-1} = 0 \\
\beta_i &= \omega \sqrt{\frac{\mu_i \varepsilon_i}{2}} \left[ 1 + \left( \frac{\sigma_i}{\omega \varepsilon_i} \right)^2 + 1 \right] = \omega \sqrt{\frac{\mu_i \varepsilon_i}{2}} \left[ 1 + \left( \frac{0}{\omega \varepsilon_i} \right)^2 + 1 \right] = \omega \sqrt{\mu_i \varepsilon_i}
\end{align*}
\]
Revised Expressions for Our Waves

Given the reflection coefficient $r$ and phase constant $\beta$, we can rewrite our wave expressions as

**Incident Wave**

\[
\tilde{E}_i(z) = E_{0,i} e^{-j\beta z} \hat{a}_x \\
\tilde{H}_i(z) = \frac{E_{0,i}}{\eta_i} e^{-j\beta z} \hat{a}_y
\]

**Reflected Wave**

\[
\tilde{E}_r(z) = rE_{0,i} e^{+j\beta z} \hat{a}_x \\
\tilde{H}_r(z) = \frac{rE_{0,i}}{\eta_i} e^{+j\beta z} \hat{a}_y
\]

Frequency-domain Standing Waves ($r = -1$)

On the reflection side, the total electromagnetic field is the sum of both the incident and reflected wave.

\[
\tilde{E}_i(z) = \tilde{E}_i(z) + \tilde{E}_r(z) = E_{0,i} e^{-j\beta z} \hat{a}_x + rE_{0,i} e^{+j\beta z} \hat{a}_x = E_{0,i} \left( e^{-j\beta z} - e^{+j\beta z} \right) \hat{a}_x
\]

\[
\tilde{H}_i(z) = \tilde{H}_i(z) + \tilde{H}_r(z) = \frac{E_{0,i}}{\eta_i} e^{-j\beta z} \hat{a}_y + \frac{E_{0,i}}{\eta_i} e^{+j\beta z} \hat{a}_y = \frac{E_{0,i}}{\eta_i} \left( e^{-j\beta z} + e^{+j\beta z} \right) \hat{a}_y
\]

The expressions in parentheses containing complex exponentials are the sine and cosine function.

\[
\tilde{E}_i(z) = -j2E_{0,i} \sin(\beta z) \hat{a}_x \\
\tilde{H}_i(z) = \frac{2E_{0,i}}{\eta_i} \cos(\beta z) \hat{a}_y
\]
Time-Domain Standing Waves ($r = -1$)

Converting our standing wave equations to the time-domain, we get

\[
\begin{align*}
\vec{E}_1(z,t) &= \text{Re}[-j2E_{0,1}\sin(\beta_zz)\hat{x} \cdot e^{j\omega t}] \\
&= -2E_{0,1}\sin(\beta_zz)\hat{x} \cdot \text{Re}[j(\cos \omega t + j \sin \omega t)] \\
&= -2E_{0,1}\sin(\beta_zz)\hat{x} \cdot \text{Re}[j \cos \omega t - \sin \omega t] \\
&= 2E_{0,1}\sin(\beta_zz)\sin(\omega t)\hat{x} \\

\vec{H}_1(z,t) &= \text{Re}[2\frac{E_{0,i}}{\eta_i}\cos(\beta_zz)\hat{y} \cdot e^{j\omega t}] \\
&= 2\frac{E_{0,i}}{\eta_i}\cos(\beta_zz)\hat{y} \cdot \text{Re}[e^{j\omega t}] \\
&= 2\frac{E_{0,i}}{\eta_i}\cos(\beta_zz)\hat{y} \cdot \text{Re}[\cos \omega t + j \sin \omega t] \\
&= 2\frac{E_{0,i}}{\eta_i}\cos(\beta_zz)\cos(\omega t)\hat{y} \\
\end{align*}
\]

Standing Waves When $r = +1$

In the frequency-domain, we have

\[
\begin{align*}
\vec{E}_1(z) &= \vec{E}_i(z) + \vec{E}_i(z) = E_{0,i}(e^{-j\beta_zz} + e^{+j\beta_zz})\hat{x} = 2E_{0,i}\cos(\beta_zz)\hat{x} \\
\vec{H}_1(z) &= \vec{H}_i(z) + \vec{H}_i(z) = \frac{E_{0,i}}{\eta_i}(e^{-j\beta_zz} - e^{+j\beta_zz})\hat{y} = -j2\frac{E_{0,i}}{\eta_i}\sin(\beta_zz)\hat{x} \\
\end{align*}
\]

In the time-domain, we have

\[
\begin{align*}
\vec{E}_1(z,t) &= 2E_{0,i}\cos(\beta_zz)\cos(\omega t)\hat{x} \\
\vec{H}_1(z,t) &= 2\frac{E_{0,i}}{\eta_i}\sin(\beta_zz)\sin(\omega t)\hat{y} \\
\end{align*}
\]
**Visualizing the Standing Waves (1 of 2)**

We will let \( r = -1 \) represent the case where \( \eta_1 > \eta_2 \).

\[
\vec{E}_1(z,t) = 2E_{01} \sin(\beta_1z) \sin(\omega t) \hat{a}_x \\
\vec{H}_1(z,t) = 2 \frac{E_{01}}{\eta_1} \cos(\beta_1z) \cos(\omega t) \hat{a}_y
\]

**Observations:**
1. 180° phase shift after reflection.
2. Max \( E \) and min \( H \) occur at the same points.
3. Min \( E \) and max \( H \) occur at the same points.
4. \( E \) is minimum at the interface and \( H \) is maximum.
5. Nodes occur a half-wavelength apart.
6. The standing wave is stationary.
7. \( \sin(\beta z) \) and \( \cos(\beta z) \) terms describe the envelope of the standing wave.

**Visualizing the Standing Waves (2 of 2)**

We will let \( r = +1 \) represent the case where \( \eta_1 < \eta_2 \).

\[
\vec{E}_1(z,t) = 2E_{01} \cos(\beta_1z) \cos(\omega t) \hat{a}_x \\
\vec{H}_1(z,t) = 2 \frac{E_{01}}{\eta_1} \sin(\beta_1z) \sin(\omega t) \hat{a}_y
\]

**Observations:**
1. 180° phase shift after reflection.
2. Max \( E \) and min \( H \) occur at the same points.
3. Min \( E \) and max \( H \) occur at the same points.
4. \( E \) is maximum at the interface and \( H \) is minimum.
5. Nodes occur a half-wavelength apart.
6. The standing wave is stationary.
7. \( \sin(\beta z) \) and \( \cos(\beta z) \) terms describe the envelope of the standing wave.
More Rigorous Visualization (1 of 2)

**Electric Field Functions**
- Standing Wave Envelope: $\cos(\beta z)$
  - Standing Wave: $E_I(z) + E_T(z)$
  - Reflected Wave: $E_T(z) = r E_0 e^{j\beta z}$
  - Incident Wave: $E_I(z) = E_0 e^{-j\beta z}$

**Magnetic Field Functions**
- Standing Wave Envelope: $\sin(\beta z)$
  - Standing Wave: $H_I(z) + H_T(z)$
  - Reflected Wave: $H_T(z) = -\frac{r E_0}{\eta} e^{j\beta z}$
  - Incident Wave: $H_I(z) = \frac{E_0}{\eta} e^{-j\beta z}$

More Rigorous Visualization (2 of 2)

**Electric Field Functions**
- Standing Wave Envelope: $\cos(\beta z)$
  - Standing Wave: $E_I(z) + E_T(z)$
  - Reflected Wave: $E_T(z) = r E_0 e^{j\beta z}$
  - Incident Wave: $E_I(z) = E_0 e^{-j\beta z}$

**Magnetic Field Functions**
- Standing Wave Envelope: $\sin(\beta z)$
  - Standing Wave: $H_I(z) + H_T(z)$
  - Reflected Wave: $H_T(z) = -\frac{r E_0}{\eta} e^{j\beta z}$
  - Incident Wave: $H_I(z) = \frac{E_0}{\eta} e^{-j\beta z}$
Standing Wave Ratio (SWR)

We wish to have a metric to quantify the severity of the standing wave. To do this, we define the standing wave ratio (SWR) as the maximum electric field observed in the standing wave divided by the minimum electric field observed in the standing wave.

\[
\text{SWR} = \frac{|E|_{\text{max}}}{|E|_{\text{min}}}
\]

Electric Field Functions
- Standing Wave Envelope: \( \cos(\beta z) \)
- Standing Wave: \( E_{i}(z) + E_{r}(z) \)
- Reflected Wave: \( E_{r}(z) = r E_{0} e^{j\beta z} \)
- Incident Wave: \( E_{i}(z) = E_{0} e^{-j\beta z} \)

\[ r = 0.33 \quad \eta_{1} \quad \eta_{2} = 2 \]
**Derivation of Standing Wave Ratio (SWR)**

Let’s examine our expression for the electric field when we have counter propagating waves.

\[ \vec{E}_i(z) = \vec{E}_i(z) + \vec{E}_i(z) = E_{0,i}(e^{-\gamma z} + re^{\gamma z}) \hat{a}_x \]

This expression has the following maximum and minimum.

\[
\begin{align*}
\max |\vec{E}_i| &= E_{0,i}(1 + |r|) \\
\min |\vec{E}_i| &= E_{0,i}(1 - |r|)
\end{align*}
\]

Substituting these into our definition of SWR gives

\[
\text{SWR} = \frac{\max |\vec{E}_i|}{\min |\vec{E}_i|} = \frac{E_{0,i}(1 + |r|)}{E_{0,i}(1 - |r|)} \rightarrow \text{SWR} = \frac{1 + |r|}{1 - |r|}
\]

**Derivation in Terms of Magnetic Field**

Let’s examine our expression for the magnetic field when we have counter propagating waves.

\[ \vec{H}_i(z) = \vec{H}_i(z) + \vec{H}_i(z) = \frac{E_{0,i}}{\eta}(e^{-\gamma z} - re^{\gamma z}) \hat{a}_y \]

This expression has the following maximum and minimum.

\[
\begin{align*}
\max |\vec{H}_i| &= \frac{E_{0,i}}{\eta}(1 + |r|) \\
\min |\vec{H}_i| &= \frac{E_{0,i}}{\eta}(1 - |r|)
\end{align*}
\]

Dividing these shows that we get the same expression for SWR

\[
\frac{\max |\vec{H}_i|}{\min |\vec{H}_i|} = \frac{\frac{E_{0,i}}{\eta}(1 + |r|)}{\frac{E_{0,i}}{\eta}(1 - |r|)} = \frac{1 + |r|}{1 - |r|} \rightarrow \text{SWR} = \frac{\vec{H}_i_{\max}}{\vec{H}_i_{\min}} = \frac{1 + |r|}{1 - |r|}
\]
SWR in Decibel Scale

Very often the SWR is given on a decibel scale.

\[ \text{SWR}_{\text{dB}} = 20 \log_{10} (\text{SWR}) \]

Given the SWR in dB, we can calculate the SWR on a linear scale.

\[ \text{SWR} = 10^{\frac{\text{SWR}_{\text{dB}}}{20}} \]

Usefulness of SWR

The standing wave ratio (SWR) is something that we can directly measure. Given the SWR, we can calculate the magnitude of the reflection coefficient.

\[ |r| = \frac{\text{SWR} - 1}{\text{SWR} + 1} \]

Derivation:

\[
\begin{align*}
\text{SWR} &= \frac{1 + |r|}{1 - |r|} \\
\text{SWR} - |r| &= \text{SWR} - 1 + |r| \\
|r| &= \frac{\text{SWR} - 1}{\text{SWR} + 1}
\end{align*}
\]
Notes About the SWR

- Since $0 \leq |r| \leq 1$, we conclude that $1 \leq \text{SWR} \leq \infty$.
- SWR is very large when the reflection is very strong.
- SWR = 1 (SWR$_{dB}$ = 0)
  - Zero standing wave
  - $|r| = 0$
  - No backward wave.
- SWR = $\infty$ (SWR$_{dB}$ = $\infty$)
  - Does NOT imply infinite amplitude standing wave
  - Standing wave has a perfect null (amplitude goes to zero)
  - $|r| = 1$
  - Forward and backward waves have equal amplitude.

Example

Suppose we have a wave inside of a $50 \, \Omega$ medium that is incident onto a second medium with impedance $120 \, \Omega$. What fraction of power is reflected? What is the standing wave ratio (SWR)?

Solution

The reflection coefficient at the interface is

$$r = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} = \frac{120 \, \Omega - 50 \, \Omega}{120 \, \Omega + 50 \, \Omega} = \frac{70}{170} = 0.4118$$

The fraction of power reflected is the reflectance.

$$R = |r|^2 = |0.4118|^2 = 0.1696 = 16.96\%$$

The SWR is

$$\text{SWR} = \frac{1+r}{1-r} = \frac{1 + 0.4118}{1 - 0.4118} = \frac{1.4118}{0.5882} = 2.4$$

SWR$_{dB}$ = $20\log_{10}$ (SWR) = $20\log_{10}$ (2.4) = 7.6 dB
Scattering From a Dielectric Slab

Analysis of a Dielectric Slab

\[ t_{12}E_0 \]
\[ r_{12}E_0 e^{-j2\theta} \]
\[ t_{21}r_{12}^2 f_{12}E_0 e^{-j2\theta} \]
\[ r_{33} f_{12}E_0 e^{-j3\theta} \]
\[ t_{21}r_{33}^3 f_{33}E_0 e^{-j3\theta} \]
\[ \ldots \text{and so on.} \]

Total Reflection

Total Transmission

\[ \psi = k_0 n_2 d \]

Standing Waves & Multiple Scattering
**Generalizations**

Normal incidence, no loss: \( \psi = k_0 n_2 d \)

Oblique incidence, no loss: \( \psi = k_0 n_2 d \cos \theta \)

Normal incidence, lossy: \( \psi = k_0 \tilde{n}_2 d, \quad \tilde{n}_2 = n_o - j \kappa \)

Oblique incidence, lossy: \( \psi = k_0 \tilde{n}_2 d \cos \theta, \quad \tilde{n}_2 = n_o - j \kappa \)

---

**Overall Reflection, \( r \) (1 of 3)**

The overall reflection from the slab is the sum of all the individual reflected waves.

\[
r = r_{12} + t_{21}r_{23}^2t_{12}e^{-j2\psi} + t_{21}r_{23}^2t_{12}^2e^{-j4\psi} + t_{21}r_{23}^3t_{12}e^{-j4\psi} + \cdots
\]

All of these terms arise due to multiple reflections within the slab. They can be written as a summation.

\[
\sum_{n=0}^{\infty} t_{21}r_{23}^n t_{12} e^{-j2(n+1)\psi}
\]

Now we put the summation back into our expression for overall reflection \( r \).

\[
r = r_{12} + r_{23}t_{21}t_{12}e^{-j2\psi} \sum_{n=0}^{\infty} \left(r_{21}r_{23}e^{-j2\psi}\right)^n
\]
Overall Reflection, $r$ (2 of 3)

Recall the closed-form expression for a geometric series.

$$\sum_{n=0}^{\infty} ax^n = \frac{a}{1-x} \quad \text{for } |x| < 1$$

The summation in our expression for overall reflection is a geometric series that we can now write in closed form.

$$r = r_{12} + r_{23} t_{12} e^{-j2\theta} \sum_{n=0}^{\infty} (r_{23} t_{23} e^{-j2\theta})^n$$

$$= r_{12} + r_{23} t_{12} e^{-j2\theta} \frac{1}{1 - r_{23} t_{23} e^{-j2\theta}}$$

$$= r_{12} + \frac{r_{23} t_{12} e^{-j2\theta}}{1 - r_{23} t_{23} e^{-j2\theta}}$$

Overall Reflection, $r$ (3 of 3)

Recall how our local reflection and transmission parameters were related.

$$r_{21} = -r_{12} \quad t_{12} = 1 - r_{12} \quad t_{21} = 1 + r_{12} \quad t_{23} = 1 - r_{23}$$

This let’s us express $r$ just in terms of $r_{12}, r_{23},$ and $\theta$.

$$r = r_{12} + \frac{r_{23} t_{12} e^{-j2\theta}}{1 - r_{23} t_{23} e^{-j2\theta}}$$

$$= r_{12} + \frac{r_{23} (1 + r_{12})(1 - r_{12}) e^{-j2\theta}}{1 - (-r_{12}) r_{23} e^{-j2\theta}}$$

$$= r_{12} \left(1 + r_{12} r_{23} e^{-j2\theta}\right) \frac{1}{1 + r_{12} r_{23} e^{-j2\theta}} + r_{23} e^{-j2\theta} - r_{12} r_{23} e^{-j2\theta}$$

$$\rightarrow r = \frac{r_{12} + r_{23} e^{-j2\theta}}{1 + r_{12} r_{23} e^{-j2\theta}}$$
Overall Transmission, $t$ (1 of 2)

The overall transmission through the slab is the sum of all the individual transmitted waves.

$$ t = t_{21}t_{12}e^{-j\omega} + t_{21}r_{21}t_{12}e^{-j3\omega} + t_{23}r_{23}^2t_{12}e^{-j5\omega} + \cdots $$

This can be written as a summation.

$$ t = \sum_{n=0}^{\infty} t_{21}r_{23}^n t_{12}e^{-j(2n+1)\omega} $$

Now we factor out some terms from the summation.

$$ t = t_{21}t_{12}e^{-j\omega} \sum_{n=0}^{\infty} (r_{12}r_{23}e^{-j2\omega})^n $$

The summation in this expression is a geometric series and can be written in closed form.

$$ t = t_{21}t_{12}e^{-j\omega} \frac{1}{1-r_{12}r_{23}e^{-j2\omega}} $$

Overall Transmission, $t$ (2 of 2)

Recall how our local reflection and transmission parameters were related.

$$ r_{21} = -r_{12} \quad t_{12} = 1-r_{12} \quad t_{21} = 1+r_{12} \quad t_{23} = 1-r_{23} $$

The let's us express $t$ just in terms of $r_{12}$, $r_{23}$, and $\theta$.

$$ t = \frac{t_{21}t_{12}e^{-j\omega}}{1-r_{12}r_{23}e^{-j2\omega}} \quad \rightarrow \quad t = \frac{(1-r_{23})(1-r_{12})e^{-j\theta}}{1+(1-r_{12}r_{23})e^{-j2\omega}} $$
**Relation Between \( r \) and \( t \)**

We solve our two expressions for \( r \) and \( t \) for \((1 + r_{12}r_{23}e^{j2\theta})\).

\[
1 + r_{12}r_{23}e^{-j2\theta} = \frac{r_{12} + r_{23}e^{-j2\theta}}{r} \\
1 + r_{12}r_{23}e^{-j2\theta} = \frac{(1-r_{23})(1-r_{12})e^{-j\omega}}{t}
\]

The expressions on the right-hand side of these equations must be equal.

\[
\frac{(1-r_{23})(1-r_{12})e^{-j\omega}}{t} = \frac{r_{12} + r_{23}e^{-j2\theta}}{r}
\]

The relation between \( r \) and \( t \) is therefore

\[
\frac{t}{r} = \frac{(1-r_{23})(1-r_{12})}{r_{12} + r_{23}e^{-j2\theta}}
\]

Note: This is NOT the same relation that we saw for a single interface.

**Plots of \( r \) and \( t \)**

**Small Reflections**
- (low finesse)
- The response resembles a cosine function and is usually approximated as such.

**Small Reflections**
- (high finesse)
- The response resembles a comb filter.
Low Finesse (1 of 2)

To understand low finesse, assume that we have a symmetric slab (i.e. $r_{12} = -r_{23}$) and that these reflection coefficients are small.

$$r = \frac{r_{12} + (-r_{12})e^{-j2\psi}}{1 + r_{12}(-r_{12})e^{-j2\psi}} = \frac{r_{12}(1-e^{-j2\psi})}{1-r_{12}^2e^{-j2\psi}}$$

For small reflections, $1-r_{12}^2e^{-j2\psi} \approx 1$

$$t = \frac{(1-(-r_{12}))(1-r_{12})e^{-j\psi}}{1 + r_{12}(-r_{12})e^{-j2\psi}} = \frac{(1+r_{12})(1-r_{12})e^{-j\psi}}{1-r_{12}^2e^{-j2\psi}}$$

Our expressions for $r$ and $t$ reduce to

$$r = r_{12}(1-e^{-j2\psi})$$

$$t = (1-r_{12}^2)e^{-j\psi}$$

Low Finesse (2 of 2)

The magnitude of $r$ gives the sine wave response we were expecting.

$$|r| = |r_{12}(1-e^{-j2\psi})| = 2|r_{12}| \sin \psi$$
Example: Do Windows Block Wifi? (1 of 2)

Windows are typically made of fused silica \((n = 1.52)\) and are around 3 mm thick.

**Solution**

Transmission through a slab of dielectric is calculated using

\[
t = \frac{(1-r_{23})(1-r_{12})e^{-j\psi}}{1+r_{12}r_{23}e^{-j\psi}}
\]

The parameters in this equation are

\[
\begin{align*}
\begin{array}{c}
r_{12} = \frac{n_2-n_1}{n_2+n_1} = \frac{1.0-1.52}{1.0+1.52} = -0.2063 \\
r_{23} = \frac{n_3-n_2}{n_3+n_2} = \frac{1.52-1.0}{1.52+1.0} = +0.2063 \\
\psi = k_dnd = \frac{2\pi f}{c_0} = \frac{2\pi(2.4 \times 10^9 \text{ Hz})}{(3.0 \times 10^8 \text{ m/s})}(1.52)(0.003 \text{ m}) = 0.2292
\end{array}
\end{align*}
\]

Example: Do Windows Block Wifi? (2 of 2)

Windows are typically made of fused silica \((n = 1.52)\) and are around 3 mm thick.

**Solution cont’d**

Substituting our values into the transmission equation gives

\[
t = \frac{(1-0.2063)(1+0.2063)e^{-j0.2292}}{1+(-0.2063)(0.2063)e^{-j0.2292}} = 0.9646 - j0.2450
\]

Total power transmitted is

\[
T = |t|^2 = |0.9646 - j0.2450|^2 = 99.52\%
\]

CONCLUSION → Windows do nothing to block Wifi.
Example: Oil on Water

Oil on water is an example of thin film interference.

\[ r(\theta, \lambda_0) \]

\[ d \]

Oil, \( n \approx 1.5 \)

Water, \( n \approx 1.33 \)

Antireflection Layer
Problem Setup (1 of 2)

Suppose we have an interface between two materials.

\[ r = \frac{n_2 - n_1}{n_2 + n_1} \]

This will produce reflections according to

How can we prevent a reflection at this interface?

Standing Waves & Multiple Scattering Slide 41

Problem Setup (2 of 2)

We insert an intermediate layer that we will call an anti-reflection layer.

How can we choose \( n_{\text{air}}, \eta_{\text{air}}, \) and \( L \) such that we get zero reflections from this interface?
How to Get \( r = 0 \) from a Slab (1 of 4)

Recall the overall reflection from a dielectric slab.

\[
    r = \frac{r_{12} + r_{23} e^{-j2\theta}}{1 + r_{12} r_{23} e^{-j2\theta}}
\]

To get \( r = 0 \), the numerator of this expression must be zero.

\[
    r_{12} + r_{23} e^{-j2\theta} = 0
\]

The reflection coefficients \( r_{12} \) and \( r_{23} \) arise from the materials in the problem that we do not wish to adjust. The trick must be in the \( e^{-j2\theta} \) term. Solving this for \( \theta \) we get

\[
    e^{-j2\theta} = -\frac{r_{12}}{r_{23}} \rightarrow \theta = \frac{1}{j2} \ln \left( \frac{r_{23}}{r_{12}} \right) - \pi m
\]

\( m = \) any integer

How to Get \( r = 0 \) from a Slab (2 of 4)

Recall that

\[
    r_{12} = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1}, \quad r_{23} = \frac{\eta_2 - \eta_{ar}}{\eta_2 + \eta_{ar}}
\]

Our expression for \( \theta \) becomes

\[
    \theta = \frac{1}{j2} \ln \left[ \frac{(\eta_2 + \eta_1)(\eta_{ar} - \eta_2)}{(\eta_{ar} - \eta_1)(\eta_{ar} + \eta_2)} \right] - \pi m
\]
How to Get $r = 0$ from a Slab (3 of 4)

Recall that $\theta = k_0 n_{ar} d$. We substitute this into our design equation.

$$k_0 n_{ar} d = \frac{1}{j2} \ln \left( \frac{(\eta_{ar} + \eta_1)(\eta_{ar} - \eta_2)}{(\eta_{ar} - \eta_1)(\eta_{ar} + \eta_2)} \right) - \pi m$$

Using $k_0 = 2\pi \lambda_0$ and solving this for $d$ gives

$$k_0 n_{ar} d = \frac{1}{j2} \ln \left( \frac{(\eta_{ar} + \eta_1)(\eta_{ar} - \eta_2)}{(\eta_{ar} - \eta_1)(\eta_{ar} + \eta_2)} \right) - \pi m$$

Downwards

$$d = \frac{\lambda_0}{4n_{ar}} \frac{1}{j\pi} \ln \left( \frac{(\eta_{ar} + \eta_1)(\eta_{ar} - \eta_2)}{(\eta_{ar} - \eta_1)(\eta_{ar} + \eta_2)} \right) - m \frac{\lambda_0}{2n_{ar}}$$

This is the most general design equation and provides more freedom than the simpler one we are about to derive.

How to Get $r = 0$ from a Slab (4 of 4)

Our design equation is complicated and we would like to simplify it.

To figure out a way to do this, we multiply out the expression inside of the natural logarithm function.

$$\frac{(\eta_{ar} + \eta_1)(\eta_{ar} - \eta_2)}{(\eta_{ar} - \eta_1)(\eta_{ar} + \eta_2)} = \frac{\eta_{ar}^2 - \eta_{ar}(\eta_2 - \eta_1) - \eta_1 \eta_2}{\eta_{ar}^2 + \eta_{ar}(\eta_2 - \eta_1) - \eta_1 \eta_2}$$

This simplifies when $\eta_{ar}^2 = \eta_1 \eta_2$. In fact, it reduces to just -1

Recognizing that $\ln(-1) = j\pi$, our simplified expression for $d$ is now

$$d = \frac{\lambda_0}{4n_{ar}} - m \frac{\lambda_0}{2n_{ar}}$$

for $\eta_{ar}^2 = \eta_1 \eta_2$
**Interpretation of Design Equation**

When $\eta_{ar}^2 = \eta_1 \eta_2$

$$d = \frac{\lambda_0}{4n_{ar}} - m \frac{\lambda_0}{2n_{ar}}$$

$m = \text{any integer}$

The second term tells us that we can adjust the length $d$ by any integer multiple of a half-wavelength.

$$m \frac{\lambda_0}{2n_{ar}} = m \frac{\lambda}{2}$$

We interpret this first term as a quarter wavelength slab of dielectric.

$$\frac{\lambda_0}{4n_{ar}} = \frac{\lambda}{4}$$

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**Design Procedure**

**Step 1** – Choose an antireflection material so that

$$\eta_{ar} = \sqrt{\eta_1 \eta_2}$$

We have a bit of freedom here.

However, when we wish to use only dielectric materials, only one choice is possible.

$$e_{ar} = \sqrt{e_1 e_2} \quad \text{or} \quad n_{ar} = \sqrt{n_1 n_2}$$

**Step 2** – Calculate thickness based on refractive index.

$$d = \frac{\lambda_0}{4n_{ar}} - m \frac{\lambda_0}{2n_{ar}}$$

$m = \text{any integer}$

$m = 0$ is the most common choice.
Example

It is desired to maximize the light through a lens. The lens is made of glass with $n = 1.52$ and resides in air with $n = 1.0$. Design an anti-reflection coating to maximize transmission at the center of the visible spectrum, $\lambda_0 = 500$ nm.

Solution

Step 1: At optical frequencies, materials cannot have a significant magnetic response. Therefore, we will design the anti-reflection layer through the refractive index $n_{ar}$.

$$n_u = \sqrt{n_ar} = \sqrt{(1.0)(1.52)} \quad \Rightarrow \quad n_u = 1.2329$$

Step 2: The thickness of the anti-reflection layer is

$$d = \frac{500 \text{ nm}}{4(1.2329)} - m \frac{500 \text{ nm}}{2(1.2329)} = 101.4 \text{ nm} - m(202.8 \text{ nm})$$

Choose the $m = 0$ solution.

$d = 101.4 \text{ nm}$

Bragg Gratings
What is a Bragg Grating?

![Diagram of a Bragg Grating](image)

**Design**

\[ L_1 = \frac{\lambda_0}{4n_1} \]  

Quarter wave layers

\[ L_2 = \frac{\lambda_0}{4n_2} \]  

Quarter wave layers

**Transmittance**

Stop band: \( \Delta \lambda \propto |n_1 - n_2| \)