EE 4347

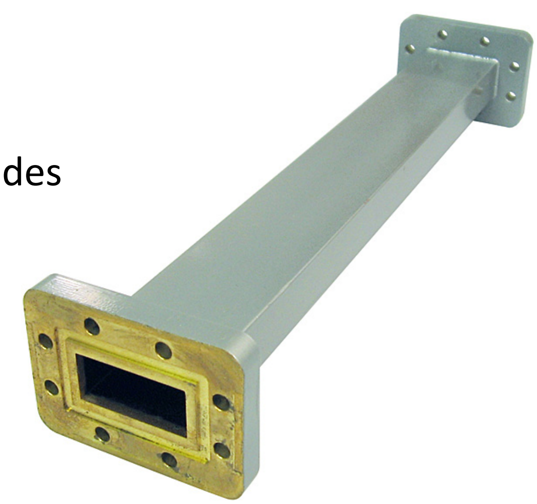
Applied Electromagnetics

Topic 5c

Rectangular Waveguide

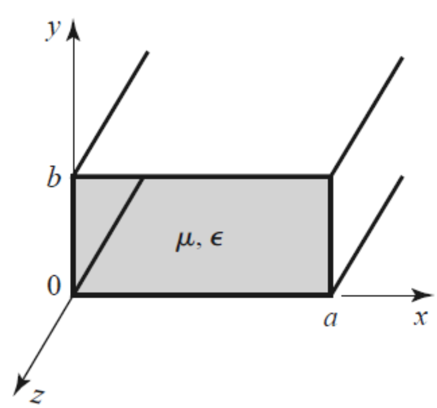
Lecture Outline

- What is a rectangular waveguide?
- TEM Analysis
- TM Analysis
- TE Analysis
- Visualization of Modes
- Conclusions



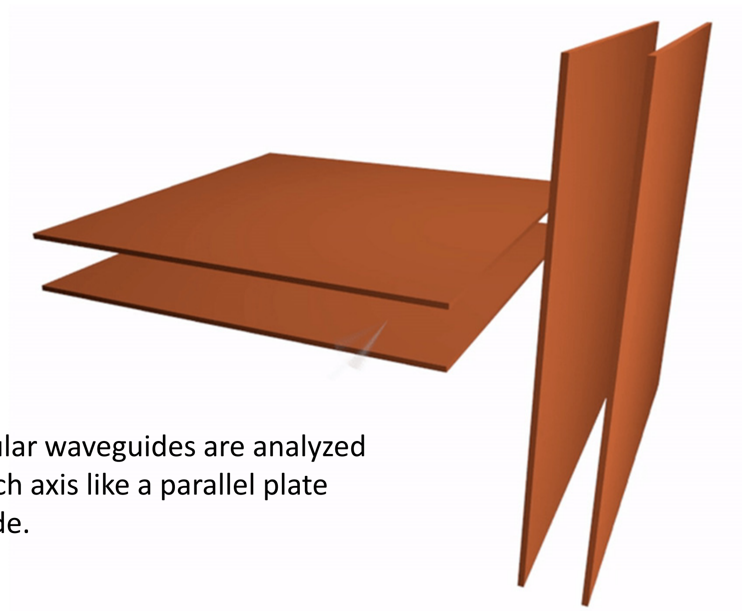
What is a Rectangular Waveguide?

Geometry of Rectangular Waveguide



Standard convention: $a \geq b$

Analysis of Rectangular Waveguide



Rectangular waveguides are analyzed along each axis like a parallel plate waveguide.

Notes on the Rectangular Waveguide

- Most classic waveguide example
- Some of the first waveguides used for microwaves
- Not a transmission because only one conductor
- Does not support a TEM mode
- Exhibits a low-frequency cutoff below which no waves will propagate

TE Analysis

Recall TE Analysis

The governing equation for TE analysis is

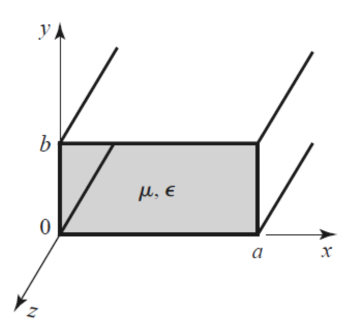
$$\frac{\partial^2 H_{0,z}}{\partial x^2} + \frac{\partial^2 H_{0,z}}{\partial y^2} - k_c^2 H_{0,z} = 0 \qquad k_c^2 = k^2 - \beta^2$$

After a solution is obtained, the remaining field components are calculated according to

$$H_{0,x} = -\frac{j\beta}{k_c^2}\frac{\partial H_{0,z}}{\partial x} \qquad E_{0,x} = -\frac{j\omega\mu}{k_c^2}\frac{\partial H_{0,z}}{\partial y}$$

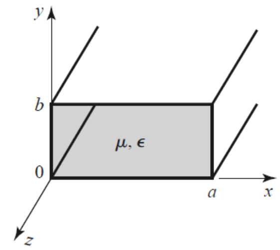
$$H_{0,y} = -\frac{j\beta}{k_c^2}\frac{\partial H_{0,z}}{\partial y} \qquad E_{0,y} = \frac{j\omega\mu}{k_c^2}\frac{\partial H_{0,z}}{\partial x}$$

$$E_{0,z} = 0$$



General Form of the Solution

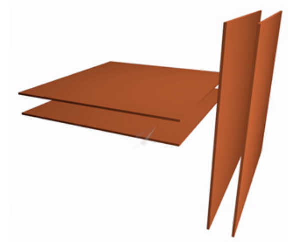
From the geometry of the waveguide, we can immediate write the form of the solution as



$$H_z(x,y,z) = H_{0,z}(x,y)e^{-j\beta z}$$

Viewing the rectangular waveguide as the combination of two parallel plate waveguides, we can apply separation of variables to write $H_{0,z}(x,y)$ as the product of two functions.

$$H_{0,z}(x,y) = X(x)Y(y)$$



Separation of Variables (1 of 3)

We write our solution as the product of two 1D functions and substitute that back into our differential equation.

$H_{0,z}(x,y) = X(x)Y(y)$

$$\frac{\partial^2 H_{0,z}}{\partial x^2} + \frac{\partial^2 H_{0,z}}{\partial y^2} - k_c^2 H_{0,z} = 0$$

$$\Downarrow$$

$$\frac{\partial^2 XY}{\partial x^2} + \frac{\partial^2 XY}{\partial y^2} - k_c^2 XY = 0$$

$$\frac{\partial^2 X}{\partial x^2} Y + X\frac{\partial^2 Y}{\partial y^2} - k_c^2 XY = 0$$

$$\frac{1}{X}\frac{d^2 X}{dx^2} + \frac{1}{Y}\frac{d^2 Y}{dy^2} - k_c^2 = 0$$

We have ordinary derivatives because $X(x)$ and $Y(y)$ have only one independent variable each.

Separation of Variables (2 of 3)

First, we focus our attention on the x-dependence in our differential equation.

$$\underbrace{\frac{1}{X}\frac{d^2X}{dx^2}}+\underbrace{\frac{1}{Y}\frac{d^2Y}{dy^2}}_{-k_x^2}-k_c^2=0$$

This definition will let us write the differential equation as a wave equation.

$$\frac{d^2X}{dx^2}-k_x^2X=0$$

Second, we focus our attention on the y-dependence in our differential equation.

$$\underbrace{\frac{1}{X}\frac{d^2X}{dx^2}}_{-k_y^2}-k_c^2+\frac{1}{Y}\frac{d^2Y}{dy^2}=0$$

This definition will let us write the differential equation as a wave equation.

$$\frac{d^2Y}{dy^2}-k_y^2Y=0$$

Separation of Variables (3 of 3)

We would like to be able to add our two new differential equations together to get the original differential equation.

$$\frac{d^2X}{dx^2}-k_x^2X=0$$
$$\frac{d^2Y}{dy^2}-k_y^2Y=0$$

$\Longrightarrow$

$$\frac{1}{X}\frac{d^2X}{dx^2}-k_x^2=0$$
$$+\ \frac{1}{Y}\frac{d^2Y}{dy^2}-k_y^2=0$$
$$\frac{1}{X}\frac{d^2X}{dx^2}+\frac{1}{Y}\frac{d^2Y}{dy^2}-k_x^2-k_y^2=0$$

We get our original differential equation back if

$$k_c^2=k_x^2+k_y^2$$

General Solution

We now have two differential equations to solve.

$$\frac{d^2X}{dx^2} - k_x^2 X = 0 \qquad \frac{d^2Y}{dy^2} - k_y^2 Y = 0$$

These are essentially the same differential equation so their solution has the same general form.

$$\frac{d^2X}{dx^2} - k_x^2 X = 0 \quad \rightarrow \quad X(x) = A\cos(k_x x) + B\sin(k_x x)$$ Slab waveguide along x

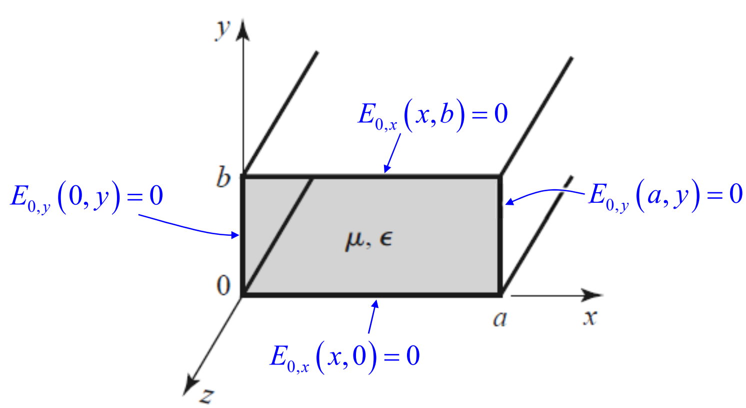
$$\frac{d^2Y}{dy^2} - k_y^2 Y = 0 \quad \rightarrow \quad Y(y) = C\cos(k_y y) + D\sin(k_y y)$$ Slab waveguide along y

The overall solution is the product of $X(x)$ and $Y(y)$.

$$H_{0,z}(x,y) = X(x)Y(y) = \left[A\cos(k_x x) + B\sin(k_x x)\right]\left[C\cos(k_y y) + D\sin(k_y y)\right]$$

Electromagnetic Boundary Conditions

Boundary conditions required that the tangential component of the electric field be zero at the boundary with a perfect conductor.



$E_{0,x}$ and $E_{0,y}$

In order to apply the boundary conditions, we must derive the electric field components $E_{0,x}$ and $E_{0,y}$ from our expression for $H_{0,z}$.

$$E_{0,x}(x,y) = -\frac{j\omega\mu}{k_c^2}\frac{\partial H_{0,z}}{\partial y}$$
$$= -\frac{j\omega\mu}{k_c^2}\frac{\partial}{\partial y}\left[A\cos(k_x x) + B\sin(k_x x)\right]\left[C\cos(k_y y) + D\sin(k_y y)\right]$$
$$= -\frac{j\omega\mu}{k_c^2}k_y\left[A\cos(k_x x) + B\sin(k_x x)\right]\left[-C\sin(k_y y) + D\cos(k_y y)\right]$$

$$E_{0,y}(x,y) = \frac{j\omega\mu}{k_c^2}\frac{\partial H_{0,z}}{\partial x}$$
$$= \frac{j\omega\mu}{k_c^2}\frac{\partial}{\partial x}\left[A\cos(k_x x) + B\sin(k_x x)\right]\left[C\cos(k_y y) + D\sin(k_y y)\right]$$
$$= \frac{j\omega\mu}{k_c^2}k_x\left[-A\sin(k_x x) + B\cos(k_x x)\right]\left[C\cos(k_y y) + D\sin(k_y y)\right]$$

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Apply Boundary Conditions (1 of 2)

At the $x = 0$ boundary, we have

$$0 = E_{0,y}(0,y)$$
$$= \frac{j\omega\mu}{k_c^2}k_x\left[-\cancel{A\sin(0)} + B\cos(0)\right]\left[C\cos(k_y y) + D\sin(k_y y)\right]$$
$$= \frac{j\omega\mu}{k_c^2}k_x\left[B\right]\left[C\cos(k_y y) + D\sin(k_y y)\right] \longrightarrow B = 0$$

At the $x = a$ boundary, we have

$$0 = E_{0,y}(a,y)$$
$$= \frac{j\omega\mu}{k_c^2}k_x\left[-A\sin(k_x a)\right]\left[C\cos(k_y y) + D\sin(k_y y)\right]$$

$A = 0$ leads to a trivial solution. It must be the $\sin(k_x a)$ term that enforces the BC.

$$0 = \sin(k_x a) \quad \rightarrow \quad k_x a = m\pi \quad m = 0,1,2,\ldots$$

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Apply Boundary Conditions (2 of 2)

At the $y = 0$ boundary, we have

$$0 = E_{0,x}(x,0)$$
$$= -\frac{j\omega\mu}{k_c^2} k_y \left[A\cos(k_x x) + B\sin(k_x x) \right] \left[-C\sin(0) + D\cos(0) \right]$$
$$= -\frac{j\omega\mu}{k_c^2} k_y \left[A\cos(k_x x) + B\sin(k_x x) \right] [D] \quad \longrightarrow \quad D = 0$$

At the $y = b$ boundary, we have

$$0 = E_{0,x}(x,b)$$
$$= -\frac{j\omega\mu}{k_c^2} k_y \left[A\cos(k_x x) + B\sin(k_x x) \right] \left[-C\sin(k_y b) \right]$$

$C = 0$ leads to a trivial solution. It must be the $\sin(k_y b)$ term that enforces the BC.

$$0 = \sin(k_y b) \quad \rightarrow \quad k_y b = n\pi \quad n = 0,1,2,\ldots$$

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Revised Solution for $H_{0,z}$

We have determined that $B = D = 0$ so our expression for $H_{0,z}$ becomes

$$H_{0,z}(x,y) = \boxed{AC}\cos(k_x x)\cos(k_y y)$$

We write the product AC as a single constant A_{mn}.

$$H_{0,z}(x,y) = A_{mn}\cos(k_x x)\cos(k_y y)$$

Also, recall our conditions for k_x and k_y.

$$k_x a = m\pi \quad \rightarrow \quad k_x = \frac{m\pi}{a} \qquad\qquad k_y b = n\pi \quad \rightarrow \quad k_y = \frac{n\pi}{b}$$

$$H_{0,z}(x,y) = A_{mn}\cos\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right)$$

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Entire Solution (1 of 2)

The final expression for $H_{0,z}$ is

$$H_{0,z}(x,y) = A_{mn}\cos\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right) \qquad E_{0,z}(x,y)=0$$

From this, the other field components are

$$E_{0,x}(x,y) = \frac{j\omega\mu n\pi}{k_c^2 b} A_{mn}\cos\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)$$

$$E_{0,y}(x,y) = -\frac{j\omega\mu m\pi}{k_c^2 a} A_{mn}\sin\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right)$$

$$H_{0,x}(x,y) = \frac{j\beta m\pi}{k_c^2 a} A_{mn}\sin\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right)$$

$$H_{0,y}(x,y) = \frac{j\beta n\pi}{k_c^2 b} A_{mn}\cos\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)$$

Entire Solution (2 of 2)

The overall electric and magnetic fields at any position are

$$E_x(x,y,z) = \frac{j\omega\mu n\pi}{k_c^2 b} A_{mn}\cos\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)e^{-j\beta z}$$

$$E_y(x,y,z) = -\frac{j\omega\mu m\pi}{k_c^2 a} A_{mn}\sin\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right)e^{-j\beta z}$$

$$E_z(x,y,z) = 0$$

$$H_x(x,y,z) = \frac{j\beta m\pi}{k_c^2 a} A_{mn}\sin\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right)e^{-j\beta z}$$

$$H_y(x,y,z) = \frac{j\beta n\pi}{k_c^2 b} A_{mn}\cos\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)e^{-j\beta z}$$

$$H_z(x,y,z) = A_{mn}\cos\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right)e^{-j\beta z}$$

Phase Constant, β

Recall the cutoff wave number

$$k_c^2 = k_x^2 + k_y^2$$

After analyzing the boundary conditions, this expression can be written as

$$k_c^2 = \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2$$

The phase constant β is therefore

$$k_c^2 = k^2 - \beta^2$$

$$\beta^2 = k^2 - k_c^2$$

$$\boxed{\beta_{mn} = \sqrt{k^2 - k_c^2} = \sqrt{k^2 - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2}}$$

Cutoff Frequency, f_c

Recall our expression for the phase constant

$$\beta = \sqrt{k^2 - k_c^2}$$

The phase constant must be a real number for a guided mode. This requires

$$k > k_c$$

Any time $k < k_c$, our mode is cutoff and not supported by the waveguide. From this, we can derive the cutoff frequency f_c.

$$k > k_c$$

$$\omega\sqrt{\mu\varepsilon} > k_c$$

$$2\pi f_c \sqrt{\mu\varepsilon} = k_c$$

$$\boxed{f_{c,mn} = \frac{k_c}{2\pi\sqrt{\mu\varepsilon}} = \frac{1}{2\pi\sqrt{\mu\varepsilon}}\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2}}$$

Characteristic Impedance, Z_{TE}

The characteristic impedance is

$$Z_{\text{TE}} = \frac{E_x}{H_y} = \frac{\frac{j\omega\mu n\pi}{k_c^2 b} A_{mn} \cos\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)e^{-j\beta z}}{\frac{j\beta n\pi}{k_c^2 b} A_{mn} \cos\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)e^{-j\beta z}} = \frac{\omega\mu}{\beta} = \frac{k\eta}{\beta}$$

Cutoff for First-Order TE Mode (1 of 2)

The cutoff frequency for the TE_{mn} mode was found to be

$$f_{c,mn} = \frac{1}{2\pi\sqrt{\mu\varepsilon}}\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2}$$

What about the TE_{00} mode?

$$\text{TE}_{00} \rightarrow m = n = 0$$

$$f_{c,00} = \frac{1}{2\pi\sqrt{\mu\varepsilon}}\sqrt{\left(\frac{0}{a}\right)^2 + \left(\frac{0}{b}\right)^2} = 0$$

The TE_{00} mode does not exist.

Cutoff for First-Order TE Mode (2 of 2)

What about the TE_{01} mode?

$$\text{TE}_{01} \rightarrow m=0, n=1$$

$$f_{c,01} = \frac{1}{2\pi\sqrt{\mu\varepsilon}}\sqrt{\left(\frac{0}{a}\right)^2 + \left(\frac{1\cdot\pi}{b}\right)^2} = \frac{1}{2b\sqrt{\mu\varepsilon}}$$

What about the TE_{10} mode?

$$\text{TE}_{10} \rightarrow m=1, n=0$$

$$f_{c,10} = \frac{1}{2\pi\sqrt{\mu\varepsilon}}\sqrt{\left(\frac{1\cdot\pi}{a}\right)^2 + \left(\frac{0}{b}\right)^2} = \frac{1}{2a\sqrt{\mu\varepsilon}}$$

Since $a > b$, we conclude that the first-order mode is TE_{10} because it has the lowest cutoff frequency.

CAUTION: We cannot yet say that the TE_{10} is the fundamental mode because we have not checked the cutoff frequency of the TM modes.

Single Mode Operation (1 of 2)

We wish to determine over what range of frequencies the waveguide supports only a single TE mode.

$$f_{c1} < f < f_{c2}$$

Low-Frequency Cutoff

We just found the lower frequency cutoff.

$$f_{c1} = \frac{1}{2a\sqrt{\mu\varepsilon}}$$

High-Frequency Cutoff

The high-frequency cutoff is the frequency where the second-order TE mode us supported. This could be the TE_{01}, TE_{11} or TE_{20} mode. We must consider all.

$$\text{TE}_{01}: \; f_{c,01} = \frac{1}{2\pi\sqrt{\mu\varepsilon}}\sqrt{\left(\frac{0}{a}\right)^2 + \left(\frac{1\cdot\pi}{b}\right)^2} = \frac{1}{2b\sqrt{\mu\varepsilon}}$$

$$\text{TE}_{11}: \; f_{c,11} = \frac{1}{2\pi\sqrt{\mu\varepsilon}}\sqrt{\left(\frac{\pi}{a}\right)^2 + \left(\frac{\pi}{b}\right)^2} = \frac{1}{2b\sqrt{\mu\varepsilon}}\sqrt{1+\left(\frac{b}{a}\right)^2}$$

$$\text{TE}_{20}: \; f_{c,20} = \frac{1}{2\pi\sqrt{\mu\varepsilon}}\sqrt{\left(\frac{2\cdot\pi}{a}\right)^2 + \left(\frac{0}{b}\right)^2} = \frac{1}{2b\sqrt{\mu\varepsilon}}\frac{2b}{a}$$

TE_{11} will always have a higher cutoff frequency than TE_{01}.

The second-order mode depends on our choice of a and b.

$$f_{c2} = \begin{cases} f_{c,01} & a \le 2b \\ f_{c,20} & a > 2b \end{cases}$$

Single Mode Operation (2 of 2)

APPLIED EM

Bandwidth

Typical rectangular waveguides will have $a > 2b$, so

$$f_{c1} = \frac{1}{2a\sqrt{\mu\varepsilon}} \qquad f_{c2} = \frac{1}{a\sqrt{\mu\varepsilon}}$$

$$\Delta f = f_{c2} - f_{c1} = \frac{1}{a\sqrt{\mu\varepsilon}} - \frac{1}{2a\sqrt{\mu\varepsilon}} = \frac{1}{2a\sqrt{\mu\varepsilon}}$$

Fractional Bandwidth

Continuing the assumption that we have $a > 2b$, the fractional bandwidth can be calculated from f_{c1} and f_{c2} above as follows

$$\text{FBW} = \frac{\Delta f}{f_c} = \frac{f_{c2} - f_{c1}}{\left(f_{c2} + f_{c1}\right)/2} = 2\frac{f_{c2} - f_{c1}}{f_{c2} + f_{c1}} = 2\frac{\frac{1}{a\sqrt{\mu\varepsilon}} - \frac{1}{2a\sqrt{\mu\varepsilon}}}{\frac{1}{a\sqrt{\mu\varepsilon}} + \frac{1}{2a\sqrt{\mu\varepsilon}}} = \frac{2}{3} = 66.7\%$$

Example #1 – TE Mode Analysis (1 of 4)

APPLIED EM

Suppose we have an air-filled rectangular waveguide with a = 3 cm and b = 2 cm.

What is the cutoff frequency of the waveguide?

$$f_{c1} = \frac{1}{2a\sqrt{\mu\varepsilon}} = \frac{c_0}{2a\sqrt{\mu_r\varepsilon_r}} = \frac{299792458 \text{ m/s}}{2\left(0.03 \text{ m}\right)\sqrt{\left(1.0\right)\left(1.0\right)}} = \boxed{5.0 \text{ GHz}}$$

Over what range of frequencies is the waveguide single mode?

We see that $a < 2b$, so the second-order mode is TE_{01}.

$$f_{c2} = \frac{1}{2b\sqrt{\mu\varepsilon}} = \frac{c_0}{2b\sqrt{\mu_r\varepsilon_r}} = \frac{299792458 \text{ m/s}}{2\left(0.02 \text{ m}\right)\sqrt{\left(1.0\right)\left(1.0\right)}} = 7.5 \text{ GHz}$$

$$\boxed{5.0 \text{ GHz} < f < 7.5 \text{ GHz}}$$

Example #1 – TE Mode Analysis (2 of 4)

APPLIED EM

What is the fractional bandwidth of the waveguide?

$$\text{FBW} = (100\%)\frac{\Delta f}{f} = (200\%)\frac{f_2 - f_1}{f_2 + f_1} = (200\%)\frac{7.5 - 5.0}{7.5 + 5.0} = \boxed{40\%}$$

Plot the phase constant and effective refractive index for the first-order and second-order modes from DC up to 15 GHz.

The phase constant is calculated as:

$$\beta_{mn} = \sqrt{k^2 - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2} \quad \rightarrow \quad \beta_1 = \sqrt{\left(\frac{2\pi f}{c_0}\right)^2 - \left(\frac{\pi}{a}\right)^2} \qquad \beta_2 = \sqrt{\left(\frac{2\pi f}{c_0}\right)^2 - \left(\frac{\pi}{b}\right)^2}$$

The effective refractive index is calculated as:

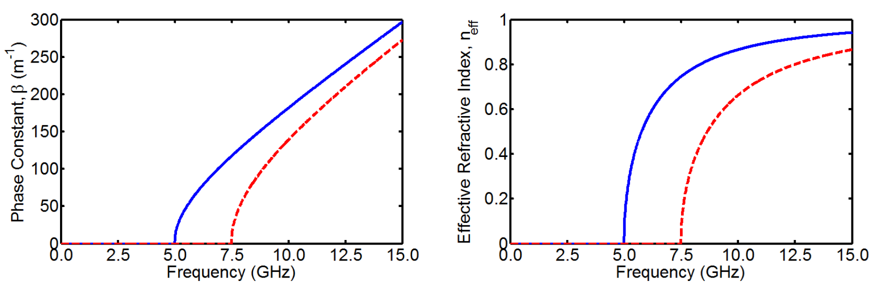
$$\beta = k_0 n_{\text{eff}} \quad \rightarrow \quad n_{\text{eff}} = \frac{\beta}{\frac{2\pi f}{c_0}} = \beta \frac{c_0}{2\pi f}$$

Example #1 – TE Mode Analysis (3 of 4)

APPLIED EM

Plot the phase constant and effective refractive index for the first-order and second-order modes from DC up to 15 GHz.

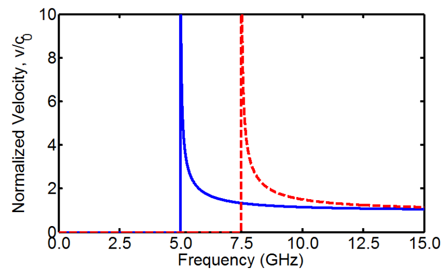
$$\beta_1 = \sqrt{\left(\frac{2\pi f}{c_0}\right)^2 - \left(\frac{\pi}{a}\right)^2} \qquad \beta_2 = \sqrt{\left(\frac{2\pi f}{c_0}\right)^2 - \left(\frac{\pi}{b}\right)^2} \qquad n_{\text{eff},1} = \beta_1 \frac{c_0}{2\pi f} \qquad n_{\text{eff},2} = \beta_2 \frac{c_0}{2\pi f}$$



Example #1 – TE Mode Analysis (4 of 4)

Plot the velocity of the modes as a function of frequency.

$$v_1 = \frac{c_0}{n_{\text{eff},1}} \qquad v_2 = \frac{c_0}{n_{\text{eff},2}}$$



Are our modes travelling faster than the speed of light?

Summary of TE Analysis

Field Solution

$$E_x(x,y,z) = \frac{j\omega\mu n\pi}{k_c^2 b} A_{mn} \cos\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)e^{-j\beta z}$$

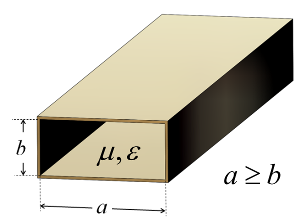
$$E_y(x,y,z) = -\frac{j\omega\mu m\pi}{k_c^2 a} A_{mn} \sin\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right)e^{-j\beta z}$$

$$E_z(x,y,z) = 0$$

$$H_x(x,y,z) = \frac{j\beta m\pi}{k_c^2 a} A_{mn} \sin\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right)e^{-j\beta z}$$

$$H_y(x,y,z) = \frac{j\beta n\pi}{k_c^2 b} A_{mn} \cos\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)e^{-j\beta z}$$

$$H_z(x,y,z) = A_{mn} \cos\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right)e^{-j\beta z}$$



- TE_{00} mode does not exist
- TE_{10} is the lowest order TE mode

Phase Constant

$$\beta_{mn} = \sqrt{k^2 - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2}$$

Same equation as for TM

Cutoff Frequency

$$f_{c,mn} = \frac{1}{2\pi\sqrt{\mu\varepsilon}}\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2}$$

Same equation as for TM

Characteristic Impedance

$$Z_{\text{TE},mn} = \frac{k\eta}{\beta_{mn}}$$

TM Analysis

Recall TM Analysis

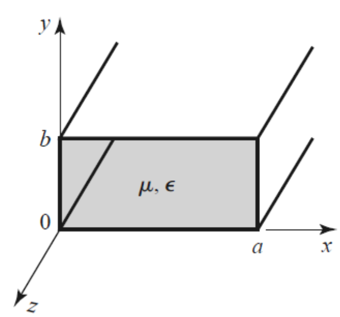
The governing equation for TM analysis is

$$\frac{\partial^2 E_{0,z}}{\partial x^2}+\frac{\partial^2 E_{0,z}}{\partial y^2}-k_c^2 E_{0,z}=0 \qquad H_{0,z}=0 \qquad k_c^2=k^2-\beta^2$$

After a solution is obtained, the remaining field components are calculated according to

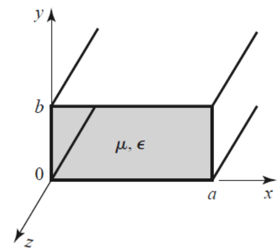
$$H_{0,x}=\frac{j\omega\varepsilon}{k_c^2}\frac{\partial E_{0,z}}{\partial y} \qquad E_{0,x}=-\frac{j\beta}{k_c^2}\frac{\partial E_{0,z}}{\partial x}$$

$$H_{0,y}=-\frac{j\omega\varepsilon}{k_c^2}\frac{\partial E_{0,z}}{\partial x} \qquad E_{0,y}=-\frac{j\beta}{k_c^2}\frac{\partial E_{0,z}}{\partial y}$$



General Form of the Solution

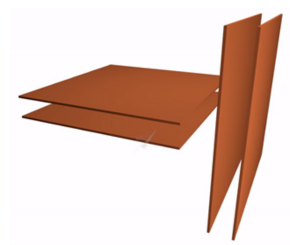
From the geometry of the waveguide, we can immediate write the form of the solution as



$$E_z(x,y,z) = E_{0,z}(x,y)e^{-j\beta z}$$

Viewing the rectangular waveguide as the combination of two parallel plate waveguides, we can apply separation of variables to write $E_{0,z}(x,y)$ as the product of two functions.

$$E_{0,z}(x,y) = X(x)Y(y)$$



Separation of Variables (1 of 3)

We write our solution as the product of two 1D functions and substitute that back into our differential equation.

$E_{0,z}(x,y) = X(x)Y(y)$

$$\frac{\partial^2 E_{0,z}}{\partial x^2} + \frac{\partial^2 E_{0,z}}{\partial y^2} - k_c^2 E_{0,z} = 0$$

$$\Downarrow$$

$$\frac{\partial^2 XY}{\partial x^2} + \frac{\partial^2 XY}{\partial y^2} - k_c^2 XY = 0$$

$$\frac{\partial^2 X}{\partial x^2} Y + X\frac{\partial^2 Y}{\partial y^2} - k_c^2 XY = 0$$

$$\frac{1}{X}\frac{d^2 X}{dx^2} + \frac{1}{Y}\frac{d^2 Y}{dy^2} - k_c^2 = 0$$

We have ordinary derivatives because $X(x)$ and $Y(y)$ have only one independent variable each.

Separation of Variables (2 of 3)

APPLIED EM

First, we focus our attention on the x-dependence in our differential equation.

$$\underbrace{\frac{1}{X}\frac{d^2X}{dx^2}}\;+\;\underbrace{\frac{1}{Y}\frac{d^2Y}{dy^2}}_{-k_x^2}-k_c^2=0$$

This definition will let us write the differential equation as a wave equation.

$$\frac{d^2X}{dx^2}-k_x^2X=0$$

Second, we focus our attention on the y-dependence in our differential equation.

$$\underbrace{\frac{1}{X}\frac{d^2X}{dx^2}-k_c^2}_{-k_y^2}+\frac{1}{Y}\frac{d^2Y}{dy^2}=0$$

This definition will let us write the differential equation as a wave equation.

$$\frac{d^2Y}{dy^2}-k_y^2Y=0$$

Separation of Variables (3 of 3)

APPLIED EM

We would like to be able to add our two new differential equations together to get the original differential equation.

$$\frac{d^2X}{dx^2}-k_x^2X=0$$
$$\frac{d^2Y}{dy^2}-k_y^2Y=0$$

$\Longrightarrow$

$$\frac{1}{X}\frac{d^2X}{dx^2}-k_x^2=0$$
$$+\;\frac{1}{Y}\frac{d^2Y}{dy^2}-k_y^2=0$$
$$\overline{\frac{1}{X}\frac{d^2X}{dx^2}+\frac{1}{Y}\frac{d^2Y}{dy^2}-k_x^2-k_y^2=0}$$

We get our original differential equation back if

$$k_c^2=k_x^2+k_y^2$$

General Solution

We now have two differential equations to solve.

$$\frac{d^2X}{dx^2} - k_x^2 X = 0 \qquad \frac{d^2Y}{dy^2} - k_y^2 Y = 0$$

These are essentially the same differential equation so their solution has the same general form.

$$\frac{d^2X}{dx^2} - k_x^2 X = 0 \rightarrow X(x) = A\cos(k_x x) + B\sin(k_x x)$$ Slab waveguide along x

$$\frac{d^2Y}{dy^2} - k_y^2 Y = 0 \rightarrow Y(y) = C\cos(k_y y) + D\sin(k_y y)$$ Slab waveguide along y

The overall solution is the product of $X(x)$ and $Y(y)$.

$$E_{0,z}(x,y) = X(x)Y(y) = \left[A\cos(k_x x) + B\sin(k_x x)\right]\left[C\cos(k_y y) + D\sin(k_y y)\right]$$

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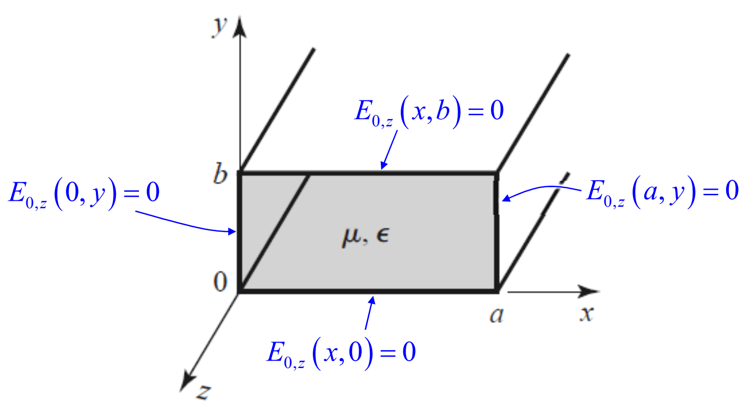
Electromagnetic Boundary Conditions

Boundary conditions required that the tangential component of the electric field be zero at the boundary with a perfect conductor. $E_{0,z}$ is tangential to all interfaces so we just use it. There is no need to calculate $E_{0,x}$ or $E_{0,y}$.



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Apply Boundary Conditions (1 of 2)

At the $x = 0$ boundary, we have

$$0 = E_{0,z}(0, y)$$
$$= \left[A\cos(0) + \cancel{B\sin(0)}\right]\left[C\cos(k_y y) + D\sin(k_y y)\right]$$
$$= [A]\left[C\cos(k_y y) + D\sin(k_y y)\right] \longrightarrow A = 0$$

At the $x = a$ boundary, we have

$$0 = E_{0,z}(a, y)$$
$$= \left[B\sin(k_x a)\right]\left[C\cos(k_y y) + D\sin(k_y y)\right]$$

$B = 0$ leads to a trivial solution. It must be the $\sin(k_x a)$ term that enforces the BC.

$$0 = \sin(k_x a) \quad \rightarrow \quad k_x a = m\pi \quad m = 1, 2, \ldots$$

$m = 0$ leads to a trivial solution.

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Apply Boundary Conditions (2 of 2)

At the $y = 0$ boundary, we have

$$0 = E_{0,x}(x, 0)$$
$$= \left[A\cos(k_x x) + B\sin(k_x x)\right]\left[C\cos(0) + \cancel{D\sin(0)}\right]$$
$$= \left[A\cos(k_x x) + B\sin(k_x x)\right][C] \longrightarrow C = 0$$

At the $y = b$ boundary, we have

$$0 = E_{0,x}(x, b)$$
$$= \left[A\cos(k_x x) + B\sin(k_x x)\right]\left[D\sin(k_y b)\right]$$

$D = 0$ leads to a trivial solution. It must be the $\sin(k_y b)$ term that enforces the BC.

$$0 = \sin(k_y b) \quad \rightarrow \quad k_y b = n\pi \quad n = 1, 2, \ldots$$

$n = 0$ leads to a trivial solution.

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Revised Solution for $H_{0,z}$

We have determined that $A = C = 0$ so our expression for $E_{0,z}$ becomes

$$E_{0,z}(x,y) = BD\sin(k_x x)\sin(k_y y)$$

We write the product BD as a single constant B_{mn}.

$$E_{0,z}(x,y) = B_{mn}\sin(k_x x)\sin(k_y y)$$

Also, recall our conditions for k_x and k_y.

$$k_x a = m\pi \quad \rightarrow \quad k_x = \frac{m\pi}{a} \qquad\qquad k_y b = n\pi \quad \rightarrow \quad k_y = \frac{n\pi}{b}$$

$$E_{0,z}(x,y) = B_{mn}\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)$$

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Entire Solution (1 of 2)

The final expression for $E_{0,z}$ is

$$E_{0,z}(x,y) = B_{mn}\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right) \qquad\qquad H_{0,z}(x,y) = 0$$

From this, the other field components are

$$E_{0,x}(x,y) = -\frac{j\beta m\pi}{k_c^2 a}B_{mn}\cos\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)$$

$$E_{0,y}(x,y) = -\frac{j\beta n\pi}{k_c^2 b}B_{mn}\sin\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right)$$

$$H_{0,x}(x,y) = \frac{j\omega\varepsilon n\pi}{k_c^2 b}B_{mn}\sin\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right)$$

$$H_{0,y}(x,y) = -\frac{j\omega\varepsilon m\pi}{k_c^2 a}B_{mn}\cos\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)$$

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Entire Solution (2 of 2)

The overall electric and magnetic fields at any position are

$$E_x(x,y,z) = -\frac{j\beta m\pi}{k_c^2 a} B_{mn} \cos\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)e^{-j\beta z}$$

$$E_y(x,y,z) = -\frac{j\beta n\pi}{k_c^2 b} B_{mn} \sin\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right)e^{-j\beta z}$$

$$E_z(x,y,z) = B_{mn} \sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)e^{-j\beta z}$$

$$H_x(x,y,z) = \frac{j\omega\varepsilon n\pi}{k_c^2 b} B_{mn} \sin\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right)e^{-j\beta z}$$

$$H_y(x,y,z) = -\frac{j\omega\varepsilon m\pi}{k_c^2 a} B_{mn} \cos\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)e^{-j\beta z}$$

$$H_z(x,y,z) = 0$$

Phase Constant, β

Recall the cutoff wave number

$$k_c^2 = k_x^2 + k_y^2$$

After analyzing the boundary conditions, this expression can be written as

$$k_c^2 = \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2$$

The phase constant β is therefore

$$k_c^2 = k^2 - \beta^2$$

$$\beta^2 = k^2 - k_c^2$$

$$\boxed{\beta_{mn} = \sqrt{k^2 - k_c^2} = \sqrt{k^2 - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2}}$$

Cutoff Frequency, f_c

Recall our expression for the phase constant

$$\beta = \sqrt{k^2 - k_c^2}$$

The phase constant must be a real number for a guided mode. This requires

$$k > k_c$$

Any time $k < k_c$, our mode is cutoff and not supported by the waveguide. From this, we can derive the cutoff frequency f_c.

$$k > k_c$$
$$\omega\sqrt{\mu\varepsilon} > k_c$$
$$2\pi f_c \sqrt{\mu\varepsilon} = k_c$$

$$\boxed{f_{c,mn} = \frac{k_c}{2\pi\sqrt{\mu\varepsilon}} = \frac{1}{2\pi\sqrt{\mu\varepsilon}}\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2}}$$

This is the same equation as for the TE modes.

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Characteristic Impedance, Z_{TM}

The characteristic impedance is

$$Z_{TM} = \frac{E_x}{H_y} = \frac{-\frac{j\beta m\pi}{k_c^2 a} B_{mn} \cos\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)e^{-j\beta z}}{-\frac{j\omega\varepsilon m\pi}{k_c^2 a} B_{mn} \cos\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)e^{-j\beta z}} = \frac{\beta}{\omega\varepsilon} = \frac{\beta\eta}{k}$$

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Cutoff for First-Order TM Mode (1 of 2)

The cutoff frequency for the TM_{mn} mode was found to be

$$f_{c,mn} = \frac{1}{2\pi\sqrt{\mu\varepsilon}}\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2}$$

Note, we cannot have $n = 0$ or $m = 0$ for the TM mode. So…

The TM_{00} mode does not exist.
The TM_{01} mode does not exist.
The TM_{10} mode does not exist.
The TM_{02} mode does not exist.
The TM_{20} mode does not exist.
The TM_{03} mode does not exist.
The TM_{30} mode does not exist.

etc.

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Cutoff for First-Order TM Mode (2 of 2)

What combination of m and n minimizes f_c?

$$f_{c,mn} = \frac{1}{2\pi\sqrt{\mu\varepsilon}}\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2}$$

Since $a > b$, the TM_{11} mode will have the lowest cutoff frequency.

$$m = 1, n = 1$$

$$f_{c1} = \frac{1}{2\pi\sqrt{\mu\varepsilon}}\sqrt{\left(\frac{1\cdot\pi}{a}\right)^2 + \left(\frac{1\cdot\pi}{b}\right)^2} = \frac{1}{2\sqrt{\mu\varepsilon}}\sqrt{\left(\frac{1}{a}\right)^2 + \left(\frac{1}{b}\right)^2}$$

CAUTION: We cannot yet say that the TM_{11} is the fundamental mode because we have not checked this against the TE modes.

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Example #2 – TM Mode Analysis (1 of 3)

Suppose we have an air-filled rectangular waveguide with a = 3 cm and b = 2 cm.

What is the cutoff frequency of the waveguide?

$$f_{c1} = \frac{1}{2\pi\sqrt{\mu\varepsilon}}\sqrt{\left(\frac{\pi}{a}\right)^2 + \left(\frac{\pi}{b}\right)^2}$$

$$= \frac{1}{2\pi\sqrt{\mu_0\mu_r\varepsilon_0\varepsilon_r}}\sqrt{\left(\frac{\pi}{a}\right)^2 + \left(\frac{\pi}{b}\right)^2}$$

$$= \frac{1}{2\pi\sqrt{\mu_0\varepsilon_0}\sqrt{\mu_r\varepsilon_r}}\sqrt{\left(\frac{\pi}{a}\right)^2 + \left(\frac{\pi}{b}\right)^2}$$

Recall that $c_0 = \frac{1}{\sqrt{\mu_0\varepsilon_0}}$

$$= \frac{c_0}{2\sqrt{\mu_r\varepsilon_r}}\sqrt{\left(\frac{1}{a}\right)^2 + \left(\frac{1}{b}\right)^2}$$

$$= \frac{299792458 \text{ m/s}}{2\sqrt{(1.0)(1.0)}}\sqrt{\left(\frac{1}{0.03 \text{ m}}\right)^2 + \left(\frac{1}{0.02 \text{ m}}\right)^2} = \boxed{9.0 \text{ GHz}}$$

Example #2 – TM Mode Analysis (2 of 3)

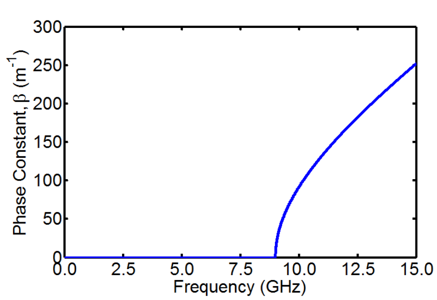
Plot the phase constant and effective refractive index for the first-order mode from DC up to 15 GHz.

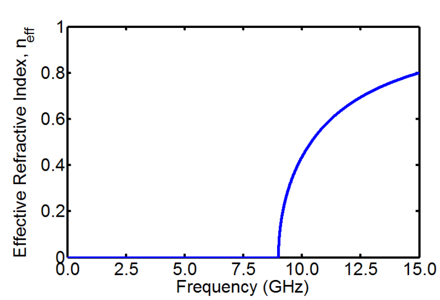
The phase constant is calculated as:

$$\beta = \sqrt{k^2 - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2} \quad \rightarrow \quad \beta = \sqrt{\left(\frac{2\pi f}{c_0}\right)^2 - \left(\frac{\pi}{a}\right)^2}$$

The effective refractive index is calculated as:

$$\beta = k_0 n_{\text{eff}} \quad \rightarrow \quad n_{\text{eff}} = \frac{\beta}{\frac{2\pi f}{c_0}} = \beta\frac{c_0}{2\pi f}$$

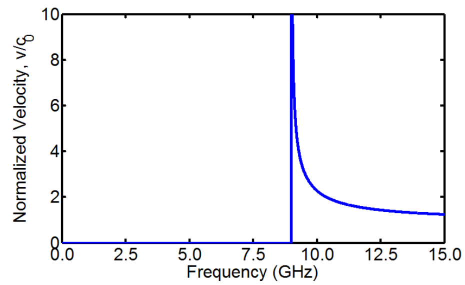




Example #2 – TM Mode Analysis (3 of 3)

Plot the velocity of the modes as a function of frequency.

$$v = \frac{c_0}{n_{\text{eff}}}$$



Are our modes travelling faster than the speed of light?

Summary of TM Analysis

Field Solution

$$E_x(x,y,z) = -\frac{j\beta m\pi}{k_c^2 a} B_{mn} \cos\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)e^{-j\beta z}$$

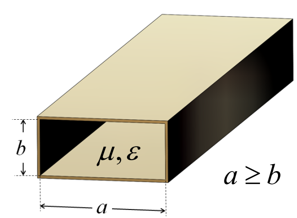
$$E_y(x,y,z) = -\frac{j\beta n\pi}{k_c^2 b} B_{mn} \sin\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right)e^{-j\beta z}$$

$$E_z(x,y,z) = B_{mn} \sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)e^{-j\beta z}$$

$$H_x(x,y,z) = \frac{j\omega\varepsilon n\pi}{k_c^2 b} B_{mn} \sin\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right)e^{-j\beta z}$$

$$H_y(x,y,z) = -\frac{j\omega\varepsilon m\pi}{k_c^2 a} B_{mn} \cos\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)e^{-j\beta z}$$

$$H_z(x,y,z) = 0$$



- $m \neq 0$ and $n \neq 0$, so TM_{00}, TM_{01}, TM_{02}, TM_{10}, TM_{20}, etc. are not supported modes.
- TM_{11} is the lowest order TM mode

Phase Constant

$$\beta_{mn} = \sqrt{k^2 - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2}$$

Same equation as for TE

Cutoff Frequency

$$f_{c,mn} = \frac{1}{2\pi\sqrt{\mu\varepsilon}}\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2}$$

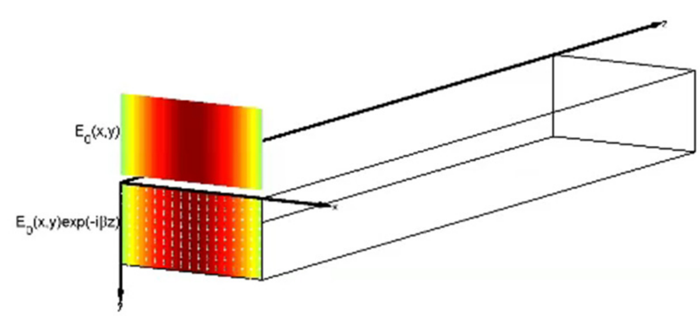
Same equation as for TE

Characteristic Impedance

$$Z_{\text{TM},mn} = \frac{\eta\beta_{mn}}{k}$$

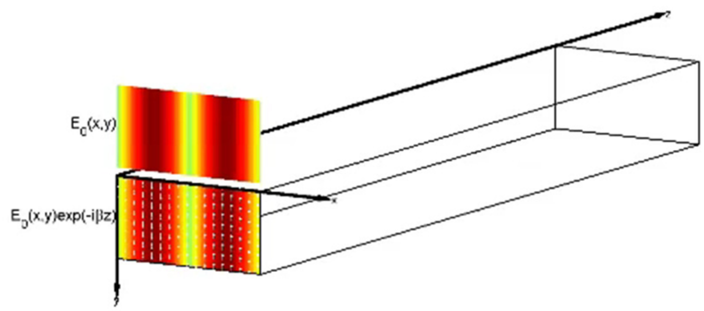
Visualization of Modes

Animation of TE_{10}



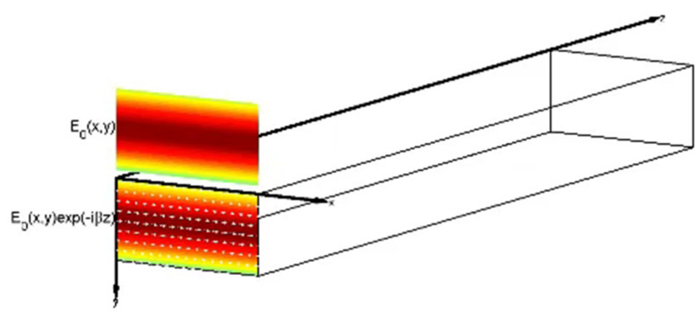
Notice one bright spot along x and zero along y. ($m = 1, n = 0$)

Animation of TE_{20}



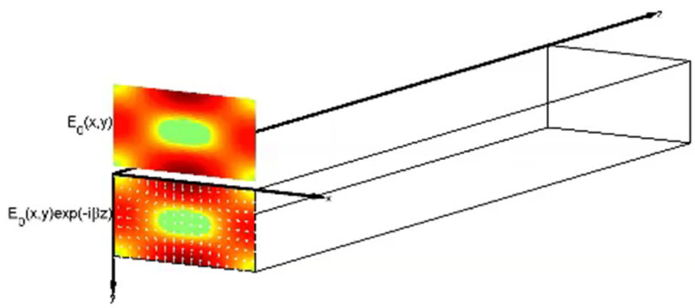
Notice two bright spots along x and zero along y. ($m = 2$, $n = 0$)

Animation of TE_{01}



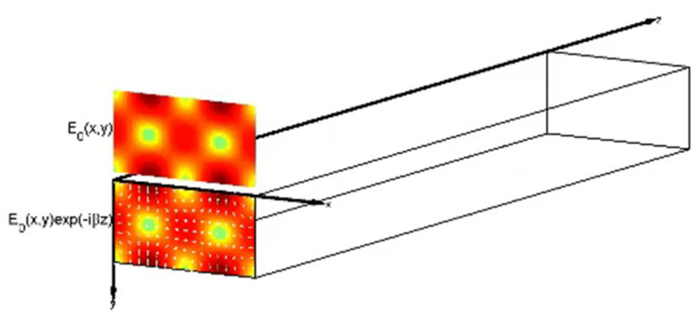
Notice zero bright spots along x and one along y. ($m = 0$, $n = 1$)

Animation of TE_{11}



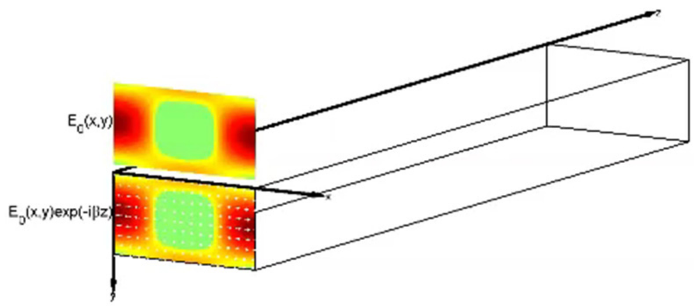
Notice one bright spot along x and one along y. ($m = 1$, $n = 1$)

Animation of TE_{21}



Notice two bright spots along x and one along y. ($m = 2$, $n = 1$)

Animation of TM_{11}



Conclusions

The Fundamental Mode

The fundamental mode is the mode which has the lowest cutoff frequency. This is either the TE_{01} or the TM_{11} mode.

$$f_{c,TE} = \frac{1}{2\sqrt{\mu\varepsilon}}\sqrt{\left(\frac{1}{a}\right)^2}$$

$$f_{c,TM} = \frac{1}{2\sqrt{\mu\varepsilon}}\sqrt{\left(\frac{1}{a}\right)^2 + \left(\frac{1}{b}\right)^2}$$

We can see that the TE_{01} mode will have the lowest cutoff frequency.

We conclude that the TE_{01} mode is the fundamental mode of the waveguide.

This is also called the dominant mode. When multiple modes are excited, usually most of the power ends up in the fundamental mode.

Example #3 – Mode Analysis (1 of 3)

Suppose we have an air-filled rectangular waveguide with a = 4 cm and b = 2 cm.

<u>Over what range of frequencies is this waveguide single mode?</u>

An easy way to do this is to calculate a table using a desktop computer.

m	n	TE Cutoff	TM Cutoff
0	0		
1	0	3.74 GHz	
2	0	7.48 GHz	
3	0	11.22 GHz	
4	0	14.96 GHz	
0	1	7.48 GHz	
1	1	8.37 GHz	8.37 GHz
2	1	10.58 GHz	10.58 GHz
3	1	13.49 GHz	13.49 GHz
4	1	16.73 GHz	16.73 GHz
0	2	14.96 GHz	
1	2	15.43 GHz	15.43 GHz
2	2	16.73 GHz	16.73 GHz
3	2	18.71 GHz	18.71 GHz
4	2	21.16 GHz	21.16 GHz
0	3	22.45 GHz	
1	3	22.76 GHz	22.76 GHz
2	3	23.66 GHz	23.66 GHz
3	3	25.10 GHz	25.10 GHz
4	3	26.98 GHz	26.98 GHz
0	4	29.93 GHz	
1	4	30.16 GHz	30.16 GHz
2	4	30.85 GHz	30.85 GHz
3	4	31.96 GHz	31.96 GHz
4	4	33.46 GHz	33.46 GHz

Example #3 – Mode Analysis (2 of 3)

Suppose we have an air-filled rectangular waveguide with a = 4 cm and b = 2 cm.

Over what range of frequencies is this waveguide single mode?

An easy way to do this is to calculate a table using a desktop computer.

Then sort the table in order of increasing cutoff frequency.

Mode	Cutoff
TE10	3.74 GHz
TE20	7.48 GHz
TE01	7.48 GHz
TE11	8.37 GHz
TM11	8.37 GHz
TE21	10.58 GHz
TM21	10.58 GHz
TE30	11.22 GHz
TE31	13.49 GHz
TM31	13.49 GHz
TE40	14.96 GHz
TE02	14.96 GHz
TE12	15.43 GHz
TM12	15.43 GHz
TE41	16.73 GHz

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Example #3 – Mode Analysis (3 of 3)

Suppose we have an air-filled rectangular waveguide with a = 4 cm and b = 2 cm.

Over what range of frequencies is this waveguide single mode?

An easy way to do this is to calculate a table using a desktop computer.

Then sort the table in order of increasing cutoff frequency.

Mode	Cutoff
TE10	3.74 GHz
TE20	7.48 GHz
TE01	7.48 GHz
TE11	8.37 GHz
TM11	8.37 GHz
TE21	10.58 GHz
TM21	10.58 GHz
TE30	11.22 GHz
TE31	13.49 GHz
TM31	13.49 GHz
TE40	14.96 GHz
TE02	14.96 GHz
TE12	15.43 GHz
TM12	15.43 GHz
TE41	16.73 GHz

We immediately see that the TE_{10} mode is the fundamental mode with the lowest cutoff frequency of 3.74 GHz.

The second-order mode is taken from the table to be either the TE_{20} or TE_{01} mode because both of these have the same cutoff frequency of 7.48 GHz.

The overall range of frequencies for single-mode operation is therefore

$$\boxed{3.74\text{ GHz} < f < 7.48\text{ GHz}}$$

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Key Points

- The rectangular waveguide is not a transmission line because it has less than two conductors.
- When filled with a homogeneous dielectric, the rectangular waveguide supports TE and TM modes, but not TEM.
- The cutoff frequencies for TE and TM modes are the same.
- The TE_{00} mode does not exist.
- For TM modes, $m \neq 0$ and $n \neq 0$.
- The TE_{10} is the dominant mode because the TM_{10} mode does not exist.
- It is only the phase velocity that exceeds the vacuum speed of light.

Summary from Text Book

Quantity	TE_{mn} Mode	TM_{mn} Mode
k	$\omega\sqrt{\mu\epsilon}$	$\omega\sqrt{\mu\epsilon}$
k_c	$\sqrt{(m\pi/a)^2 + (n\pi/b)^2}$	$\sqrt{(m\pi/a)^2 + (n\pi/b)^2}$
β	$\sqrt{k^2 - k_c^2}$	$\sqrt{k^2 - k_c^2}$
λ_c	$\frac{2\pi}{k_c}$	$\frac{2\pi}{k_c}$
λ_g	$\frac{2\pi}{\beta}$	$\frac{2\pi}{\beta}$
v_p	$\frac{\omega}{\beta}$	$\frac{\omega}{\beta}$
α_d	$\frac{k^2 \tan\delta}{2\beta}$	$\frac{k^2 \tan\delta}{2\beta}$
E_z	0	$B \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b} e^{-j\beta z}$
H_z	$A \cos\frac{m\pi x}{a} \cos\frac{n\pi y}{b} e^{-j\beta z}$	0
E_x	$\frac{j\omega\mu n\pi}{k_c^2 b} A \cos\frac{m\pi x}{a} \sin\frac{n\pi y}{b} e^{-j\beta z}$	$\frac{-j\beta m\pi}{k_c^2 a} B \cos\frac{m\pi x}{a} \sin\frac{n\pi y}{b} e^{-j\beta z}$
E_y	$\frac{-j\omega\mu m\pi}{k_c^2 a} A \sin\frac{m\pi x}{a} \cos\frac{n\pi y}{b} e^{-j\beta z}$	$\frac{-j\beta n\pi}{k_c^2 b} B \sin\frac{m\pi x}{a} \cos\frac{n\pi y}{b} e^{-j\beta z}$
H_x	$\frac{j\beta m\pi}{k_c^2 a} A \sin\frac{m\pi x}{a} \cos\frac{n\pi y}{b} e^{-j\beta z}$	$\frac{j\omega\epsilon n\pi}{k_c^2 b} B \sin\frac{m\pi x}{a} \cos\frac{n\pi y}{b} e^{-j\beta z}$
H_y	$\frac{j\beta n\pi}{k_c^2 b} A \cos\frac{m\pi x}{a} \sin\frac{n\pi y}{b} e^{-j\beta z}$	$\frac{-j\omega\epsilon m\pi}{k_c^2 a} B \cos\frac{m\pi x}{a} \sin\frac{n\pi y}{b} e^{-j\beta z}$
Z	$Z_{TE} = \frac{k\eta}{\beta}$	$Z_{TM} = \frac{\beta\eta}{k}$