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Computational Methods in EE

Topic 3a – Review of Linear Algebra

EE 4386/5301 Computational Methods in EE

Outline

Computational Methods in EE

- Solving systems of equations
- Matrix terminology and special matrices
- Matrix operations
- Common linear algebra problems

Computational Methods in EE

Solving Systems of Equations

Systems of Linear Equations

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Very often in science and engineering, problems can be reduced to a system of linear equations.

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\
&\vdots \\
a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= b_n
\end{aligned}
$$

$a_{ij} \equiv$ constant coefficient (usually known)

$x_i \equiv$ unknown values

$b_i \equiv$ constants (usually excitation)

Direct Analytical Solution

Computational Methods in EE

Suppose we wish to solve the following system of equations

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2$$
$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3$$

Step 1 – Solve first equation for x_1.

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1 \quad \rightarrow \quad x_1 = \frac{b_1}{a_{11}} - \frac{a_{12}}{a_{11}}x_2 - \frac{a_{13}}{a_{11}}x_3$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2$$
$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3$$

Direct Analytical Solution

Computational Methods in EE

Step 2 – *Forward Substitution* – Substitute this new equation into 2nd and 3rd equations to eliminate x_1.

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$$
$$a'_{22}x_2 + a'_{23}x_3 = b'_2$$
$$a'_{32}x_2 + a'_{33}x_3 = b'_3$$

$$a'_{22} = a_{22} - \frac{a_{21}a_{12}}{a_{11}} \qquad a'_{23} = a_{23} - \frac{a_{21}a_{13}}{a_{11}} \qquad b'_2 = b_2 - \frac{a_{21}b_1}{a_{11}}$$

$$a'_{32} = a_{32} - \frac{a_{31}a_{12}}{a_{11}} \qquad a'_{33} = a_{33} - \frac{a_{31}a_{13}}{a_{11}} \qquad b'_3 = b_3 - \frac{a_{31}b_1}{a_{11}}$$

Direct Analytical Solution

Computational Methods in EE

Step 3 – Solve second equation for x_2.

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$$
$$a'_{22}x_2 + a'_{23}x_3 = b'_2 \quad \rightarrow \quad x_2 = \frac{b'_2}{a'_{22}} - \frac{a'_{23}}{a'_{22}}x_3$$
$$a'_{32}x_2 + a'_{33}x_3 = b'_3$$

Direct Analytical Solution

Computational Methods in EE

Step 4 – *Forward Substitution* – Substitute this new equation into 3rd equation to eliminate x_2.

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$$
$$a'_{22}x_2 + a'_{23}x_3 = b'_2$$
$$a''_{33}x_3 = b''_3$$

$$a''_{33} = a'_{33} - \frac{a'_{32}a'_{23}}{a'_{22}} \qquad b''_3 = b'_3 - \frac{a'_{32}b'_2}{a'_{22}}$$

Direct Analytical Solution

Computational Methods in EE

Step 5 – Solve third equation for x_3. Since this is the last equation, we get the final answer for x_3.

$$x_3 = \frac{b_3''}{a_{33}''}$$

Direct Analytical Solution

Computational Methods in EE

Step 6 – *Backward Substitution* – Given x_3, calculate x_2 using equation from Step 3.

$$x_2 = \frac{b_2' - a_{23}' x_3}{a_{22}'}$$

Direct Analytical Solution

Step 7 – *Backward Substitution* – Given x_2 and x_3, calculate x_1 using equation from Step 1.

$$x_1 = \frac{b_1 - a_{12}x_2 - a_{13}x_3}{a_{11}}$$

Computational Methods in EE

Matrix Terminology & Special Matrices

Systems of Linear Equations

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Systems of equations can be written in matrix form.

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\
&\vdots \\
a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= b_n
\end{aligned}
\quad \rightarrow \quad
\underbrace{\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \\ a_{n1} & a_{n2} & & a_{nn} \end{bmatrix}}_{[A] \text{ or } \mathbf{A}}
\underbrace{\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}}_{[x] \text{ or } \mathbf{x}}
=
\underbrace{\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}}_{[b] \text{ or } \mathbf{b}}
$$

Rows, Columns, and Diagonals

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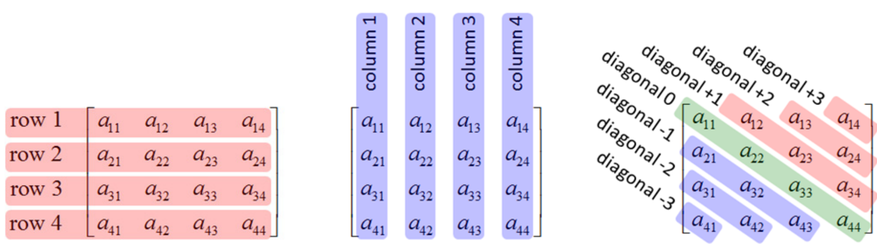

The center diagonal is usually just called *the diagonal*.

The elements along the diagonal are sometimes called the *pivot elements*.

Special Matrices (1 of 2)

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Symmetric Matrix

$$[A]=\begin{bmatrix}1 & 2 & 9 & 4\\ 2 & 6 & 5 & 8\\ 9 & 5 & 7 & 0\\ 4 & 8 & 0 & 3\end{bmatrix}$$

Diagonal Matrix

$$[A]=\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 6 & 0 & 0\\ 0 & 0 & 7 & 0\\ 0 & 0 & 0 & 3\end{bmatrix}$$

Identity Matrix

$$[I]=\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}$$

Zero Matrix

$$[0]=\begin{bmatrix}0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}$$

Special Matrices (2 of 2)

Computational Methods in EE

Upper Triangular Matrix

$$[A]=\begin{bmatrix}1 & 2 & 9 & 4\\ 0 & 6 & 5 & 8\\ 0 & 0 & 7 & 5\\ 0 & 0 & 0 & 3\end{bmatrix}$$

Triangular matrices can be thought of as “almost” solved matrices. They are very fast to solve.

Lower Triangular Matrix

$$[A]=\begin{bmatrix}1 & 0 & 0 & 0\\ 2 & 6 & 0 & 0\\ 9 & 5 & 7 & 0\\ 4 & 8 & 1 & 3\end{bmatrix}$$

Banded Matrix

$$[A]=\begin{bmatrix}1 & 2 & 0 & 0\\ 4 & 6 & 5 & 0\\ 0 & 8 & 7 & 5\\ 0 & 0 & 10 & 3\end{bmatrix}$$

Bandwidth of 3

Vandermonde Matrix

$$\begin{bmatrix}1 & x_1 & x_1^2 & & x_1^N\\ 1 & x_2 & x_2^2 & \cdots & x_2^N\\ 1 & x_3 & x_3^2 & & x_3^N\\ & \vdots & & \ddots & \\ 1 & x_{N+1} & x_{N+1}^2 & & x_{N+1}^N\end{bmatrix}$$

Arises when curve fitting to polynomials. Usually ill-conditioned for large matrices.

Block Matrices

Computational Methods in EE

Block matrices are "matrices of matrices."

$$[F]=\begin{bmatrix}[A] & [B]\\ [C] & [D]\end{bmatrix}=\begin{bmatrix}a_{11} & a_{12} & b_{11} & b_{12}\\ a_{21} & a_{22} & b_{21} & b_{22}\\ c_{11} & c_{12} & d_{11} & d_{12}\\ c_{21} & c_{22} & d_{21} & d_{22}\end{bmatrix}$$

$$[A]=\begin{bmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{bmatrix}\quad [B]=\begin{bmatrix}b_{11} & b_{12}\\ b_{21} & b_{22}\end{bmatrix}\quad [C]=\begin{bmatrix}c_{11} & c_{12}\\ c_{21} & c_{22}\end{bmatrix}\quad [D]=\begin{bmatrix}d_{11} & d_{12}\\ d_{21} & d_{22}\end{bmatrix}$$

Sparse Matrices

Computational Methods in EE

Many matrices contain 99.9% zeros.

$$H=\begin{bmatrix}1&1&1&1&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&1&1&1&1&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&1&1&1&1&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&1&1&1&1\\ 1&0&0&0&0&0&0&1&0&0&1&0&0&1&0&0\\ 0&1&0&0&1&0&0&0&0&0&0&1&0&0&1&0\\ 0&0&1&0&0&1&0&0&1&0&0&0&0&0&0&1\\ 0&0&0&1&0&0&1&0&0&1&0&0&1&0&0&0\end{bmatrix}$$

It is not efficient use of memory to store all these zeros. Instead, we store only the non-zero elements along with their positions in the matrix.

The opposite of a "sparse" matrix is a "full" matrix.

Matrix Problem Size

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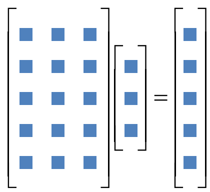
 # Equations > # Unknowns

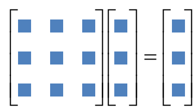


Usually occurs when the equations are derived from samples.

Solution is obtained as a *best fit* and is not exact.

Applications
- Curve fitting

 # Equations = # Unknowns

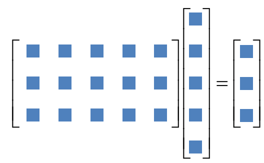


Most usual case.

Many standard algorithms exist to obtain an *exact* solution.

Applications
- Circuit theory
- Solving ODEs

 # Equations < # Unknowns



Usually occurs when little is known about the problem or solution.

Solution is obtained by *optimization* and is not exact.

Applications
- Topology optimization

Health of a Matrix (1 of 3)

Computational Methods in EE

Is this system of equations solvable?

$$\begin{aligned} x+2y+z&=8 \\ x+2y+z&=8 \\ 3x-y+z&=4 \end{aligned} \qquad \begin{bmatrix} 1 & 2 & 1 \\ 1 & 2 & 1 \\ 3 & -1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 8 \\ 8 \\ 4 \end{bmatrix}$$

No!
The 1st and 2nd equations are the same.
The 2nd equation does not provide any new information to the problem.

$$\begin{aligned} x+2y+z&=8 \\ 2x+4y+2z&=16 \\ 3x-y+z&=4 \end{aligned} \qquad \begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 2 \\ 3 & -1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 8 \\ 16 \\ 4 \end{bmatrix}$$

No!
The 2nd equation is just 2× the 1st equation.
The 2nd equation is still not providing any new information.

$$\begin{aligned} x+2y+z&=8 \\ 4x+y+2z&=12 \\ 3x-y+z&=4 \end{aligned} \qquad \begin{bmatrix} 1 & 2 & 1 \\ 4 & 1 & 2 \\ 3 & -1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 8 \\ 12 \\ 4 \end{bmatrix}$$

No!
The 2nd equation is the sum of the 1st and 3rd equation, thus the 2nd equation still does not provide any new information.

Health of a Matrix (2 of 3)

Computational Methods in EE

Is this system of equations solvable?

$$\begin{aligned} x+z&=8 \\ x+2z&=7 \\ 3x+z&=4 \end{aligned} \qquad \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 2 \\ 3 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 8 \\ 7 \\ 4 \end{bmatrix}$$

No!
None of these equations contain any information about y.

So how do we know if a problem is solvable?

- All rows must be linearly independent – this ensures they provide new information to the problem.
- No rows can be all zeros – This would not provide any information.
- No columns can be all zeros – This would be ignoring information from one of the unknowns.

$$[A][x]=[b] \text{ is solvable if } \det[A] \neq 0$$

Health of a Matrix (3 of 3)

Computational Methods in EE

Is the following system of equations solvable?

$$\begin{aligned} x+2y+z&=8 \\ 1.0001x+2y+z&=8.0001 \\ 3x-y+z&=4 \end{aligned} \qquad \begin{bmatrix} 1 & 2 & 1 \\ 1.0001 & 2 & 1 \\ 3 & -1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 8 \\ 8.0001 \\ 4 \end{bmatrix}$$

Technically yes, but we would expect the solution to be somewhat "touchy" and unstable. This is an *ill-conditioned* matrix.

Condition Number of a Matrix

The condition number $\kappa(\mathbf{A})$ of matrix $\mathbf{A}$ is a measure of how numerically "stable" it is.

Matrices with high condition numbers are less stable. Small changes in the element values of $\mathbf{A}$ will result in large changes in the elements of $\mathbf{b}$.

$$\kappa(\mathbf{A}) = \left| \frac{\sigma_{\max}(\mathbf{A})}{\sigma_{\min}(\mathbf{A})} \right|$$

$\sigma_{\min}(\mathbf{A}) \equiv$ smallest singular value of $\mathbf{A}$

$\sigma_{\max}(\mathbf{A}) \equiv$ largest singular value of $\mathbf{A}$

Example: Condition Number

What is the condition number?

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 2 & 1 \\ 3 & -1 & 1 \end{bmatrix} \qquad \kappa(\mathbf{A}) = 5.84 \times 10^{16}$$

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 0 & 1 \\ 3 & -1 & 1 \end{bmatrix} \qquad \kappa(\mathbf{A}) = 7.76$$

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 1 \\ 1.0001 & 2 & 1 \\ 3 & -1 & 1 \end{bmatrix} \qquad \kappa(\mathbf{A}) = 1.4 \times 10^{5}$$

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 1 \\ 1.01 & 2 & 1 \\ 3 & -1 & 1 \end{bmatrix} \qquad \kappa(\mathbf{A}) = 1.4 \times 10^{3}$$

Computational Methods in EE

Matrix Operations

Matrix Math (1 of 4)

Computational Methods in EE

Addition:

$$[A]+[B]=\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}+\begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}=\begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} & a_{13}+b_{13} \\ a_{21}+b_{21} & a_{22}+b_{22} & a_{23}+b_{23} \\ a_{31}+b_{31} & a_{32}+b_{32} & a_{33}+b_{33} \end{bmatrix}$$

Subtraction:

$$[A]-[B]=\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}-\begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}=\begin{bmatrix} a_{11}-b_{11} & a_{12}-b_{12} & a_{13}-b_{13} \\ a_{21}-b_{21} & a_{22}-b_{22} & a_{23}-b_{23} \\ a_{31}-b_{31} & a_{32}-b_{32} & a_{33}-b_{33} \end{bmatrix}$$

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Matrix Math (2 of 4)

Computational Methods in EE

Multiplication by a Scalar:

$$s[A]=s\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}=\begin{bmatrix} sa_{11} & sa_{12} & sa_{13} \\ sa_{21} & sa_{22} & sa_{23} \\ sa_{31} & sa_{32} & sa_{33} \end{bmatrix}$$

Multiplication by a Matrix

$$[A][B]=\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}\begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}=\begin{bmatrix} a_{11}b_{11}+a_{12}b_{21}+a_{13}b_{31} & \# & \# \\ \# & \# & \# \\ \# & \# & \# \end{bmatrix}$$

$$[A][x]=\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} a_{11}x_1+a_{12}x_2+a_{13}x_3 \\ a_{21}x_1+a_{22}x_2+a_{23}x_3 \\ a_{31}x_1+a_{32}x_2+a_{33}x_3 \end{bmatrix}$$

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Matrix Math (3 of 4)

Computational Methods in EE

Matrix Transpose:

$$[A]^T = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}^T = \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{bmatrix}$$

$$a'_{ij} = a_{ji}$$

Animation of Transpose Operation

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}^T = \begin{bmatrix} a_{11} & a_{21} & a_{31} & a_{41} \\ a_{12} & a_{22} & a_{32} & a_{42} \\ a_{13} & a_{23} & a_{33} & a_{43} \\ a_{14} & a_{24} & a_{34} & a_{44} \end{bmatrix}$$

Hermitian Transpose:

$$[A]^H = \left(\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}^T \right)^* = \begin{bmatrix} a_{11}^* & a_{21}^* & a_{31}^* \\ a_{12}^* & a_{22}^* & a_{32}^* \\ a_{13}^* & a_{23}^* & a_{33}^* \end{bmatrix}$$

$$a'_{ij} = a_{ji}^*$$

Matrix Math (4 of 4)

Computational Methods in EE

Determinants:

$\det[A]$ Think of this as the "magnitude" or "volume" of a matrix.

Matrix Inverse:

$$[A]^{-1}[A] = [I]$$

Matrix Division:

$[A]^{-1}[B]$ predivide `A\B`

$[B][A]^{-1}$ postdivide `B/A`

While both expressions divide by $[A]$, these do not give the same answer.

Matrix Multiplication:

$[A][B]$ $[A]$ premultiplies $[B]$

$[B][A]$ $[A]$ postmultiplies $[B]$

Matrix Algebra (1 of 3)

Computational Methods in EE

Commutative Laws

$$[A]+[B]=[B]+[A]$$
$$[A][B]\neq[B][A]$$

$[A][B] = [B][A]$ when $[A]$ and $[B]$ are diagonal matrices.

Associative Laws

$$([A]+[B])+[C]=[A]+([B]+[C])$$
$$([A][B])[C]=[A]([B][C])$$

Matrix Inverses and Transposes

$$[A]^{-1}[A]=[A][A]^{-1}=[I]$$
$$\left([A]^{-1}\right)^{-1}=[A]$$
$$([A][B])^{-1}=[B]^{-1}[A]^{-1}$$
$$\left([A]^{T}\right)^{-1}=\left([A]^{-1}\right)^{T}$$
$$\left([A]^{T}\right)^{T}=[A]$$
$$([A]+[B])^{T}=[A]^{T}+[B]^{T}$$
$$([A][B])^{T}=[B]^{T}[A]^{T}$$

Distributive Laws

$$([A]+[B])[C]=[A][C]+[B][C]$$
$$[A]([B]+[C])=[A][B]+[A][C]$$

Matrix Algebra (2 of 3)

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Addition with a Scalar

$$\alpha+[A]=\text{doesn't make sense}$$

$$\alpha[I]+[A]=\begin{bmatrix}\alpha+a_{11} & a_{12} & \cdots & a_{1n}\\ a_{21} & \alpha+a_{22} & \cdots & a_{2n}\\ \vdots & \vdots & \ddots & \\ a_{n1} & a_{n2} & & \alpha+a_{nn}\end{bmatrix}$$

Multiplication with a Scalar

$$\alpha([A]+[B])=\alpha[A]+\alpha[B]$$
$$\alpha([A][B])=(\alpha[A])[B]=[A](\alpha[B])$$

Matrix Algebra (3 of 3)

Computational Methods in EE

Operations with Special Matrices

$$[0][A]=[A][0]=[0]$$
$$[I][A]=[A][I]=[A]$$
$$[0]+[A]=[A]+[0]=[A]$$
$$[A]-[A]=[0]$$

Example of Matrix Algebra

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Simplify the Following Equation

$$\left(\mathbf{C}^{-1}\mathbf{A}\right)^{-1}+\mathbf{D}=\mathbf{BC}+\mathbf{D}$$

Step 1 – Subtract $\mathbf{D}$ from both sides

$$\left(\mathbf{C}^{-1}\mathbf{A}\right)^{-1}+\mathbf{D}-\mathbf{D}=\mathbf{BC}+\mathbf{D}-\mathbf{D}$$
$$\left(\mathbf{C}^{-1}\mathbf{A}\right)^{-1}+\mathbf{0}=\mathbf{BC}+\mathbf{0}$$
$$\left(\mathbf{C}^{-1}\mathbf{A}\right)^{-1}=\mathbf{BC}$$

Step 2 – Inverse both sides

$$\left\{\left(\mathbf{C}^{-1}\mathbf{A}\right)^{-1}\right\}^{-1}=\left\{\mathbf{BC}\right\}^{-1}$$
$$\mathbf{C}^{-1}\mathbf{A}=\mathbf{C}^{-1}\mathbf{B}^{-1}$$

Step 3 – Premultiply both sides by $\mathbf{C}$.

$$\mathbf{CC}^{-1}\mathbf{A}=\mathbf{CC}^{-1}\mathbf{B}^{-1}$$
$$\mathbf{IA}=\mathbf{IB}^{-1}$$
$$\mathbf{A}=\mathbf{B}^{-1}$$

Computational Methods in EE

Common Linear Algebra Problems

$[A][x] = [b]$

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This problem arises when a problem $[A]$ is given some excitation $[b]$ and produces a solution $[x]$.

Examples: (1) waves scattering from an object, (2) heat through a device, (3) solving currents and voltages in a circuit.

It produces a single solution.

Step 1 – Differential equation

$$\frac{d^2 f}{dx^2} + \gamma \frac{df}{dx} + f = b$$

Step 2 – ODE is converted to system of equations using finite-differences, finite elements, etc.

$$\begin{aligned} a_{11} f_1 + a_{12} f_2 + \cdots + a_{1n} f_n &= b_1 \\ a_{21} f_1 + a_{22} f_2 + \cdots + a_{2n} f_n &= b_2 \\ &\vdots \\ a_{n1} f_1 + a_{n2} f_2 + \cdots + a_{nn} f_n &= b_n \end{aligned}$$

Step 3 – System of equations is put into matrix form.

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \\ a_{n1} & a_{n2} & & a_{nn} \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

Step 4 – Matrix problem is solved for [f]

$$[f] = [A]^{-1}[b]$$

Step 5 – [f] is post processed to learn something.

Eigen-Value Problems

Computational Methods in EE

Eigen-value problems arise when multiple solutions exist. No excitation is needed.

Examples: (1) resonating modes on a string, (2) electromagnetic modes in a waveguide, (3) electronic bands in a semiconductor.

$[A][x] = \lambda[x]$ Standard eigen-value problem

$[A][x] = \lambda[B][x]$ Generalized eigen-value problem

$[A]$ is the linear operation

$[x]$ is the unknown (eigen-vector)

λ is the eigen-value and is just a scalar number

$[B]$ is potentially another part of the linear operation

Determinants

Computational Methods in EE

The determinant is an important number associated with square matrices.

It is sort of a magnitude or volume.

Unique solutions to systems of equations do not exist when the determinant is zero.

<u>2×2 Matrices</u>

$$\det[A] = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

<u>3×3 Matrices</u>

$$\det[A] = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

This can be calculated by walking across any of the rows.

Cramer's Rule

Computational Methods in EE

Cramer's rule provides a methodical approach for calculating the unknowns of a system of equations.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$$x_1 = \frac{1}{D} \begin{vmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{vmatrix} \qquad x_2 = \frac{1}{D} \begin{vmatrix} a_{11} & b_1 & a_{13} \\ a_{21} & b_2 & a_{23} \\ a_{31} & b_3 & a_{33} \end{vmatrix} \qquad x_3 = \frac{1}{D} \begin{vmatrix} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \\ a_{31} & a_{32} & b_3 \end{vmatrix}$$

$$D = \det \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$