Computational Science:
Computational Methods in Engineering

Errors in Computation

Outline

- Errors in Computation
- Error Propagation
- Uncertainty Analysis

Errors in Computation

Types of Errors

Truncation Error – Arises when approximations are used instead of performing exact operations.

$$\frac{\partial f(x)}{\partial x} \approx \frac{f(x_b) - f(x_a)}{b - a} \qquad \sqrt{1+x} \approx 1 + \frac{x}{2}$$

Round-Off Error – Arises when limited significant figures are used to represent exact numbers

$$\pi, e, \sqrt{3}$$

```
>> 0.3/0.1 - 3

ans =
  -4.4409e-16
```

Human Error – Arises when people make mistakes.

$$1 + 1 = 3$$

EMPossible

Quantifying Error
Significant Figures

Very often quantities are limited to some number of digits.

This can happen because a computer cannot store any more digits or because the measurement is not accurate out to that many digits.

Rules: 1. All non-zero digits are significant
2. Zeros that do nothing but place the decimal point are not significant.

How many significant digits?

1.234	4 significant digits
0.00545	3 significant digits
32100	3 or 5 significant digits (ambiguous)
0.0500	3 significant digits

EMPossible 5

Quantifying Error
True Error

True Error – Actual error in a measurement.

$$\Delta = (\text{true error}) = (\text{true value}) - (\text{approximate value})$$

Example

The speed of light is exactly 299,792,458 m/s. An experiment was performed that measured the speed of light to be 299,792,445 m/s. What is the true error in this measurement?

$$(\text{true value}) = 299{,}792{,}458 \text{ m/s}$$

$$(\text{approximate value}) = 299{,}792{,}445 \text{ m/s}$$

$$\Delta = (\text{true error}) = (299{,}792{,}458 \text{ m/s}) - (299{,}792{,}445 \text{ m/s}) = 13 \text{ m/s}$$

Notes

- It is very rare to know the true value or true error of a measurement.
- Somewhat unrealistic error model because the true value must be known, but good for analyzing how errors propagate through computations.
- True error does not convey the relative severity of the error.

EMPossible 6

Quantifying Error
Relative Error

Relative Error – Actual error in a measurement relative to the size of the measurement.

$$\varepsilon_t = (\text{relative error}) = \frac{(\text{true error})}{(\text{true value})}$$

Example

The speed of light is exactly 299,792,458 m/s. An experiment was performed that measured the speed of light to be 299,792,445 m/s. What is the relative error of this measurement?

$$(\text{true error}) = 13 \text{ m/s}$$
$$(\text{true value}) = 299{,}792{,}458 \text{ m/s}$$
$$(\text{relative error}) = \frac{13 \text{ m/s}}{299{,}792{,}458 \text{ m/s}} = 4.33 \times 10^{-8} = 0.0000000433 \xrightarrow{\times 100\%} 0.00000433\%$$

Pretty good!

Notes

This is a much better metric for determining the severity of the error.

EMPossible

7

Quantifying Error
Approximate Relative Error

Approximate Relative Error – The true value of something is rarely known so errors must be calculated based on an approximate error.

$$\varepsilon_a = (\text{approximate relative error}) = \frac{(\text{approximate error})}{(\text{approximate value})}$$

Note

Quantities are correct to at least n significant digits if:

$$|\varepsilon_a| = \left(0.5 \times 10^{2-n}\right) \%$$

EMPossible

8

Error Propagation

What is Error Propagation?

In computation, it is necessary to calculate things from quantities that have some error. This results in some degree of error in the computations.

$$\tilde{x} = x + \Delta x$$

True error

True value

Our measured value with an error

How can the error Δf of $f(\tilde{x})$ be calculated from Δx?

EMPossible

Derive Error Propagation Equation
Taylor Series

The Taylor series is very often used to determine error associated with truncation.

$$f(x)=f(a)+f'(a)\cdot(x-a)+\frac{f''(a)}{2!}(x-a)^2+\frac{f'''(a)}{3!}(x-a)^3+\cdots=\sum_{k=0}^{\infty}\frac{f^{(k)}(a)}{k!}(x-a)^k$$

This is an infinite array. Retain only the first $n+1$ terms.

$$f(x)=f(a)+f'(a)\cdot(x-a)+\frac{f''(a)}{2!}(x-a)^2+\frac{f'''(a)}{3!}(x-a)^3+R_n$$

All remaining terms have been lumped into R_n.

$$R_n=\int_a^x\frac{(x-t)^n}{n!}f^{(n+1)}(t)\,dt$$

EMPossible

11

Derive Error Propagation Equation
Propagating Error

First, rewrite the Taylor series in terms of the measured value $\tilde{x}$.

$$f(x)=f(\tilde{x})+f'(\tilde{x})(x-\tilde{x})+\frac{f''(\tilde{x})}{2!}(x-\tilde{x})^2+\cdots$$

Second, bring $f(\tilde{x})$ to the left side of the equation.

$$\underbrace{f(x)-f(\tilde{x})}_{\text{This is the definition of error } \Delta f \text{ of } f(\tilde{x})}=f'(\tilde{x})(x-\tilde{x})+\frac{f''(\tilde{x})}{2!}(x-\tilde{x})^2+\cdots$$

These are both Δx.

Third, ignore all higher order terms to get an expression for error Δf.

$$\Delta f=f'(\tilde{x})(\Delta x)+\cancel{\frac{f''(\tilde{x})}{2!}(\Delta x)^2}+\cdots$$

$$\boxed{\text{error}=\Delta f=f'(\tilde{x})\Delta x}$$

Generalization to Two Variables

$$\Delta f=\left|\frac{\partial f}{\partial x}\right|\Delta x+\left|\frac{\partial f}{\partial y}\right|\Delta y$$

EMPossible

12

Example

Example 1

Suppose we have $\tilde{x} = 2.0$ with an error of $\Delta x = 0.05$.

What is the error Δf if $f(\tilde{x}) = \tilde{x}^2$?

Solution

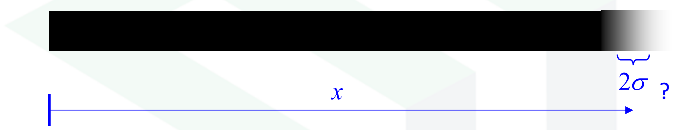
$\Delta f(\tilde{x}) = f'(\tilde{x}) \cdot \Delta x$	Start with the definition of error.
$= 2\tilde{x} \cdot \Delta x$	Derive an expression for $f'(\tilde{x})$ and substitute into equation.
$= 2(2.0) \cdot 0.05$	Substitute in values for $\tilde{x}$ and Δx.
$= 4 \cdot 0.05$	Calculate and simplify.
$= \underline{0.2}$	Final answer

EMPossible

Uncertainty Analysis

What is Uncertainty?

How long is this object?



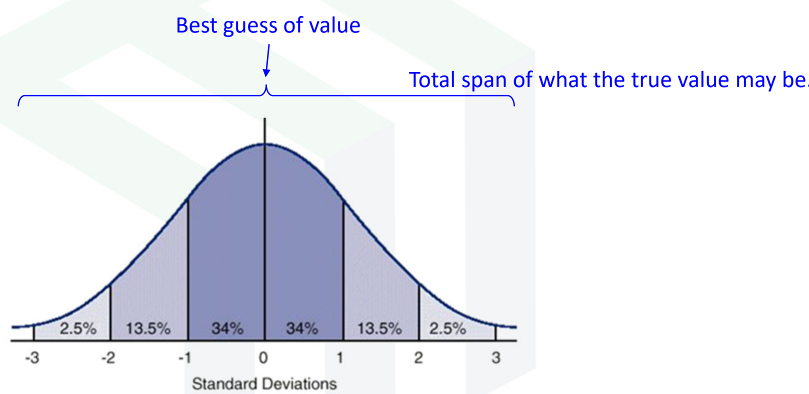
The measurement is written as $x \pm \sigma_x$

Since the measurement error is "blurred," the exact error is not certain. The uncertainty σ_x is a statistically derived quantity.

Notes

- This is a more realistic way to treat error because the true value does not have to be known.
- Uncertainty is used to predict errors in computations.

How to Interpret the Uncertainty σ



σ is one standard deviation.

There is a 68% chance that the error will be less than 1σ.
There is a 32% chance that the error will be greater than 1σ.

Uncertainly Analysis

Suppose there are multiple parameters and each has an uncertainty associated with it.

$x \pm \sigma_x \qquad y \pm \sigma_y \qquad z \pm \sigma_z$

Now suppose a new quantity f is calculated from these.

$f(x, y, z)$

What is the uncertainty σ_f of $f(x, y, z)$?

$$\sigma_f^2 = \left(\sigma_x \frac{\partial f}{\partial x}\right)^2 + \left(\sigma_y \frac{\partial f}{\partial y}\right)^2 + \left(\sigma_z \frac{\partial f}{\partial z}\right)^2$$

This is called propagating uncertainty through a calculation.

EMPossible 17

Table of Common Uncertainty Calculations

Function	Uncertainty
$f = ax$	$\sigma_f = a\sigma_x$
$f = ax \pm by$	$\sigma_f^2 = (a\sigma_x)^2 + (b\sigma_y)^2$
$f = xy$	$(\sigma_f/f)^2 = (\sigma_x/x)^2 + (\sigma_y/y)^2$
$f = x/y$	$(\sigma_f/f)^2 = (\sigma_x/x)^2 + (\sigma_y/y)^2$
$f = x^{\pm b}$	$\sigma_f/f = b\sigma_x/x$
$f = \ln(\pm bx)$	$\sigma_f = b\sigma_x/x$
$f = \log x$	$\sigma_f = b\sigma_x/(x \ln 10)$
$f = e^{\pm bx}$	$\sigma_f/f = b\sigma_x$
$f = a^{\pm bx}$	$\sigma_f/f = b\sigma_x \ln a$
$f = \sin x$	$\sigma_f = \sigma_x \cos x$
$f = \cos x$	$\sigma_f = \sigma_x \sin x$
$f = \tan x$	$\sigma_f = \sigma_x/\cos^2 x$
$f = \sin^{-1}(x)$	$\sigma_f^2 = \sigma_x^2/(1 - x^2)$
$f = \cos^{-1}(x)$	$\sigma_f^2 = \sigma_x^2/(1 - x^2)$
$f = \tan^{-1}(x)$	$\sigma_f = \sigma_x/(1 + x^2)$

EMPossible 18

Uncertainty Through Multiple Calculations

Suppose the uncertainty σ_f is to be calculated when multiple calculations are involved.

$$f(a,b,c) = \ln(a+bc) \qquad \text{Given } \sigma_a,\ \sigma_b,\text{ and } \sigma_c$$

	Value	Uncertainty
1	bc	$\left(\frac{\sigma_{bc}}{bc}\right)^2 = \left(\frac{\sigma_b}{b}\right)^2 + \left(\frac{\sigma_c}{c}\right)^2$
2	$a+bc$	$\sigma_{a+bc}^2 = \sigma_a^2 + \sigma_{bc}^2$
3	$\ln(a+bc)$	$\sigma_{\ln(a+bc)} = \frac{\sigma_{a+bc}}{a+bc}$

EMPossible 19

Example 1

What is the uncertainty of the sum of two quantities?

$$x+y$$

Solution

$$\sigma_f^2 = \left(\sigma_x \frac{\partial f}{\partial x}\right)^2 + \left(\sigma_y \frac{\partial f}{\partial y}\right)^2 \qquad f = x+y$$

$$\sigma_f^2 = \left[\sigma_x \frac{\partial}{\partial x}(x+y)\right]^2 + \left[\sigma_y \frac{\partial}{\partial y}(x+y)\right]^2 = \left[\sigma_x \cdot 1\right]^2 + \left[\sigma_y \cdot 1\right]^2$$

$$\boxed{\sigma_f = \sqrt{\sigma_x^2 + \sigma_y^2}}$$

EMPossible 20

Example 2

What is the uncertainty of a decibel quantity?

$$10\log_{10}\left(x \pm \sigma_x\right)$$

Solution

$$\sigma_f^2 = \left(\sigma_x \frac{\partial f}{\partial x}\right)^2 \qquad f\left(x\right) = 10\log_{10}\left(x\right)$$

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\left[10\log_{10}\left(x\right)\right] = 10\left[\frac{\partial}{\partial x}\log_{10}\left(x\right)\right] = 10\frac{1}{x}\frac{1}{\ln 10} = \frac{10}{x\ln 10}$$

$$\sigma_f^2 = \left(\sigma_x \frac{10}{x\ln 10}\right)^2$$

$$\boxed{\sigma_f = 4.3429\frac{\sigma_x}{x}}$$

EMPossible

21

Example 3

The signal-to-noise ratio (SNR) of a system is 50 ± 1.2. What is the SNR in decibels along with the uncertainty?

Solution

The SNR in decibels is

$$10\log_{10}\left(50\right) = 16.99 \text{ dB}$$

The uncertainty is calculated using the equation derived in the previous example.

$$\sigma_f = 4.3429\frac{\sigma_x}{x} = 4.3429\frac{1.2}{50} = 0.1$$

The final answer is

$$\boxed{\text{SNR} = 16.99 \pm 0.1 \text{ dB}}$$

EMPossible

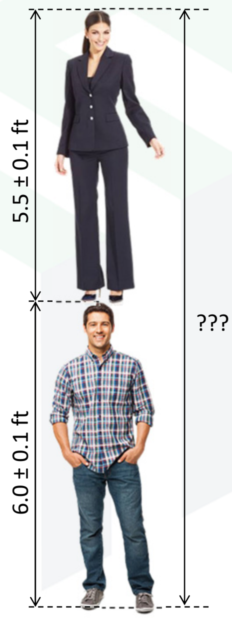
22

Example 4

The height of person 1 was measured to be 6.0 ± 0.1 ft.

The height of person 2 was measured to be 5.5±0.1 ft.

If person 2 stands on the head of person 1, what is the total height and the uncertainty of the total height?



<u>Solution</u>

Total height

$$
\begin{aligned}
h &= h_1 + h_2 \\
&= (6.0\text{ ft}) + (5.5\text{ ft}) \\
&= 11.5\text{ ft}
\end{aligned}
$$

Uncertainty

$$
\begin{aligned}
\sigma_h &= \sqrt{\sigma_1^2 + \sigma_2^2} \\
&= \sqrt{(0.1\text{ ft})^2 + (0.1\text{ ft})^2} \\
&= 0.14\text{ ft}
\end{aligned}
$$

Final Answer

$$\boxed{h = 11.5 \pm 0.14\text{ ft}}$$

EMPossible

23