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**EE 5337**

## Computational Electromagnetics

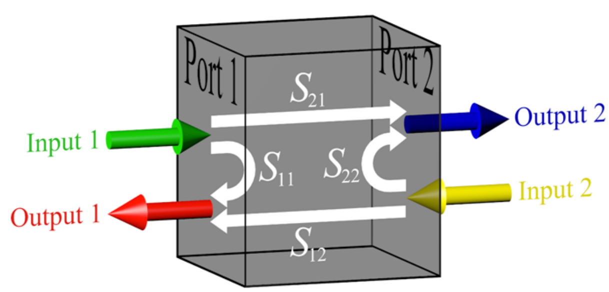
**Lecture #5a**

# Scattering Matrices for Semi-Analytical Methods



## Outline

- Scattering matrix for a single layer
- Multilayer structures
- Longitudinally periodic structures
- Dispersion analysis
- Alternatives to scattering matrices



# Scattering Matrix for a Single Layer

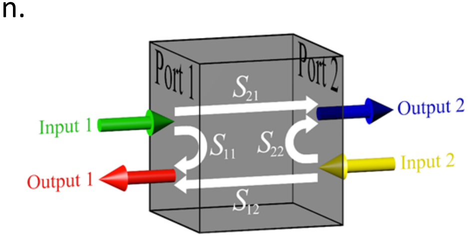
R. C. Rumpf, "Improved Formulation of Scattering Matrices for Semi-Analytical Methods That is Consistent with Convention," PIERS B, Vol. 35, pp. 241-261, 2011.



## Motivation for Scattering Matrices

Scattering matrices offer several important features and benefits:

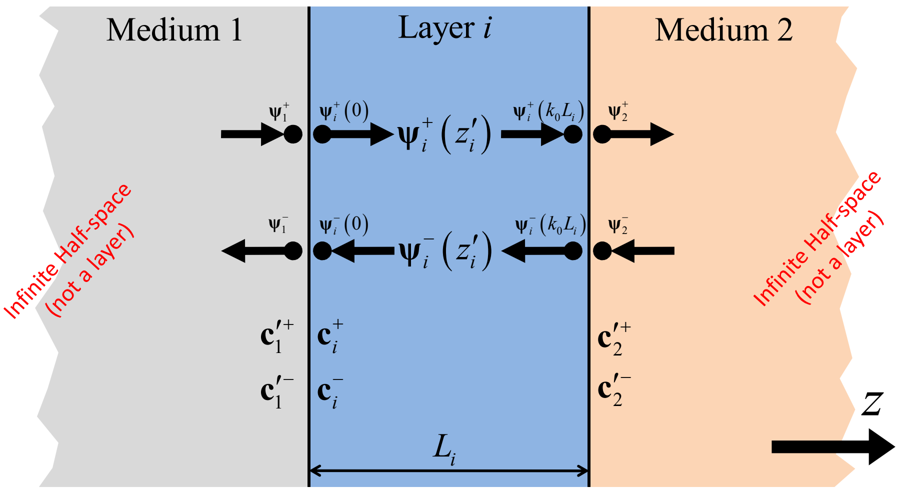
- Unconditionally stable method.
- Parameters have physical meaning.
- Parameters correspond to those measured in the lab.
- Can be used to extract dispersion.
- Very memory efficient.
- Can be used to exploit longitudinal periodicity.
- Mature and proven approach.
- Much greater wealth of literature available.




However, excellent alternatives to S-matrices do exist!

# Geometry of a Single Layer

● Indicates a point that lies on an interface, but associated with a particular side.




$\psi_i^\pm(z) \equiv$ field within $i^{\text{th}}$ layer

$\mathbf{c}_i^\pm \equiv$ mode coefficients inside $i^{\text{th}}$ layer

$\mathbf{c}_i'^\pm \equiv$ mode coefficients outside $i^{\text{th}}$ layer

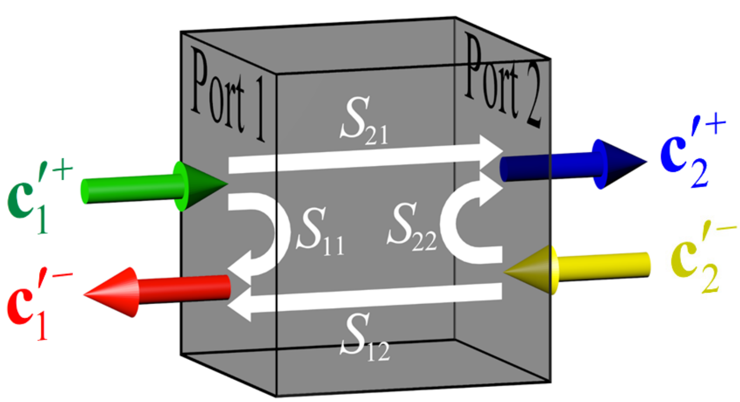


# Definition of A Scattering Matrix

$$\begin{bmatrix} \mathbf{c}_1'^- \\ \mathbf{c}_2'^+ \end{bmatrix} = \begin{bmatrix} \mathbf{S}_{11} & \mathbf{S}_{12} \\ \mathbf{S}_{21} & \mathbf{S}_{22} \end{bmatrix} \begin{bmatrix} \mathbf{c}_1'^+ \\ \mathbf{c}_2'^- \end{bmatrix}$$

$\mathbf{S}_{11} \equiv$ reflection

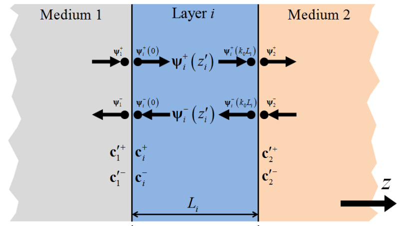
$\mathbf{S}_{21} \equiv$ transmission




This is consistent with network theory and experimental convention.

## Field Relations

Field inside the $i^{\text{th}}$ layer:




$$\boldsymbol{\psi}_i(z_i') = \begin{bmatrix} E_{x,i}(z_i') \\ E_{y,i}(z_i') \\ \tilde{H}_{x,i}(z_i') \\ \tilde{H}_{y,i}(z_i') \end{bmatrix} = \begin{bmatrix} \mathbf{W}_i & \mathbf{W}_i \\ \mathbf{V}_i & -\mathbf{V}_i \end{bmatrix} \begin{bmatrix} e^{\boldsymbol{\lambda}_i z_i'} & \mathbf{0} \\ \mathbf{0} & e^{-\boldsymbol{\lambda}_i z_i'} \end{bmatrix} \begin{bmatrix} \mathbf{c}_i^+ \\ \mathbf{c}_i^- \end{bmatrix}$$

Boundary conditions at the first interface:

$$\boldsymbol{\psi}_1 = \boldsymbol{\psi}_i(0)$$

$$\begin{bmatrix} \mathbf{W}_1 & \mathbf{W}_1 \\ \mathbf{V}_1 & -\mathbf{V}_1 \end{bmatrix} \begin{bmatrix} \mathbf{c}_1'^+ \\ \mathbf{c}_1'^- \end{bmatrix} = \begin{bmatrix} \mathbf{W}_i & \mathbf{W}_i \\ \mathbf{V}_i & -\mathbf{V}_i \end{bmatrix} \begin{bmatrix} \mathbf{c}_i^+ \\ \mathbf{c}_i^- \end{bmatrix}$$

Boundary conditions at the second interface:

$$\boldsymbol{\psi}_i(k_0 L_i) = \boldsymbol{\psi}_2$$

$$\begin{bmatrix} \mathbf{W}_i & \mathbf{W}_i \\ \mathbf{V}_i & -\mathbf{V}_i \end{bmatrix} \begin{bmatrix} e^{\boldsymbol{\lambda}_i k_0 L_i} & \mathbf{0} \\ \mathbf{0} & e^{-\boldsymbol{\lambda}_i k_0 L_i} \end{bmatrix} \begin{bmatrix} \mathbf{c}_i^+ \\ \mathbf{c}_i^- \end{bmatrix} = \begin{bmatrix} \mathbf{W}_2 & \mathbf{W}_2 \\ \mathbf{V}_2 & -\mathbf{V}_2 \end{bmatrix} \begin{bmatrix} \mathbf{c}_2'^+ \\ \mathbf{c}_2'^- \end{bmatrix}$$

Note: $k_0$ has been incorporated to normalize $L_i$.



## Derivation of the Scattering Matrix

Solve both boundary condition equations for the intermediate mode coefficients $\mathbf{c}_i^+$ and $\mathbf{c}_i^-$.

$$\begin{bmatrix} \mathbf{c}_i^+ \\ \mathbf{c}_i^- \end{bmatrix} = \begin{bmatrix} \mathbf{W}_i & \mathbf{W}_i \\ \mathbf{V}_i & -\mathbf{V}_i \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{W}_1 & \mathbf{W}_1 \\ \mathbf{V}_1 & -\mathbf{V}_1 \end{bmatrix} \begin{bmatrix} \mathbf{c}_1'^+ \\ \mathbf{c}_1'^- \end{bmatrix} \qquad \begin{bmatrix} \mathbf{c}_i^+ \\ \mathbf{c}_i^- \end{bmatrix} = \begin{bmatrix} e^{-\boldsymbol{\lambda}_i k_0 L_i} & \mathbf{0} \\ \mathbf{0} & e^{\boldsymbol{\lambda}_i k_0 L_i} \end{bmatrix} \begin{bmatrix} \mathbf{W}_i & \mathbf{W}_i \\ \mathbf{V}_i & -\mathbf{V}_i \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{W}_2 & \mathbf{W}_2 \\ \mathbf{V}_2 & -\mathbf{V}_2 \end{bmatrix} \begin{bmatrix} \mathbf{c}_2'^+ \\ \mathbf{c}_2'^- \end{bmatrix}$$

Both of these equations have the term

$$\begin{bmatrix} \mathbf{W}_i & \mathbf{W}_i \\ \mathbf{V}_i & -\mathbf{V}_i \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{W}_j & \mathbf{W}_j \\ \mathbf{V}_j & -\mathbf{V}_j \end{bmatrix} = \frac{1}{2} \begin{bmatrix} \mathbf{A}_{ij} & \mathbf{B}_{ij} \\ \mathbf{B}_{ij} & \mathbf{A}_{ij} \end{bmatrix} \qquad \begin{aligned} \mathbf{A}_{ij} &= \mathbf{W}_i^{-1}\mathbf{W}_j + \mathbf{V}_i^{-1}\mathbf{V}_j \\ \mathbf{B}_{ij} &= \mathbf{W}_i^{-1}\mathbf{W}_j - \mathbf{V}_i^{-1}\mathbf{V}_j \end{aligned}$$

See HW2

We set substitute this result into the first two equations and then set them equal to eliminate the intermediate mode coefficients.

$$\frac{1}{2} \begin{bmatrix} \mathbf{A}_{i1} & \mathbf{B}_{i1} \\ \mathbf{B}_{i1} & \mathbf{A}_{i1} \end{bmatrix} \begin{bmatrix} \mathbf{c}_1'^+ \\ \mathbf{c}_1'^- \end{bmatrix} = \frac{1}{2} \begin{bmatrix} e^{-\boldsymbol{\lambda}_i k_0 L_i} & \mathbf{0} \\ \mathbf{0} & e^{\boldsymbol{\lambda}_i k_0 L_i} \end{bmatrix} \begin{bmatrix} \mathbf{A}_{i2} & \mathbf{B}_{i2} \\ \mathbf{B}_{i2} & \mathbf{A}_{i2} \end{bmatrix} \begin{bmatrix} \mathbf{c}_2'^+ \\ \mathbf{c}_2'^- \end{bmatrix}$$

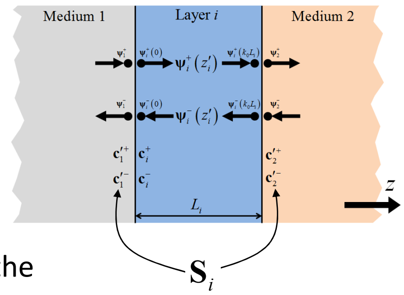
We write this as two matrix equations and rearrange the terms until they have the form of a scattering matrix.

$$\begin{bmatrix} \mathbf{c}_1'^- \\ \mathbf{c}_2'^+ \end{bmatrix} = \begin{bmatrix} ? & ? \\ ? & ? \end{bmatrix} \begin{bmatrix} \mathbf{c}_1'^+ \\ \mathbf{c}_2'^- \end{bmatrix}$$

## The Scattering Matrix

The scattering matrix $\mathbf{S}_i$ of the $i^{\text{th}}$ layer is defined as:

$$\begin{bmatrix} \mathbf{c}_1'^- \\ \mathbf{c}_2'^+ \end{bmatrix} = \mathbf{S}^{(i)} \begin{bmatrix} \mathbf{c}_1'^+ \\ \mathbf{c}_2'^- \end{bmatrix} \qquad \mathbf{S}^{(i)} = \begin{bmatrix} \mathbf{S}_{11}^{(i)} & \mathbf{S}_{12}^{(i)} \\ \mathbf{S}_{21}^{(i)} & \mathbf{S}_{22}^{(i)} \end{bmatrix}$$




After some algebra, the components of the scattering matrix are computed according to

$$\mathbf{S}_{11}^{(i)} = \left(\mathbf{A}_{i1} - \mathbf{X}_i\mathbf{B}_{i2}\mathbf{A}_{i2}^{-1}\mathbf{X}_i\mathbf{B}_{i1}\right)^{-1}\left(\mathbf{X}_i\mathbf{B}_{i2}\mathbf{A}_{i2}^{-1}\mathbf{X}_i\mathbf{A}_{i1} - \mathbf{B}_{i1}\right)$$

$$\mathbf{S}_{12}^{(i)} = \left(\mathbf{A}_{i1} - \mathbf{X}_i\mathbf{B}_{i2}\mathbf{A}_{i2}^{-1}\mathbf{X}_i\mathbf{B}_{i1}\right)^{-1}\mathbf{X}_i\left(\mathbf{A}_{i2} - \mathbf{B}_{i2}\mathbf{A}_{i2}^{-1}\mathbf{B}_{i2}\right)$$

$$\mathbf{S}_{21}^{(i)} = \left(\mathbf{A}_{i2} - \mathbf{X}_i\mathbf{B}_{i1}\mathbf{A}_{i1}^{-1}\mathbf{X}_i\mathbf{B}_{i2}\right)^{-1}\mathbf{X}_i\left(\mathbf{A}_{i1} - \mathbf{B}_{i1}\mathbf{A}_{i1}^{-1}\mathbf{B}_{i1}\right)$$

$$\mathbf{S}_{22}^{(i)} = \left(\mathbf{A}_{i2} - \mathbf{X}_i\mathbf{B}_{i1}\mathbf{A}_{i1}^{-1}\mathbf{X}_i\mathbf{B}_{i2}\right)^{-1}\left(\mathbf{X}_i\mathbf{B}_{i1}\mathbf{A}_{i1}^{-1}\mathbf{X}_i\mathbf{A}_{i2} - \mathbf{B}_{i2}\right)$$

$$\mathbf{A}_{ij} = \mathbf{W}_i^{-1}\mathbf{W}_j + \mathbf{V}_i^{-1}\mathbf{V}_j$$

$$\mathbf{B}_{ij} = \mathbf{W}_i^{-1}\mathbf{W}_j - \mathbf{V}_i^{-1}\mathbf{V}_j$$

$$\mathbf{X}_i = e^{\lambda_i k_0 L_i}$$

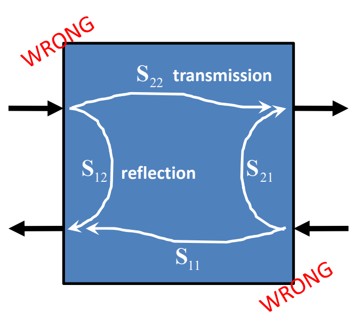
*i* is the layer number.
*j* is either 1 or 2 depending on which external medium is being referenced.

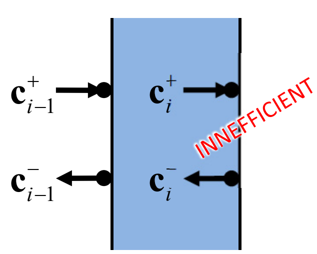


## WARNING: Scattering Matrices in the Literature

For some reason, the computational electromagnetics community has: (1) deviated from convention, and (2) formulated inefficient scattering matrices.

$$\begin{bmatrix} \mathbf{c}_{i-1}^- \\ \mathbf{c}_i^+ \end{bmatrix} = \begin{bmatrix} \mathbf{S}_{11} & \mathbf{S}_{12} \\ \mathbf{S}_{21} & \mathbf{S}_{22} \end{bmatrix} \begin{bmatrix} \mathbf{c}_i^- \\ \mathbf{c}_{i-1}^+ \end{bmatrix}$$



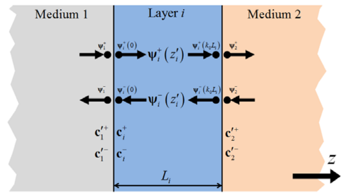




- Here $s_{11}$ is not reflection. Instead. it is backward transmission!
- Here $s_{21}$ is not transmission. Instead, it is a reflection parameter!
- Scattering matrices can not be interchanged.
- Scattering matrices are not symmetric so they take twice the memory to store and are more time-consuming to calculate.

R. C. Rumpf, "Improved formulation of scattering matrices for semi-analytical methods that is consistent with convention," PIERS B, Vol. 35, 241-261, 2011.

## Limitation of Conventional S-Matrix Formulation

Note that the elements of a scattering matrix are a function of materials outside of the layer.

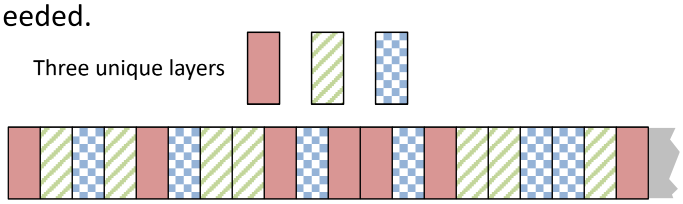



This makes it difficult to interchange scattering matrices arbitrarily.

For example, there are only three unique layers in the multilayer structure below, yet 20 separate computations of scattering matrices are needed.

Three unique layers

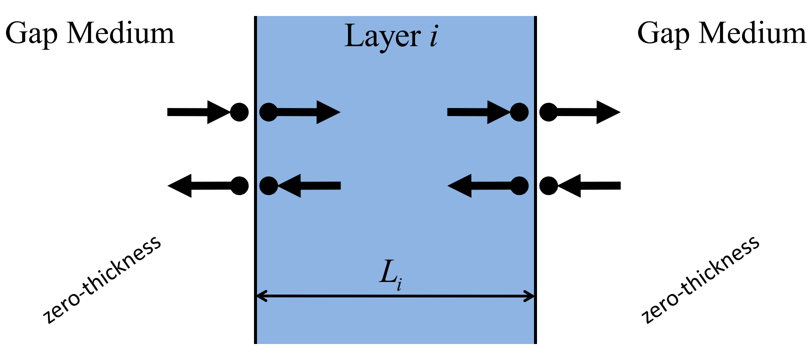
20 layer stack





## Solution

To get around this, we will surround each layer with external regions of zero thickness. This lets us connect the scattering matrices in any order because they all calculate fields that exist outside of the layers in the same medium. This will have no effect electromagnetically as long as we make the external regions have zero thickness between layers.

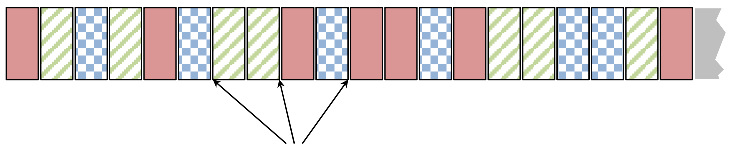


# Visualization of the Technique

We calculate the scattering matrices for just the unique layers.

Three unique layers

Then we just manipulate these same three scattering matrices to "build" the global scattering matrix.

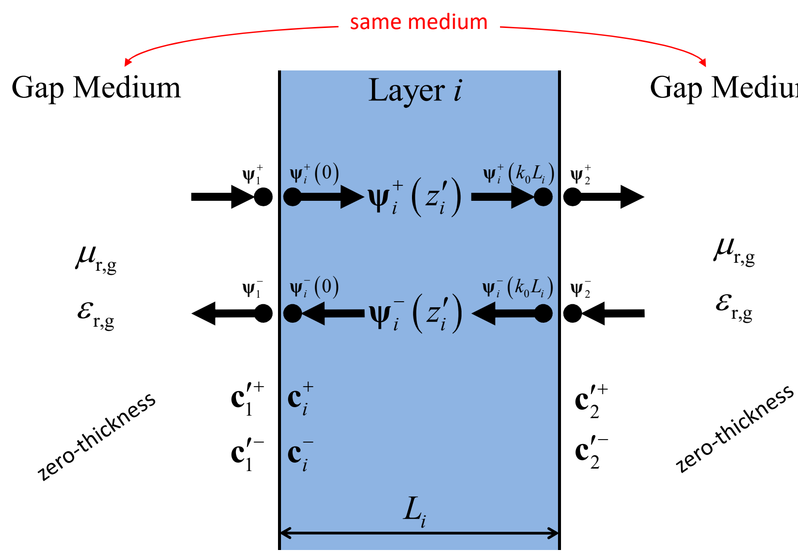


Gaps between the layers are made to have zero thickness so they have no effect electromagnetically.

*Faster! Simpler! Less memory needed!*

Lecture 5a — Slide 13

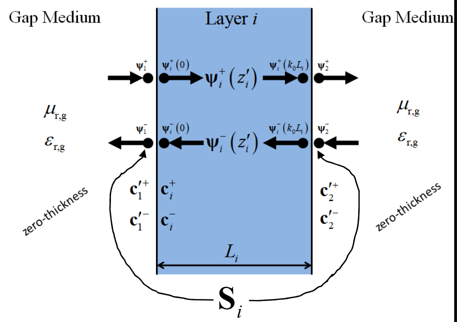
# Revised Geometry of a Single Layer




Lecture 5a — Slide 14

# Calculating Revised Scattering Matrices

The scattering matrix $\mathbf{S}_i$ of the $i^{\text{th}}$ layer is still defined as:




$$\begin{bmatrix} \mathbf{c}_1'^- \\ \mathbf{c}_2'^+ \end{bmatrix} = \mathbf{S}^{(i)} \begin{bmatrix} \mathbf{c}_1'^+ \\ \mathbf{c}_2'^- \end{bmatrix} \qquad \mathbf{S}^{(i)} = \begin{bmatrix} \mathbf{S}_{11}^{(i)} & \mathbf{S}_{12}^{(i)} \\ \mathbf{S}_{21}^{(i)} & \mathbf{S}_{22}^{(i)} \end{bmatrix}$$

But the equations to calculate the elements reduce to

$$\mathbf{S}_{11}^{(i)} = \left(\mathbf{A}_i - \mathbf{X}_i\mathbf{B}_i\mathbf{A}_i^{-1}\mathbf{X}_i\mathbf{B}_i\right)^{-1}\left(\mathbf{X}_i\mathbf{B}_i\mathbf{A}_i^{-1}\mathbf{X}_i\mathbf{A}_i - \mathbf{B}_i\right)$$

$$\mathbf{S}_{12}^{(i)} = \left(\mathbf{A}_i - \mathbf{X}_i\mathbf{B}_i\mathbf{A}_i^{-1}\mathbf{X}_i\mathbf{B}_i\right)^{-1}\mathbf{X}_i\left(\mathbf{A}_i - \mathbf{B}_i\mathbf{A}_i^{-1}\mathbf{B}_i\right)$$

$$\mathbf{A}_i = \mathbf{W}_i^{-1}\mathbf{W}_{\text{g}} + \mathbf{V}_i^{-1}\mathbf{V}_{\text{g}}$$

$$\mathbf{B}_i = \mathbf{W}_i^{-1}\mathbf{W}_{\text{g}} - \mathbf{V}_i^{-1}\mathbf{V}_{\text{g}}$$

$$\mathbf{S}_{21}^{(i)} = \mathbf{S}_{12}^{(i)} \qquad \mathbf{S}_{22}^{(i)} = \mathbf{S}_{11}^{(i)}$$

- Layers are symmetric so the scattering matrix elements have redundancy.
- Scattering matrix equations are simplified.
- Fewer calculations.
- Less memory storage.

$$\mathbf{X}_i = e^{\boldsymbol{\lambda}_i k_0 L_i}$$

Lecture 5a — Slide 15

# Layers in TMM are Actually Four-Port Networks

We have written the scattering matrices as 2×2 block matrices.

$$\begin{bmatrix} \mathbf{S}_{11} & \mathbf{S}_{12} \\ \mathbf{S}_{21} & \mathbf{S}_{22} \end{bmatrix}$$

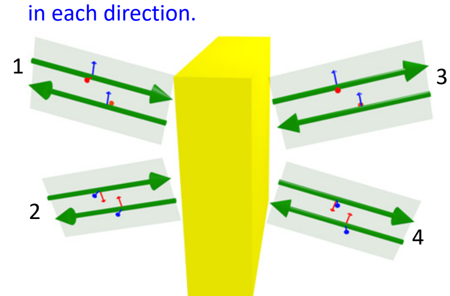
For TMM, this actually expands to a 4×4 element scattering matrix.

$$\begin{bmatrix} \mathbf{S}_{11} & \mathbf{S}_{12} \\ \mathbf{S}_{21} & \mathbf{S}_{22} \end{bmatrix} = \begin{bmatrix} s_{11} & s_{12} & s_{13} & s_{14} \\ s_{21} & s_{22} & s_{23} & s_{24} \\ s_{31} & s_{32} & s_{33} & s_{34} \\ s_{41} & s_{42} & s_{43} & s_{44} \end{bmatrix}$$

$$\mathbf{S}_{11} = \begin{bmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{bmatrix} \qquad \mathbf{S}_{12} = \begin{bmatrix} s_{13} & s_{14} \\ s_{23} & s_{24} \end{bmatrix}$$

$$\mathbf{S}_{21} = \begin{bmatrix} s_{31} & s_{32} \\ s_{41} & s_{42} \end{bmatrix} \qquad \mathbf{S}_{22} = \begin{bmatrix} s_{33} & s_{34} \\ s_{43} & s_{44} \end{bmatrix}$$

Each mode provides an I/O mechanism and there are two modes on each side in each direction.




Lecture 5a — Slide 16

## Scattering Matrices of Lossless Media

If a scattering matrix is composed of materials that have no loss and no gain, the scattering matrix must conserve power. That is, all incident power must either reflect or transmit.

This implies that the scattering matrix is *unitary*.

If the scattering matrix is unitary, it must obey the following rules:

$$\mathbf{S}^H = \mathbf{S}^{-1}$$

$$\mathbf{S}^H\mathbf{S} = \mathbf{S}\mathbf{S}^H = \mathbf{S}^{-1}\mathbf{S} = \mathbf{S}\mathbf{S}^{-1} = \mathbf{I}$$

Note: If the regions external to the layer are different from each other, the scattering matrices will not be unitary. This is because the field amplitudes will be different even though the field carries the same amount of power.



## Hints About Stability in These Formulations

- Diagonal elements $\mathbf{S}_{11}$ and $\mathbf{S}_{22}$ tend to be the largest numbers. Divide by these instead of any off-diagonal elements for best numerical stability.

$$\mathbf{S} = \begin{bmatrix} \mathbf{S}_{11} & \mathbf{S}_{12} \\ \mathbf{S}_{21} & \mathbf{S}_{22} \end{bmatrix}$$

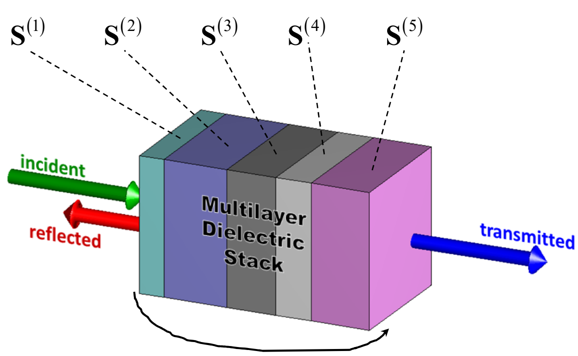
- $\mathbf{X}$ describes propagation through an entire layer. Don't divide by $\mathbf{X}$ or your code can become unstable.

# Multilayer Structures



## Solution Using Scattering Matrices

The scattering matrix method consists of working through the device one layer at a time and calculating an overall scattering matrix.




$$\mathbf{S}^{(\text{device})} = \mathbf{S}^{(1)} \otimes \mathbf{S}^{(2)} \otimes \mathbf{S}^{(3)} \otimes \mathbf{S}^{(4)} \otimes \mathbf{S}^{(5)}$$

Redheffer star product.
NOT matrix multiplication!

## Derivation of the Redheffer Star Product

We start with the equations for the two adjacent scattering matrices.

$$\begin{bmatrix} \mathbf{c}_1'^- \\ \mathbf{c}_2'^+ \end{bmatrix} = \begin{bmatrix} \mathbf{S}_{11}^{(A)} & \mathbf{S}_{12}^{(A)} \\ \mathbf{S}_{21}^{(A)} & \mathbf{S}_{22}^{(A)} \end{bmatrix} \begin{bmatrix} \mathbf{c}_1'^+ \\ \mathbf{c}_2'^- \end{bmatrix} \qquad \begin{bmatrix} \mathbf{c}_2'^- \\ \mathbf{c}_3'^+ \end{bmatrix} = \begin{bmatrix} \mathbf{S}_{11}^{(B)} & \mathbf{S}_{12}^{(B)} \\ \mathbf{S}_{21}^{(B)} & \mathbf{S}_{22}^{(B)} \end{bmatrix} \begin{bmatrix} \mathbf{c}_2'^+ \\ \mathbf{c}_3'^- \end{bmatrix}$$

We expand these into four matrix equations.

$$\mathbf{c}_1'^- = \mathbf{S}_{11}^{(A)}\mathbf{c}_1'^+ + \mathbf{S}_{12}^{(A)}\mathbf{c}_2'^- \quad \text{Eq. } (1) \qquad \mathbf{c}_2'^- = \mathbf{S}_{11}^{(B)}\mathbf{c}_2'^+ + \mathbf{S}_{12}^{(B)}\mathbf{c}_3'^- \quad \text{Eq. } (3)$$

$$\mathbf{c}_2'^+ = \mathbf{S}_{21}^{(A)}\mathbf{c}_1'^+ + \mathbf{S}_{22}^{(A)}\mathbf{c}_2'^- \quad \text{Eq. } (2) \qquad \mathbf{c}_3'^+ = \mathbf{S}_{21}^{(B)}\mathbf{c}_2'^+ + \mathbf{S}_{22}^{(B)}\mathbf{c}_3'^- \quad \text{Eq. } (4)$$

We substitute Eq. (2) into Eq. (3) to get an equation with only $\mathbf{c}_2'^-$.
We substitute Eq. (3) into Eq. (2) to get an equation with only $\mathbf{c}_2'^+$.

$$\left(\mathbf{I} - \mathbf{S}_{11}^{(B)}\mathbf{S}_{22}^{(A)}\right)\mathbf{c}_2'^- = \mathbf{S}_{11}^{(B)}\mathbf{S}_{21}^{(A)}\mathbf{c}_1'^+ + \mathbf{S}_{12}^{(B)}\mathbf{c}_3'^- \quad \text{Eq. } (5)$$

$$\left(\mathbf{I} - \mathbf{S}_{22}^{(A)}\mathbf{S}_{11}^{(B)}\right)\mathbf{c}_2'^+ = \mathbf{S}_{21}^{(A)}\mathbf{c}_1'^+ + \mathbf{S}_{22}^{(A)}\mathbf{S}_{12}^{(B)}\mathbf{c}_3'^- \quad \text{Eq. } (6)$$

We eliminate $\mathbf{c}_2'^-$ and $\mathbf{c}_2'^+$ by substituting these equations into Eq. (1) and (4). We then rearrange terms into the form of a scattering matrix.

$$\begin{bmatrix} \mathbf{c}_1'^- \\ \mathbf{c}_3'^+ \end{bmatrix} = \begin{bmatrix} ? & ? \\ ? & ? \end{bmatrix} \begin{bmatrix} \mathbf{c}_1'^+ \\ \mathbf{c}_3'^- \end{bmatrix}$$

Overall, this is just algebra. We start with 4 equations and 6 unknowns and reduce it to 2 equations with 4 unknowns.



## Redheffer Star Product

Two scattering matrices may be combined into a single scattering matrix using Redheffer's star product.

$$\mathbf{S}^{(AB)} = \mathbf{S}^{(A)} \otimes \mathbf{S}^{(B)} \qquad \mathbf{S}^{(A)} = \begin{bmatrix} \mathbf{S}_{11}^{(A)} & \mathbf{S}_{12}^{(A)} \\ \mathbf{S}_{21}^{(A)} & \mathbf{S}_{22}^{(A)} \end{bmatrix} \qquad \mathbf{S}^{(B)} = \begin{bmatrix} \mathbf{S}_{11}^{(B)} & \mathbf{S}_{12}^{(B)} \\ \mathbf{S}_{21}^{(B)} & \mathbf{S}_{22}^{(B)} \end{bmatrix}$$

The combined scattering matrix is then

$$\mathbf{S}^{(AB)} = \begin{bmatrix} \mathbf{S}_{11}^{(AB)} & \mathbf{S}_{12}^{(AB)} \\ \mathbf{S}_{21}^{(AB)} & \mathbf{S}_{22}^{(AB)} \end{bmatrix}$$

$$\mathbf{S}_{11}^{(AB)} = \mathbf{S}_{11}^{(A)} + \mathbf{S}_{12}^{(A)}\left[\mathbf{I} - \mathbf{S}_{11}^{(B)}\mathbf{S}_{22}^{(A)}\right]^{-1}\mathbf{S}_{11}^{(B)}\mathbf{S}_{21}^{(A)}$$

$$\mathbf{S}_{12}^{(AB)} = \mathbf{S}_{12}^{(A)}\left[\mathbf{I} - \mathbf{S}_{11}^{(B)}\mathbf{S}_{22}^{(A)}\right]^{-1}\mathbf{S}_{12}^{(B)}$$

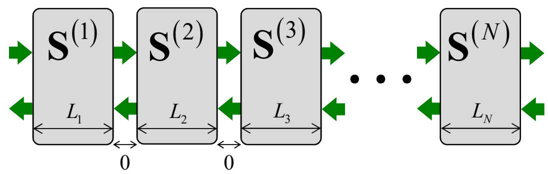
$$\mathbf{S}_{21}^{(AB)} = \mathbf{S}_{21}^{(B)}\left[\mathbf{I} - \mathbf{S}_{22}^{(A)}\mathbf{S}_{11}^{(B)}\right]^{-1}\mathbf{S}_{21}^{(A)}$$

$$\mathbf{S}_{22}^{(AB)} = \mathbf{S}_{22}^{(B)} + \mathbf{S}_{21}^{(B)}\left[\mathbf{I} - \mathbf{S}_{22}^{(A)}\mathbf{S}_{11}^{(B)}\right]^{-1}\mathbf{S}_{22}^{(A)}\mathbf{S}_{12}^{(B)}$$

R. Redheffer, "Difference equations and functional equations in transmission-line theory," *Modern Mathematics for the Engineer*, Vol. 12, pp. 282-337, McGraw-Hill, New York, 1961.

## Putting it All Together (1 of 2)

First, we calculate the *device scattering matrix* by iterating through each layer of the device and combining the scattering matrices using the Redheffer star product.

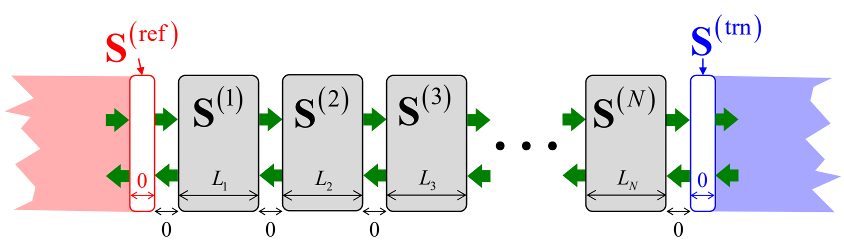



$$\mathbf{S}^{(\text{device})} = \mathbf{S}^{(1)} \otimes \mathbf{S}^{(2)} \otimes \mathbf{S}^{(3)} \otimes \cdots \otimes \mathbf{S}^{(N)}$$



## Putting it All Together (2 of 2)

Second, we must connect the device scattering matrix to the external regions to get the global scattering matrix. We use *connection scattering matrices* to do this.




$$\mathbf{S}^{(\text{global})} =$$

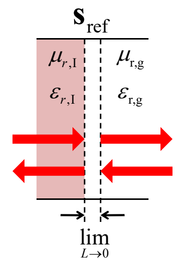
$$\mathbf{S}^{(\text{ref})} \otimes \mathbf{S}^{(1)} \otimes \mathbf{S}^{(2)} \otimes \mathbf{S}^{(3)} \otimes \cdots \otimes \mathbf{S}^{(N)} \otimes \mathbf{S}^{(\text{trn})}$$

## Reflection/Transmission Side Scattering Matrices

The reflection-side scattering matrix is

$$\mathbf{S}_{11}^{(\mathrm{ref})} = -\mathbf{A}_{\mathrm{ref}}^{-1}\mathbf{B}_{\mathrm{ref}}$$
$$\mathbf{S}_{12}^{(\mathrm{ref})} = 2\mathbf{A}_{\mathrm{ref}}^{-1}$$
$$\mathbf{S}_{21}^{(\mathrm{ref})} = 0.5\left(\mathbf{A}_{\mathrm{ref}} - \mathbf{B}_{\mathrm{ref}}\mathbf{A}_{\mathrm{ref}}^{-1}\mathbf{B}_{\mathrm{ref}}\right)$$
$$\mathbf{S}_{22}^{(\mathrm{ref})} = \mathbf{B}_{\mathrm{ref}}\mathbf{A}_{\mathrm{ref}}^{-1}$$

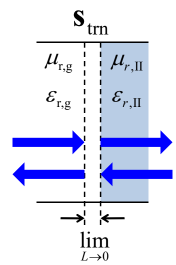
$$\mathbf{A}_{\mathrm{ref}} = \mathbf{W}_{\mathrm{g}}^{-1}\mathbf{W}_{\mathrm{ref}} + \mathbf{V}_{\mathrm{g}}^{-1}\mathbf{V}_{\mathrm{ref}}$$
$$\mathbf{B}_{\mathrm{ref}} = \mathbf{W}_{\mathrm{g}}^{-1}\mathbf{W}_{\mathrm{ref}} - \mathbf{V}_{\mathrm{g}}^{-1}\mathbf{V}_{\mathrm{ref}}$$




The transmission-side scattering matrix is

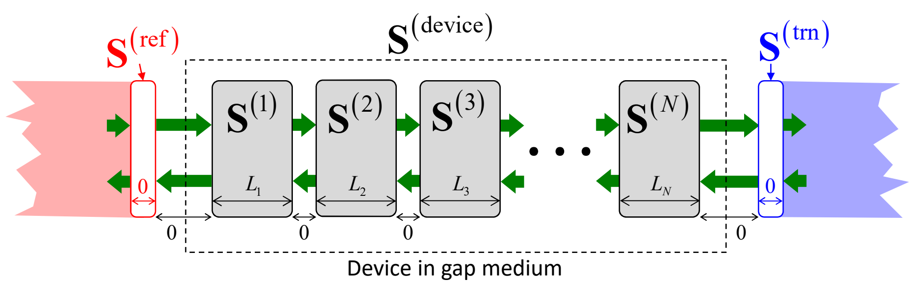
$$\mathbf{S}_{11}^{(\mathrm{trn})} = \mathbf{B}_{\mathrm{trn}}\mathbf{A}_{\mathrm{trn}}^{-1}$$
$$\mathbf{S}_{12}^{(\mathrm{trn})} = 0.5\left(\mathbf{A}_{\mathrm{trn}} - \mathbf{B}_{\mathrm{trn}}\mathbf{A}_{\mathrm{trn}}^{-1}\mathbf{B}_{\mathrm{trn}}\right)$$
$$\mathbf{S}_{21}^{(\mathrm{trn})} = 2\mathbf{A}_{\mathrm{trn}}^{-1}$$
$$\mathbf{S}_{22}^{(\mathrm{trn})} = -\mathbf{A}_{\mathrm{trn}}^{-1}\mathbf{B}_{\mathrm{trn}}$$

$$\mathbf{A}_{\mathrm{trn}} = \mathbf{W}_{\mathrm{g}}^{-1}\mathbf{W}_{\mathrm{trn}} + \mathbf{V}_{\mathrm{g}}^{-1}\mathbf{V}_{\mathrm{trn}}$$
$$\mathbf{B}_{\mathrm{trn}} = \mathbf{W}_{\mathrm{g}}^{-1}\mathbf{W}_{\mathrm{trn}} - \mathbf{V}_{\mathrm{g}}^{-1}\mathbf{V}_{\mathrm{trn}}$$






## Summary of Using Scattering Matrices



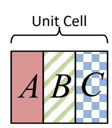

$$\mathbf{S}^{(\mathrm{global})} = \mathbf{S}^{(\mathrm{ref})} \otimes \underbrace{\left[\mathbf{S}^{(1)} \otimes \mathbf{S}^{(2)} \otimes \cdots \otimes \mathbf{S}^{(N)}\right]}_{\mathbf{S}^{(\mathrm{device})}} \otimes \mathbf{S}^{(\mathrm{trn})}$$

# Longitudinally Periodic Devices

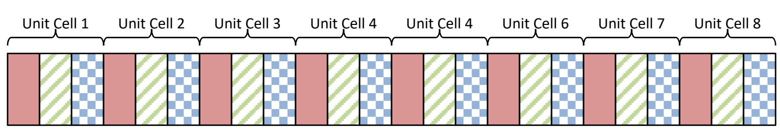


## Longitudinally Periodic Devices

Suppose we just calculated the scattering matrix for the unit cell of a longitudinally periodic device.




$$\mathbf{S}^{(\times 1)} = \mathbf{S}^{(A)} \otimes \mathbf{S}^{(B)} \otimes \mathbf{S}^{(C)}$$




There exists a very efficient way of calculating the global scattering matrix of a longitudinally periodic device without calculating and combining all the individual scattering matrices.

$$\mathbf{S}^{(\times 8)} = \mathbf{S}^{(A)} \otimes \mathbf{S}^{(B)} \otimes \mathbf{S}^{(C)} \otimes \mathbf{S}^{(A)} \otimes \mathbf{S}^{(B)} \otimes \mathbf{S}^{(C)} \otimes \mathbf{S}^{(A)} \otimes \mathbf{S}^{(B)} \otimes \mathbf{S}^{(C)} \otimes \mathbf{S}^{(A)} \otimes \mathbf{S}^{(B)} \otimes \mathbf{S}^{(C)} \otimes \mathbf{S}^{(A)} \otimes \mathbf{S}^{(B)} \otimes \mathbf{S}^{(C)} \otimes \mathbf{S}^{(A)} \otimes \mathbf{S}^{(B)} \otimes \mathbf{S}^{(C)} \otimes \mathbf{S}^{(A)} \otimes \mathbf{S}^{(B)} \otimes \mathbf{S}^{(C)} \otimes \mathbf{S}^{(A)} \otimes \mathbf{S}^{(B)} \otimes \mathbf{S}^{(C)}$$

$$\mathbf{S}^{(\times 8)} = \mathbf{S}^{(\times 1)} \otimes \mathbf{S}^{(\times 1)} \otimes \mathbf{S}^{(\times 1)} \otimes \mathbf{S}^{(\times 1)} \otimes \mathbf{S}^{(\times 1)} \otimes \mathbf{S}^{(\times 1)} \otimes \mathbf{S}^{(\times 1)} \otimes \mathbf{S}^{(\times 1)}$$

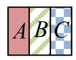
Both are inefficient*!!!*

## Cascading and Doubling

We can quickly build an overall scattering matrix that describes hundreds and thousands of unit cells.

We start by calculating the scattering matrix for a single unit cell.

$$\mathbf{S}^{(\times 1)} = \mathbf{S}^{(A)} \otimes \mathbf{S}^{(B)} \otimes \mathbf{S}^{(C)}$$

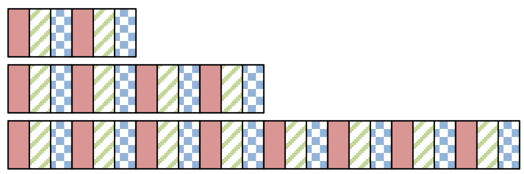



Next, we keep connecting the scattering matrix to itself to keep doubling the number of unit cells it describes.

$$\mathbf{S}^{(\times 2)} = \mathbf{S}^{(\times 1)} \otimes \mathbf{S}^{(\times 1)}$$

$$\mathbf{S}^{(\times 4)} = \mathbf{S}^{(\times 2)} \otimes \mathbf{S}^{(\times 2)}$$

$$\mathbf{S}^{(\times 8)} = \mathbf{S}^{(\times 4)} \otimes \mathbf{S}^{(\times 4)}$$

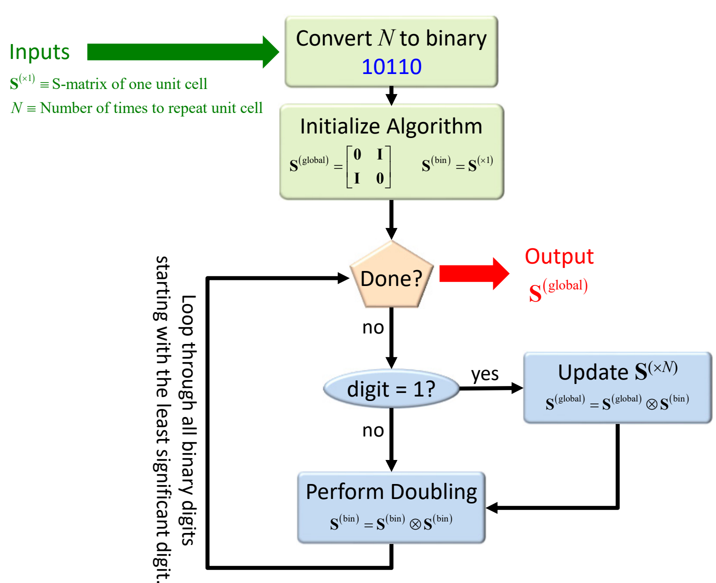


⋮ and so on… ⋮

What if you have a non-integer power of 2 number of layers?



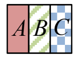
## Block Diagram for Modified Cascading and Doubling Algorithm



## Example of Cascading and Doubling Algorithm

**Step 0 – Calculate scattering matrix for one unit cell**

$\mathbf{S}^{(\times 1)} = \mathbf{S}^{(A)} \otimes \mathbf{S}^{(B)} \otimes \mathbf{S}^{(C)}$




**Inputs to algorithm:**

$\mathbf{S}^{(\times 1)} \equiv$ scattering matrix for a single unit cell

$N = 22 \equiv$ number of unit cells to combine

**Step 1 – Convert $N$ to binary**

| | 16's | 8's | 4's | 2's | 1's |
|---|---|---|---|---|---|
| $22 \rightarrow$ | 1 | 0 | 1 | 1 | 0 |

**Step 2 – Initialize binary and global scattering matrices**

$\mathbf{S}^{(\text{bin})} = \mathbf{S}^{(\times 1)}$

$\mathbf{S}_{11}^{(\text{global})} \equiv \mathbf{0}$

$\mathbf{S}_{12}^{(\text{global})} \equiv \mathbf{I}$

$\mathbf{S}_{21}^{(\text{global})} \equiv \mathbf{I}$

$\mathbf{S}_{22}^{(\text{global})} \equiv \mathbf{0}$



## Example of Cascading and Doubling Algorithm

**Step 3 – Loop through binary digits** $22 \rightarrow 10110$

1's digit = 0
- → Do not update $\mathbf{S}^{(\text{global})}$ — $\mathbf{S}^{(\text{global})}$ still encompasses 0 unit cells
- → Double $\mathbf{S}^{(\text{bin})}$ — $\mathbf{S}^{(\text{bin})} \equiv \mathbf{S}^{(\text{bin})} \otimes \mathbf{S}^{(\text{bin})}$ — $\mathbf{S}^{(\text{bin})}$ now represents 2 unit cells

2's digit = 1
- → Update $\mathbf{S}^{(\text{global})}$ — $\mathbf{S}^{(\text{global})} \equiv \mathbf{S}^{(\text{global})} \otimes \mathbf{S}^{(\text{bin})}$ — $\mathbf{S}^{(\text{global})}$ now encompasses 2 unit cells
- → Double $\mathbf{S}^{(\text{bin})}$ — $\mathbf{S}^{(\text{bin})} \equiv \mathbf{S}^{(\text{bin})} \otimes \mathbf{S}^{(\text{bin})}$ — $\mathbf{S}^{(\text{bin})}$ now represents 4 unit cells

4's digit = 1
- → Update $\mathbf{S}^{(\text{global})}$ — $\mathbf{S}^{(\text{global})} \equiv \mathbf{S}^{(\text{global})} \otimes \mathbf{S}^{(\text{bin})}$ — $\mathbf{S}^{(\text{global})}$ now encompasses 6 unit cells
- → Double $\mathbf{S}^{(\text{bin})}$ — $\mathbf{S}^{(\text{bin})} \equiv \mathbf{S}^{(\text{bin})} \otimes \mathbf{S}^{(\text{bin})}$ — $\mathbf{S}^{(\text{bin})}$ now represents 8 unit cells

8's digit = 0
- → Do not update $\mathbf{S}^{(\text{global})}$ — $\mathbf{S}^{(\text{global})}$ still encompasses 6 unit cells
- → Double $\mathbf{S}^{(\text{bin})}$ — $\mathbf{S}^{(\text{bin})} \equiv \mathbf{S}^{(\text{bin})} \otimes \mathbf{S}^{(\text{bin})}$ — $\mathbf{S}^{(\text{bin})}$ now represents 16 unit cells

16's digit = 1
- → Update $\mathbf{S}^{(\text{global})}$ — $\mathbf{S}^{(\text{global})} \equiv \mathbf{S}^{(\text{global})} \otimes \mathbf{S}^{(\text{bin})}$ — $\mathbf{S}^{(\text{global})}$ now encompasses 22 unit cells
- → Double $\mathbf{S}^{(\text{bin})}$ — $\mathbf{S}^{(\text{bin})} \equiv \mathbf{S}^{(\text{bin})} \otimes \mathbf{S}^{(\text{bin})}$ — $\mathbf{S}^{(\text{bin})}$ now represents 32 unit cells

Oops! This algorithm performs one unnecessary doubling operation.
How can we fix this?

# Dispersion Analysis



## Dispersion Analysis (1 of 2)

An overall scattering matrix is calculated that describes the unit cell.

$$\begin{bmatrix} \mathbf{c}_0^- \\ \mathbf{c}_{N+1}^+ \end{bmatrix} = \begin{bmatrix} \mathbf{S}_{11}^{(\text{uc})} & \mathbf{S}_{12}^{(\text{uc})} \\ \mathbf{S}_{21}^{(\text{uc})} & \mathbf{S}_{22}^{(\text{uc})} \end{bmatrix} \begin{bmatrix} \mathbf{c}_0^+ \\ \mathbf{c}_{N+1}^- \end{bmatrix}$$

$\mathbf{S}^{(\text{uc})}$ same as $\mathbf{S}^{(\times 1)}$ on previous slide

The terms are rearranged in "almost" the form of a transfer matrix.

$$\begin{bmatrix} \mathbf{0} & -\mathbf{S}_{12}^{(\text{uc})} \\ \mathbf{I} & -\mathbf{S}_{22}^{(\text{uc})} \end{bmatrix} \begin{bmatrix} \mathbf{c}_{N+1}^+ \\ \mathbf{c}_{N+1}^- \end{bmatrix} = \begin{bmatrix} \mathbf{S}_{11}^{(\text{uc})} & -\mathbf{I} \\ \mathbf{S}_{21}^{(\text{uc})} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{c}_0^+ \\ \mathbf{c}_0^- \end{bmatrix}$$

If the device is infinitely periodic in the $z$ direction, then the following periodic boundary condition must hold.

$$\begin{bmatrix} \mathbf{c}_{N+1}^+ \\ \mathbf{c}_{N+1}^- \end{bmatrix} = e^{jk_z \Lambda_z} \begin{bmatrix} \mathbf{c}_0^+ \\ \mathbf{c}_0^- \end{bmatrix}$$

Here $k_z$ is the "effective" propagation constant of the mode.

## Dispersion Analysis (2 of 2)

We substitute the periodic boundary condition into our rearranged equation to get

$$\begin{bmatrix} \mathbf{S}_{11}^{(\mathrm{uc})} & -\mathbf{I} \\ \mathbf{S}_{21}^{(\mathrm{uc})} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{c}_0^+ \\ \mathbf{c}_0^- \end{bmatrix} = e^{jk_z\Lambda_z} \begin{bmatrix} \mathbf{0} & -\mathbf{S}_{12}^{(\mathrm{uc})} \\ \mathbf{I} & -\mathbf{S}_{22}^{(\mathrm{uc})} \end{bmatrix} \begin{bmatrix} \mathbf{c}_0^+ \\ \mathbf{c}_0^- \end{bmatrix}$$

This is a generalized eigen-value problem.

$$\mathbf{Ax} = \lambda \mathbf{Bx}$$

$$\mathbf{A} = \begin{bmatrix} \mathbf{S}_{11}^{(\mathrm{uc})} & -\mathbf{I} \\ \mathbf{S}_{21}^{(\mathrm{uc})} & \mathbf{0} \end{bmatrix} \qquad \mathbf{x} = \begin{bmatrix} \mathbf{c}_0^+ \\ \mathbf{c}_0^- \end{bmatrix}$$

$$\mathbf{B} = \begin{bmatrix} \mathbf{0} & -\mathbf{S}_{12}^{(\mathrm{uc})} \\ \mathbf{I} & -\mathbf{S}_{22}^{(\mathrm{uc})} \end{bmatrix} \qquad \lambda = e^{jk_z\Lambda_z}$$

```
[V,D] = eig(A,B);
```

Eigen vectors
Bloch modes

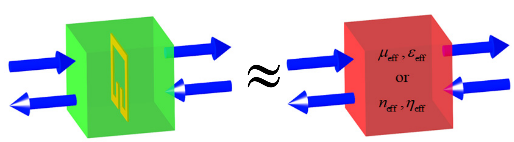
Eigen values
$k_z$'s



## Who Cares?

Given $k_z$, we can

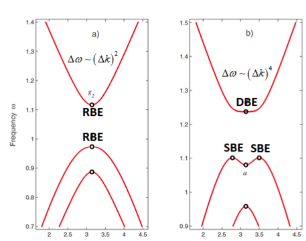
1. Calculate the effective properties of the unit cell.




$$k_z = \omega\sqrt{\mu_{\mathrm{r,eff}}\varepsilon_{\mathrm{r,eff}}}$$

This is an over simplification and beyond the scope of this course.

2. Construct band diagrams.



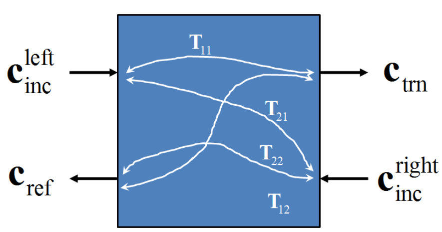
# Alternatives to Scattering Matrices



## Transmittance Matrices (T-Matrices)

The *T*-matrix method is the transfer matrix method where forward and backward waves are distinguished.

$$\begin{bmatrix} \mathbf{c}_{\text{trn}} \\ \mathbf{c}_{\text{inc}}^{\text{right}} \end{bmatrix} = \begin{bmatrix} \mathbf{T}_{11} & \mathbf{T}_{12} \\ \mathbf{T}_{21} & \mathbf{T}_{22} \end{bmatrix} \begin{bmatrix} \mathbf{c}_{\text{inc}}^{\text{left}} \\ \mathbf{c}_{\text{ref}} \end{bmatrix}$$




<u>Benefits</u>
- Much faster (5 to 10 times)
- Unconditionally stable

<u>Drawbacks</u>
- Less memory efficient
- Cannot exploit longitudinal periodicity
- Less popular in the literature

M. G. Moharam, Drew A. Pommet, Eric B. Grann, “Stable implementation of the rigorous coupled-wave analysis for surface-relief gratings: enhanced transmittance matrix approach,” J. Opt. Soc. Am. A, Vol. 12, No. 5, pp. 1077-1086, 1995.

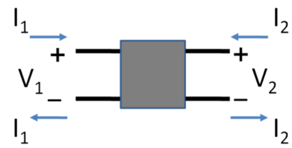
# Hybrid Matrices (H-Matrices)

The *h*-matrix method is borrowed from electrical two-port networks.

$$\begin{bmatrix} V_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{bmatrix} \begin{bmatrix} I_1 \\ V_2 \end{bmatrix}$$

$$h_{11} \equiv \left.\frac{V_1}{I_2}\right|_{V_2=0} \qquad h_{12} \equiv \left.\frac{V_1}{V_2}\right|_{I_1=0}$$

$$h_{21} \equiv \left.\frac{I_2}{I_1}\right|_{V_2=0} \qquad h_{22} \equiv \left.\frac{I_2}{V_2}\right|_{I_1=0}$$




In the framework of fields, the *h*-matrix is defined as

$$\begin{bmatrix} E_{x,i-1} \\ E_{y,i-1} \\ \tilde{H}_{x,i} \\ \tilde{H}_{y,i} \end{bmatrix} = \begin{bmatrix} \mathbf{H}_{11}^{(i)} & \mathbf{H}_{12}^{(i)} \\ \mathbf{H}_{21}^{(i)} & \mathbf{H}_{22}^{(i)} \end{bmatrix} \begin{bmatrix} \tilde{H}_{x,i-1} \\ \tilde{H}_{y,i-1} \\ E_{x,i} \\ E_{y,i} \end{bmatrix}$$

Claimed Benefits
- Improved numerical stability
- More concise formulation
- Simpler to implement
- Improved numerical efficiency (~30% better than ETM)
- Unconditionally stable

Eng L. Tan, "Hybrid-matrix algorithm for rigorous coupled-wave analysis of multilayered diffraction gratings," J. Mod. Opt., Vol. 53, No. 4, pp. 417-428, 2006.

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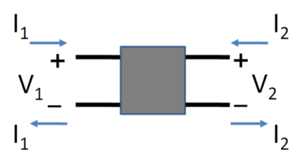
# R-Matrices

The *R*-matrix method is essentially the impedance matrix framework borrowed from electrical two-port networks.

$$\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix}$$

$$z_{11} \equiv \left.\frac{V_1}{I_1}\right|_{I_2=0} \qquad z_{12} \equiv \left.\frac{V_1}{I_2}\right|_{I_1=0}$$

$$z_{21} \equiv \left.\frac{V_2}{I_1}\right|_{I_2=0} \qquad z_{22} \equiv \left.\frac{V_2}{I_2}\right|_{I_1=0}$$




In the framework of fields, the *h*-matrix is defined as

$$\begin{bmatrix} E_{x,i-1} \\ E_{y,i-1} \\ E_{x,i} \\ E_{y,i} \end{bmatrix} = \begin{bmatrix} \mathbf{R}_{11}^{(i)} & \mathbf{R}_{12}^{(i)} \\ \mathbf{R}_{21}^{(i)} & \mathbf{R}_{22}^{(i)} \end{bmatrix} \begin{bmatrix} \tilde{H}_{x,i-1} \\ \tilde{H}_{y,i-1} \\ \tilde{H}_{x,i} \\ \tilde{H}_{y,i} \end{bmatrix}$$

Claimed Benefits
- Unconditionally stable
- Improved numerical efficiency

Lifeng Li, "Bremmer series, *R*-matrix propagation algorithm, and numerical modeling of diffraction gratings," J. Opt. Soc. Am. A, Vol. 11, No. 11, pp. 2829-2836, 1994.

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