Computational Science:
Computational Methods in Engineering

Errors in Computation

Outline

• Errors in Computation
• Error Propagation
• Uncertainty Analysis
Errors in Computation

Types of Errors

**Truncation Error** – Arises when approximations are used instead of performing exact operations.

\[
\frac{\partial f(x)}{\partial x} \approx \frac{f(x_b) - f(x_a)}{b - a}
\]

\[
\sqrt{1 + x} \approx 1 + \frac{x}{2}
\]

**Round-Off Error** – Arises when limited significant figures are used to represent exact numbers.

\[
\pi, e, \sqrt{3}
\]

\[
>> 0.3/0.1 - 3
\]

ans =

\[-4.4409e-16\]

**Human Error** – Arises when people make mistakes.

\[1+1=3\]
Quantifying Error  
*Significant Figures*

Very often quantities are limited to some number of digits. This can happen because a computer cannot store any more digits or because the measurement is not accurate out to that many digits.

**Rules:**  
1. All non-zero digits are significant  
2. Zeros that do nothing but place the decimal point are not significant.

**How many significant digits?**

- 1.234 \( \rightarrow \) 4 significant digits  
- 0.00545 \( \rightarrow \) 3 significant digits  
- 32100 \( \rightarrow \) 3 or 5 significant digits (ambiguous)  
- 0.0500 \( \rightarrow \) 3 significant digits

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Quantifying Error  
*True Error*

**True Error** – Actual error in a measurement.  
\[
\Delta = (\text{true error}) = (\text{true value}) - (\text{approximate value})
\]

**Example**

The speed of light is exactly 299,792,458 m/s. An experiment was performed that measured the speed of light to be 299,792,445 m/s. What is the true error in this measurement?

- \(\text{(true value)} = 299,792,458 \text{ m/s}\)  
- \(\text{(approximate value)} = 299,792,445 \text{ m/s}\)  
- \(\Delta = (\text{true error}) = (299,792,458 \text{ m/s}) - (299,792,445 \text{ m/s}) = 13 \text{ m/s}\)

**Notes**

- It is very rare to know the true value or true error of a measurement.  
- Somewhat unrealistic error model because the true value must be known, but good for analyzing how errors propagate through computations.  
- True error does not convey the relative severity of the error.
Quantifying Error

**Relative Error**

**Relative Error** – Actual error in a measurement relative to the size of the measurement.

\[
\varepsilon_r = \text{relative error} = \frac{\text{true error}}{\text{true value}}
\]

**Example**

The speed of light is exactly 299,792,458 m/s. An experiment was performed that measured the speed of light to be 299,792,445 m/s. What is the relative error of this measurement?

- (true error) = 13 m/s
- (true value) = 299,792,458 m/s
- (relative error) = \[\frac{13 \text{ m/s}}{299,792,458 \text{ m/s}} = 4.33 \times 10^{-4} = 0.000000433 = 0.00000433\%\]

**Notes**

This is a much better metric for determining the severity of the error.

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Quantifying Error

**Approximate Relative Error**

**Approximate Relative Error** – The true value of something is rarely known so errors must be calculated based on an approximate error.

\[
\varepsilon_a = \text{(approximate relative error)} = \frac{\text{(approximate error)}}{\text{(approximate value)}}
\]

**Note**

Quantities are correct to at least \( n \) significant digits if:

\[
|\varepsilon_a| \leq \left( 0.5 \times 10^{-n} \right) \%
\]
Error Propagation

What is Error Propagation?

In computation, it is necessary to calculate things from quantities that have some error. This results in some degree of error in the computations.

\[ \tilde{x} = x + \Delta x \]

How can the error \( \Delta f \) of \( f(\tilde{x}) \) be calculated from \( \Delta x \)?
Derive Error Propagation Equation

**Taylor Series**

The Taylor series is very often used to determine error associated with truncation.

\[ f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k \]

This is an infinite array. Retain only the first \( n + 1 \) terms.

\[ f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + R_n \]

All remaining terms have been lumped into \( R_n \).

\[ R_n = \int_a^x \frac{t^n}{n!} f^{(n+1)}(t) \, dt \]

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**Propagating Error**

First, rewrite the Taylor series in terms of the measured value \( \tilde{x} \).

\[ f(x) = f(\tilde{x}) + f'(\tilde{x})(x-\tilde{x}) + \frac{f''(\tilde{x})}{2!}(x-\tilde{x})^2 + \cdots \]

Second, bring \( f(\tilde{x}) \) to the left side of the equation.

\[ f(x) - f(\tilde{x}) = f'(\tilde{x})(x-\tilde{x}) + \frac{f''(\tilde{x})}{2!}(x-\tilde{x})^2 + \cdots \]

This is the definition of error \( \Delta f \) of \( f(\tilde{x}) \).

These are both \( \Delta x \).

Third, ignore all higher order terms to get an expression for error \( \Delta f \).

\[ \Delta f = f'(\tilde{x}) \Delta x + \frac{f''(\tilde{x})}{2!} (\Delta x)^2 + \cdots \]

**Generalization to Two Variables**

\[ \Delta f = \frac{\partial f}{\partial x} \Delta x + \frac{\partial f}{\partial y} \Delta y \]

\[ \text{error} = \Delta f = f'(\tilde{x}) \Delta x \]
Example

Example 1
Suppose we have $\bar{x} = 2.0$ with an error of $\Delta x = 0.05$.
What is the error $\Delta f$ if $f(\bar{x}) = \bar{x}^2$?

Solution

$$
\Delta f(\bar{x}) = f'(\bar{x}) \cdot \Delta x
$$

Start with the definition of error.

$$
= 2\bar{x} \cdot \Delta x
$$

Derive an expression for $f'(\bar{x})$ and substitute into equation.

$$
= 2(2.0) \cdot 0.05
$$

Substitute in values for $\bar{x}$ and $\Delta x$.

$$
= 4 \cdot 0.05
$$

Calculate and simplify.

$$
= 0.2
$$

Final answer

Uncertainty Analysis
What is Uncertainty?

How long is this object?

The measurement is written as $x \pm \sigma_x$

Since the measurement error is “blurred,” the exact error is not certain. The uncertainty $\sigma_x$ is a statistically derived quantity.

Notes
- This is a more realistic way to treat error because the true value does not have to be known.
- Uncertainty is used to predict errors in computations.

How to Interpret the Uncertainty $\sigma$

$\sigma$ is one standard deviation.

There is a 68% chance that the error will be less than $1\sigma$.
There is a 32% chance that the error will be greater than $1\sigma$. 
Uncertainty Analysis

Suppose there are multiple parameters and each has an uncertainty associated with it.

\[ x \pm \sigma_x, \quad y \pm \sigma_y, \quad z \pm \sigma_z \]

Now suppose a new quantity \( f \) is calculated from these.

\[ f(x, y, z) \]

What is the uncertainty \( \sigma_f \) of \( f(x, y, z) \)?

\[ \sigma_f^2 = \left( \frac{\partial f}{\partial x} \sigma_x \right)^2 + \left( \frac{\partial f}{\partial y} \sigma_y \right)^2 + \left( \frac{\partial f}{\partial z} \sigma_z \right)^2 \]

This is called propagating uncertainty through a calculation.

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Table of Common Uncertainty Calculations

<table>
<thead>
<tr>
<th>Function</th>
<th>Uncertainty</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f = ax )</td>
<td>( \sigma_f = \sigma_a )</td>
</tr>
<tr>
<td>( f = ax \pm by )</td>
<td>( \sigma_f = \sqrt{(\sigma_a)^2 + (b \sigma_y)^2} )</td>
</tr>
<tr>
<td>( f = ax \cdot y )</td>
<td>( \sigma_f = \sqrt{(\sigma_a)^2 + (\sigma_y)^2} )</td>
</tr>
<tr>
<td>( f = \sin x )</td>
<td>( \sigma_f = \sigma_x \cdot \cos x )</td>
</tr>
<tr>
<td>( f = \cos x )</td>
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<tr>
<td>( f = \tan x )</td>
<td>( \sigma_f = \sigma_x \cdot \sec^2 x )</td>
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<tr>
<td>( f = \sin^{-1} x )</td>
<td>( \sigma_f = \sigma_x / (1-x^2)^{1/2} )</td>
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<tr>
<td>( f = \tan^{-1} x )</td>
<td>( \sigma_f = \sigma_x / (1+x^2)^{1/2} )</td>
</tr>
</tbody>
</table>
Uncertainty Through Multiple Calculations

Suppose the uncertainty $\sigma_f$ is to be calculated when multiple calculations are involved.

$$f(a, b, c) = \ln(a + bc)$$

Given $\sigma_a$, $\sigma_b$, and $\sigma_c$

<table>
<thead>
<tr>
<th>Value</th>
<th>Uncertainty</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 $bc$</td>
<td>$\left(\frac{\sigma_{bc}}{bc}\right)^2 = \left(\frac{\sigma_b}{b}\right)^2 + \left(\frac{\sigma_c}{c}\right)^2$</td>
</tr>
<tr>
<td>2 $a + bc$</td>
<td>$\sigma_{a+bc}^2 = \sigma_a^2 + \sigma_{bc}^2$</td>
</tr>
<tr>
<td>3 $\ln(a + bc)$</td>
<td>$\sigma_{\ln(a+bc)} = \frac{\sigma_{a+bc}}{a + bc}$</td>
</tr>
</tbody>
</table>

Example 1

What is the uncertainty of the sum of two quantities?

$x + y$

Solution

$$\sigma_f^2 = \left(\sigma_x \frac{\partial f}{\partial x}\right)^2 + \left(\sigma_y \frac{\partial f}{\partial y}\right)^2$$

$$f = x + y$$

$$\sigma_f^2 = \left[\sigma_x \frac{\partial}{\partial x}(x + y)\right]^2 + \left[\sigma_y \frac{\partial}{\partial y}(x + y)\right]^2 = [\sigma_x \cdot 1]^2 + [\sigma_y \cdot 1]^2$$

$$\sigma_f = \sqrt{\sigma_x^2 + \sigma_y^2}$$
Example 2

What is the uncertainty of a decibel quantity?

\[ 10 \log_{10} (x \pm \sigma_x) \]

Solution

\[ \sigma_f^2 = \left( \sigma_x \frac{\partial f}{\partial x} \right)^2 \]

\[ f(x) = 10 \log_{10} (x) \]

\[ \frac{\partial f}{\partial x} = \frac{1}{\ln 10} \left( \frac{10 \log_{10}(x)}{x} \right) = \frac{10}{x \ln 10} \]

\[ \sigma_f^2 = \left( \sigma_x \frac{10}{x \ln 10} \right)^2 \]

\[ \sigma_f = 4.3429 \frac{\sigma_x}{x} \]

Example 3

The signal-to-noise ratio (SNR) of a system is 50 \( \pm 1.2 \). What is the SNR in decibels along with the uncertainty?

Solution

The SNR in decibels is

\[ 10 \log_{10} (50) = 16.99 \text{ dB} \]

The uncertainty is calculated using the equation derived in the previous example.

\[ \sigma_f = 4.3429 \frac{\sigma_x}{x} = 4.3429 \frac{1.2}{50} = 0.1 \]

The final answer is

\[ \text{SNR} = 16.99 \pm 0.1 \text{ dB} \]
Example 4

The height of person 1 was measured to be 6.0 ± 0.1 ft.

The height of person 2 was measured to be 5.5±0.1 ft.

If person 2 stands on the head of person 1, what is the total height and the uncertainty of the total height?

Solution

Total height

\[ h = h_1 + h_2 \]

\[ = (6.0 \text{ ft}) + (5.5 \text{ ft}) \]

\[ = 11.5 \text{ ft} \]

Uncertainty

\[ \sigma_h = \sqrt{\sigma_1^2 + \sigma_2^2} \]

\[ = \sqrt{(0.1 \text{ ft})^2 + (0.1 \text{ ft})^2} \]

\[ = 0.14 \text{ ft} \]

Final Answer

\[ h = 11.5 \pm 0.14 \text{ ft} \]