EE 4347
Applied Electromagnetics

Topic 4c

Multi Segment Transmission Lines

Lecture Outline

• Quarter-Wave Transformer
• Impedance Matching
• Circuit/Wave Equivalence
• Analysis of Multi Segment Transmission Lines
• Digital Filter Analogy
• Stubs
• Scattering Parameters
Quarter-Wave Transformer

A quarter-wave transformer is a section of line that is a $\lambda/4$ long.

When the length of the line is $\lambda/4$, then we have

$$\beta \ell = \frac{2\pi \cdot \lambda}{\lambda} = \frac{\pi}{2}$$

When this is the case, our impedance transformation equation reduces to

$$Z_{in}(-\ell) = Z_0 \frac{Z_L + jZ_0 \tan(\pi/2)}{Z_0 + jZ_L \tan(\pi/2)} = Z_0 \frac{Z_L + jZ_0 \cdot \infty}{Z_0 + jZ_L \cdot \infty}$$

$\tan(\pi/2) = \infty$. 

**Quarter-Wave Transformer (2 of 2)**

Since both the numerator and denominator are $\infty$, we must apply L'Hopital's rule.

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = \frac{f'(x_0)}{g'(x_0)}$$

Applying this to our impedance transformation equation, we get

$$Z_{in}(-\ell) = \lim_{\beta \to \infty} \frac{Z_L + jZ_0 \tan(\beta \ell)}{Z_0 + jZ_L \tan(\beta \ell)} = \frac{Z_0 - jZ_L \sec^2(\beta \ell)}{jZ_0 \sec^2(\beta \ell)} = \frac{Z_L^2}{Z_0^2}$$

The final equation shows that the load impedance $Z_L$ gets completely inverted. The input impedance becomes the input admittance.

$$Z_{in} \left(\frac{-\lambda}{4}\right) = \frac{Z_L^2}{Z_0^2}$$

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**Impedance Inversion (1 of 5)**

**Generator** → **Quarter-Wave Transmission Line** → **Inductive Load**

$V_g$ → $Z_g$ → $Z_{in}$ → $\gamma$, $Z_0$ → $\infty$ → $L$

$z = -\lambda/4$ → $z = 0$

**Input Impedance is Capacitive**

$V_s$ → $Z_s$ → $Z_{in}$ → $\infty$ → $L$

$z = -\lambda/4$

Inductors look like capacitors!

$Z_{in}$ of Quarter-Wave Line

$$Z_{in} = \frac{Z_L^2}{Z_0}$$

Equivalent Capacitance

$$C = \frac{L}{Z_0^2}$$
Impedance Inversion (2 of 5)

Generator
Quarter-Wave Transmission Line
Capacitive Load

\[ V_g \]

\[ Z_g \]

\[ Z_{in} \]

\[ \gamma, Z_0 \]

\[ C \]

\[ z = -\lambda/4 \]

\[ z = 0 \]

Impedance Inversion (3 of 5)

Generator
Quarter-Wave Transmission Line
Short-Circuit Load

\[ V_g \]

\[ Z_g \]

\[ Z_{in} \]

\[ \gamma, Z_0 \]

\[ Z_L = 0 \]

\[ z = -\lambda/4 \]

\[ z = 0 \]

Capacitors look like inductors!

\[ Z_{in} = \frac{Z_0^2}{Z_L} \]

Equivalent Inductance

\[ L = CZ_0^2 \]

Short circuits look like open circuits!
Impedance Inversion (4 of 5)

- Generator
- Quarter-Wave Transmission Line
- Open-Circuit Load

\[ Z_L = \infty \]

\[ Z_{\text{in}} = 4 \frac{\lambda}{4} \]

- Input Impedance is a short circuit.

\[ V_g \]

\[ Z_{\text{in}} = 0 \]

Open circuits look like short circuits!

Impedance Inversion (5 of 5)

- Generator
- Quarter-Wave Transmission Line
- Matched Load

\[ Z_L = Z_0 \]

\[ Z_{\text{in}} = \frac{Z_L^2}{Z_L} = 0 \]

- Input Impedance is \( Z_0 \).

\[ V_g \]

\[ Z_{\text{in}} = 0 \]

Matched loads are always matched!
Similar to the anti-reflection layer for waves, we can match a transmission line to a load impedance by inserting a quarter-wave section of a second transmission line.

We must perform an electromagnetic analysis of the transmission line to determine $\beta_{\text{air}}$. 

$$\ell = \frac{\lambda}{4} = \frac{\pi}{2\beta_{\text{air}}}$$
Example (1 of 3)

A 50 Ω microstrip line on FR-4 ($\varepsilon_r = 4.4$) operates at 2.4 GHz and is connected to patch antenna which has a 120 Ω input impedance. How much power is reflected? How can the circuit be improved?

Reflected Power:

$$\left| r \right| = \left| \frac{Z_L - Z_0}{Z_L + Z_0} \right| = \left| \frac{(120 \, \Omega) - (50 \, \Omega)}{(120 \, \Omega) + (50 \, \Omega)} \right| = \frac{0.4118}{1} = 17\%$$

Example (2 of 3)

A 50 Ω microstrip line on FR-4 ($\varepsilon_r = 4.4$) operates at 2.4 GHz and is connected to patch antenna which has a 120 Ω input impedance. How much power is reflected? How can the circuit be improved?

Design: $Z'' = \sqrt{Z_L Z_0} = \sqrt{(120 \, \Omega)(50 \, \Omega)} = 77.5 \, \Omega$

Perform an EM analysis to determine TL dimensions to get 50 Ω. For the TEM mode,

$$\beta = \omega \sqrt{\mu \varepsilon} = \frac{2\pi f}{c_0} \sqrt{\mu \varepsilon} = \frac{2\pi}{(3.0 \times 10^8 \, \text{m/s}) \sqrt{(1.0)(4.4)}} = 105.44 \, \text{rad/s}$$
Example (3 of 3)

A 50 Ω microstrip line on FR-4 ($\varepsilon_r = 4.4$) operates at 2.4 GHz and is connected to patch antenna which has a 120 Ω input impedance. How much power is reflected? How can the circuit be improved?

Design: Given $\beta$, the length of the line should be

$$\beta \ell = \frac{\pi}{2} \quad \rightarrow \quad \ell = \frac{\frac{\pi}{2}}{2 \beta} = \frac{\pi}{2(105.44 \text{ rad/s})} = 1.4898 \times 10^{-2} \text{ m}$$

Circuit Wave Equivalence
Here we have a “stepped-impedance” microwave circuit.

We view the circuit as a series of discrete segments that are uniform within the segment.
This is equivalent to waves propagating through multiple slabs of dielectric.

\[ n_i = \sqrt{\mu_i \epsilon_i}, \]
\[ Z_i = \eta_i = \sqrt{\frac{\mu_i}{\epsilon_i}}. \]
Impedance Transformation Method (1 of 6)

How do we calculate the reflection from this circuit?

Impedance Transformation Method (2 of 6)

Step 1 – The input impedance $Z_{in}$ at the load is simply the load impedance $Z_L$. 

$Z_{in} = Z_L$
Step 2 – We use impedance transformation to calculate the impedance looking into the fourth segment.

\[ Z'_4 = \frac{Z_4 + jZ_4 \tan(\beta_4 d_4)}{Z_4 + jZ_L \tan(\beta_4 d_4)} \]

Step 3 – We use impedance transformation to calculate the impedance looking into the third segment.

\[ Z'_3 = \frac{Z_3 Z'_4 + jZ_3 \tan(\beta_3 d_3)}{Z_3 + jZ'_4 \tan(\beta_3 d_3)} \]
Step 4 – We use impedance transformation to calculate the impedance looking into the second segment.

\[ Z'_2 = Z_2 \frac{Z'_3 + jZ_2 \tan(\beta_2d_2)}{Z_2 + jZ'_3 \tan(\beta_2d_2)} \]

Step 5 – We use impedance transformation to calculate the impedance looking into the first segment.

\[ Z'_1 = Z_1 \frac{Z'_3 + jZ_1 \tan(\beta_1d_1)}{Z_1 + jZ'_3 \tan(\beta_1d_1)} \]
Impedance Transformation Method (7 of 6)

Step 6 – We calculate the overall reflection as seen by the generator.

\[ \Gamma = \frac{Z'_1 - Z_g}{Z'_1 + Z_g} \]

It is straightforward to extend this procedure to analyze circuits composed of any number of segments.

Impedance Transformation Method for Waves

Impedance Transformation for Waves

\[ \eta' = \eta_i \frac{\eta''_{i+1} + j \eta_i \tan(\beta d_i)}{\eta_i + j \eta''_{i+1} \tan(\beta d_i)} \]

Overall Reflection

\[ r = \frac{\eta'_i - \eta_s}{\eta'_i + \eta_s} \]
\[ R = \left| r \right|^2 \]
Notes on the Impedance Transformation Method

- Very fast and simple to implement!
- Difficult to modify the method to calculate transmission when the materials have loss or gain.
- When gain and loss can be ignored, $R + T = 1$.
- Method cannot directly visualize the fields inside of the device.

Digital Filter Analogy
**Conditions for Analogy**

The reflection coefficient \( \Gamma_i \) for a wave in medium \( i \) incident on medium \( i+1 \) is given by

\[
\Gamma_i = \frac{Z_{i+1} - Z_i}{Z_{i+1} + Z_i} = \frac{\eta_{i+1} - \eta_i}{\eta_{i+1} + \eta_i}
\]

Notice this is written for both transmission lines and dielectric slabs.

The overall reflection coefficient \( \Gamma_{in} \) from a series of \( M \) segments is very complicated. However, it can be greatly simplified given two conditions:

1. The electrical length of each segment is the same.
   \[ \psi \approx \beta_i d_i \approx \cdots \approx \beta_M d_M \]

2. The reflection coefficients \( \Gamma_i \) at the interfaces are small, allowing us to ignore waves reflected more than once.
   \[ |\Gamma_i| < \sim 0.1 \]

**Expression for Overall Reflection**

Given the two conditions on the previous slide, the overall reflection coefficient \( \Gamma_{in} \) from \( M \) layers can be written as

\[
\Gamma_{in} = \Gamma_0 + \Gamma_1 e^{-j2\psi} + \Gamma_2 e^{-j4\psi} + \cdots + \Gamma_M e^{-j2M\psi}
\]

\[
\Gamma_i = \frac{Z_{i+1} - Z_i}{Z_{i+1} + Z_i} = \frac{\eta_{i+1} - \eta_i}{\eta_{i+1} + \eta_i}
\]

\[ \psi \approx \beta_i d_i \approx \cdots \approx \beta_M d_M \]
Digital FIR Filters

The general form of the transfer function for a finite impulse response (FIR) digital filter is

\[ H(z) = \sum_{k=0}^{M} h(k) z^{-k} = h(0) + h(1) z^{-1} + h(2) z^{-2} + \cdots + h(M) z^{-M} \]

If we excite the digital filter with an impulse function at \( t = 0 \), the we observe \( h(0) \) at time 0, \( h(1) \) at time 1, \( h(2) \) at time 2, and so on.

The frequency-domain response of a digital FIR filter is

\[ H(\omega) = \sum_{k=0}^{M} h(k) e^{-j2\omega k} = h(0) + h(1) e^{-j2\omega} + h(2) e^{-j4\omega} + \cdots + h(M) e^{-j2M\omega} \]

The Analogy

If we compare the overall reflection coefficient \( \Gamma_{in} \) of our multi-segment/multi-layer device, we see that it has the same form as a digital filter.

\[ \Gamma_{in} = \Gamma_0 + \Gamma_1 e^{-j2\omega} + \Gamma_2 e^{-j4\omega} + \cdots + \Gamma_M e^{-j2M\omega} \]

\[ H = h(0) + h(1) e^{-j2\omega} + h(2) e^{-j4\omega} + \cdots + h(M) e^{-j2M\omega} \]

This means we can design multi-segment/multi-layer filters just like we design a digital filter.

Much is known about designing digital filters and all of this knowledge and all of the tools can be used to design filters for electromagnetic waves!
Stubs

What is a Stub? (1 of 3)

What do short circuits look like \( \lambda/4 \) away?
What is a Stub? (2 of 2)

What do short circuits look like $\lambda/4$ away? Open circuits!

The Shorted Stub is a Band Pass Filter

The circuit is actually shorted for all frequencies other than whatever frequency has wavelength $\lambda$ inside the line. The short circuit blocks all signals.

At the frequency with wavelength $\lambda$, the circuit is not shorted and signals are allowed to pass.
Stubs in Practice

Scattering Parameters
Definition of a Scattering Matrix

The scattering matrix relates the amplitudes of the input waves to the amplitudes of the output waves.

\[
\begin{bmatrix}
V_1^- \\
V_2^- \\
\vdots \\
V_N^-
\end{bmatrix} =
\begin{bmatrix}
S_{11} & S_{12} & \cdots & S_{1N} \\
S_{21} & S_{22} & \cdots & S_{2N} \\
\vdots & \vdots & \ddots & \vdots \\
S_{N1} & S_{N2} & \cdots & S_{NN}
\end{bmatrix}
\begin{bmatrix}
V_1^+ \\
V_2^+ \\
\vdots \\
V_N^+
\end{bmatrix}
\]

\[ S_{ij} = \frac{V_i^+}{V_j^+} \] no other applied voltages

S-Matrix for Two-Port Networks

Any linear system can be reduced to a single scattering matrix that describes how it behaves.

Very often, engineers will say "S-1-1" instead of saying "reflection," and say "S-2-1" instead of saying transmission.