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Computational Methods in EE

Topic 4b – Open Methods for Root Finding

EE 4386/5301 Computational Methods in EE

Outline

Computational Methods in EE

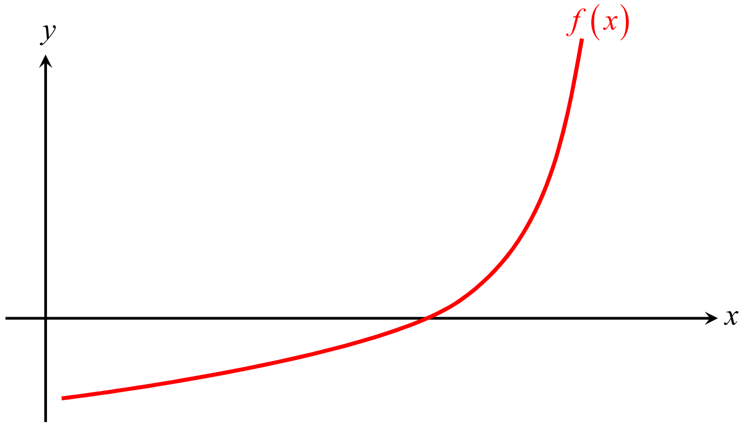
- Open Methods for Root Finding
  - Newton-Raphson Method
  - Secant Method
- Multiple Roots

Computational Methods in EE

Newton-Raphson Method

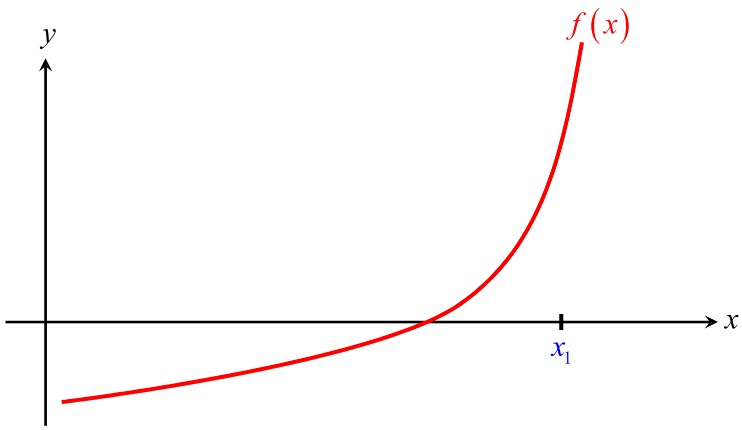
We Start with Any Function

Computational Methods in EE



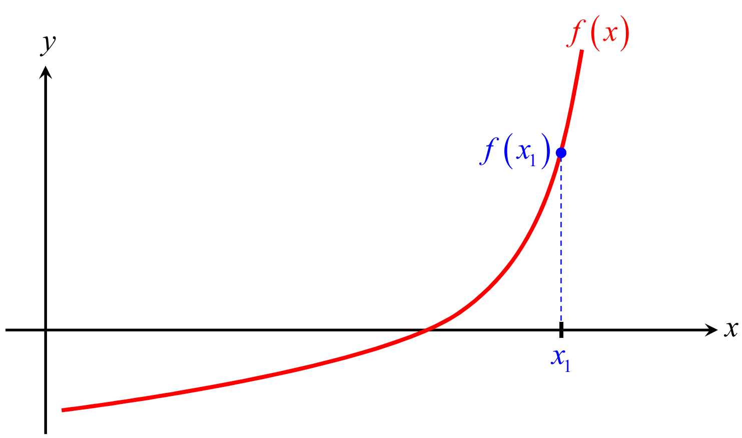
We Make an Initial Guess for the Position of the Root x_1

Computational Methods in EE



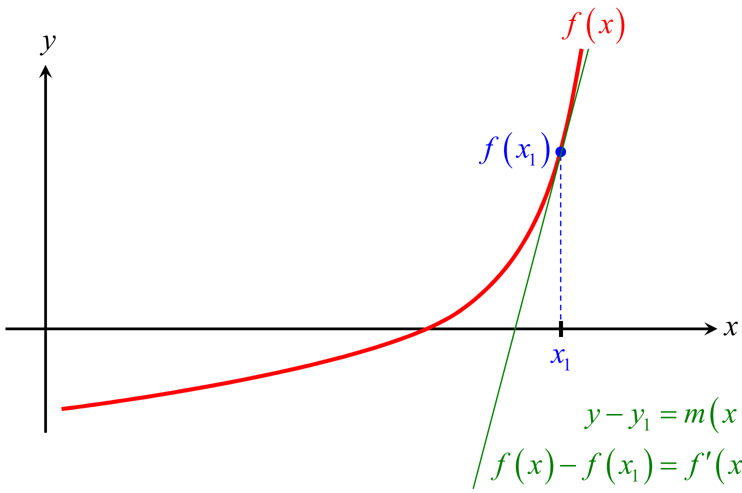
Evaluate the Function at x_1

Computational Methods in EE



Calculate the Equation of the Line Tangential to the Point on $f(x)$

Computational Methods in EE

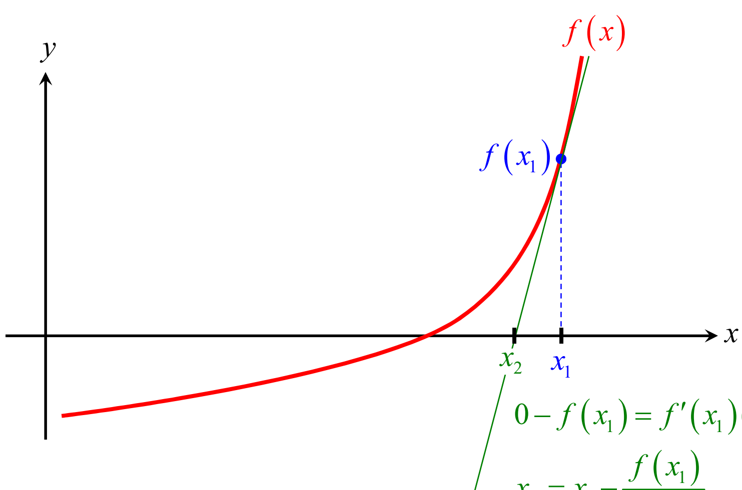


$$y - y_1 = m(x - x_1)$$

$$f(x) - f(x_1) = f'(x_1)(x - x_1)$$

Calculate Where the Line Crosses the x-Axis

Computational Methods in EE

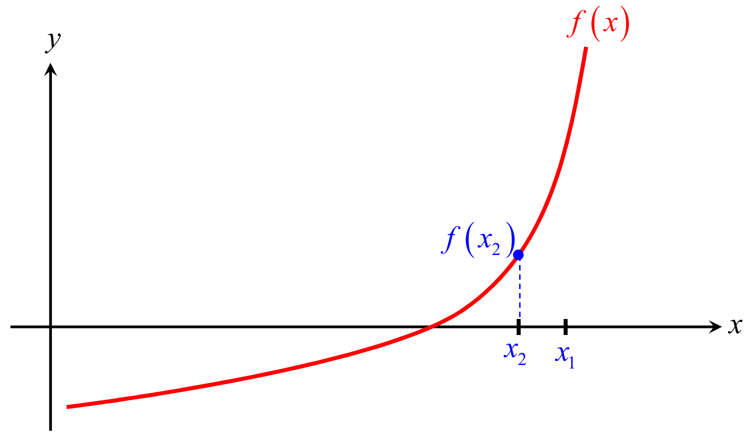


$$0 - f(x_1) = f'(x_1)(x_2 - x_1)$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

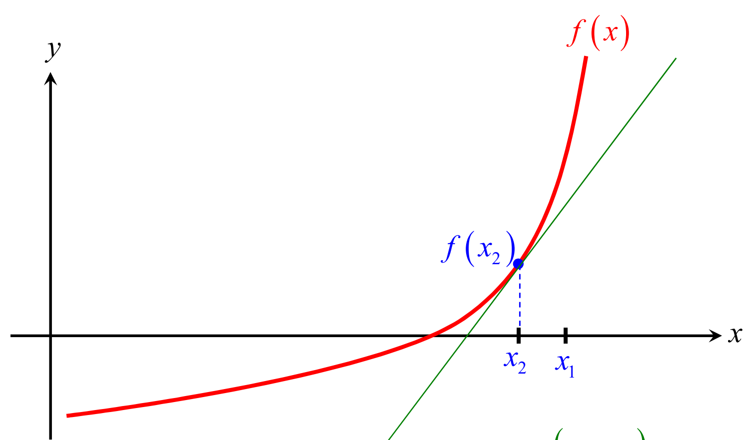
Evaluate the Function at x_2

Computational Methods in EE



Calculate the Equation of the Line Tangential to the Point on $f(x)$

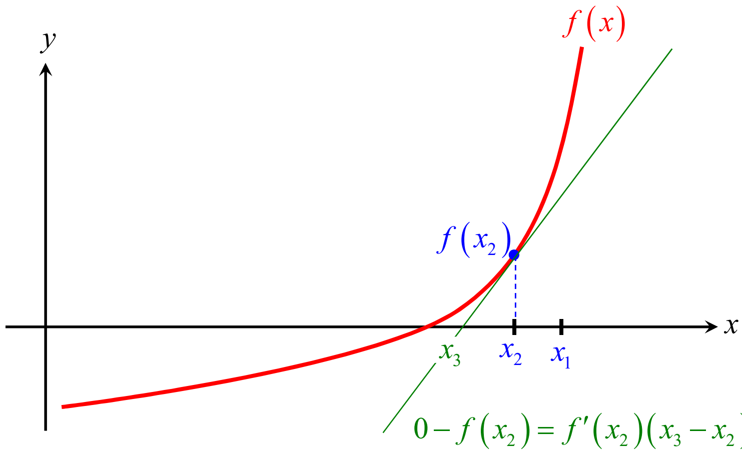
Computational Methods in EE



$$y - y_2 = m(x - x_2)$$
$$f(x) - f(x_2) = f'(x_2)(x - x_2)$$

Calculate Where the Line Crosses the x-Axis

Computational Methods in EE

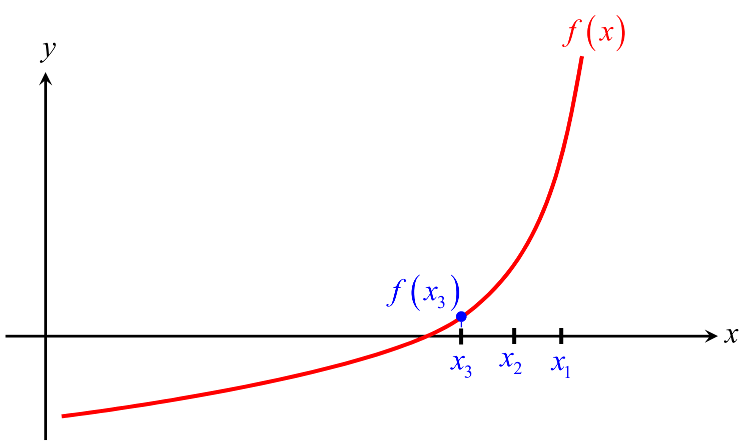


$$0 - f(x_2) = f'(x_2)(x_3 - x_2)$$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$

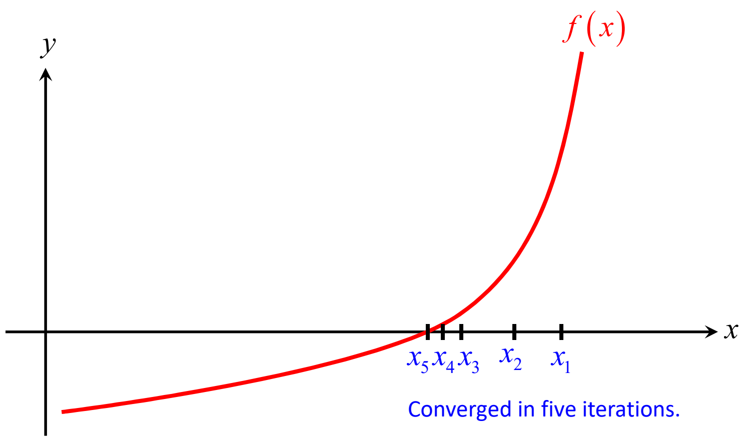
Evaluate the Function at x_3

Computational Methods in EE



And so on…

Computational Methods in EE



Converged in five iterations.

Algorithm

Computational Methods in EE

1. Derive analytical expressions for $f(x)$ and $f'(x)$, or $f(x)/f'(x)$.
2. Determine a good initial guess x_i.
3. Iterate until converged
    1. Calculate Δx_i.

       $$\Delta x_i = f(x_i)/f'(x_i)$$

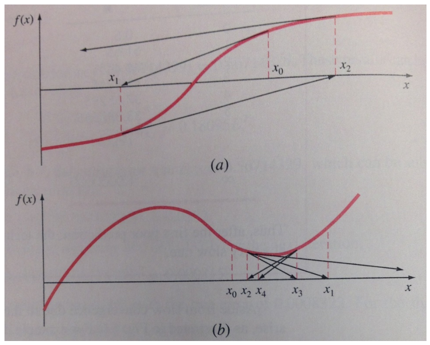
    2. Calculate new estimate for root x_{i+1}.

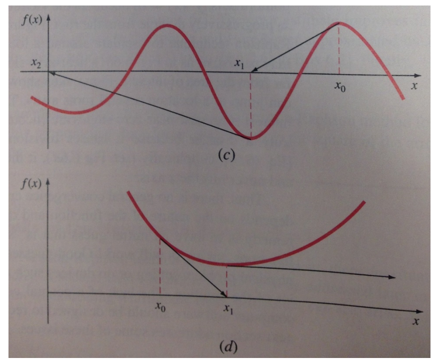
       $$x_{i+1} = x_i - \Delta x_i$$

    3. If $|\Delta x_i| <$ tolerance, Done!

Poor or Unstable Convergence

Computational Methods in EE





Notes on Newton-Raphson Method

Computational Methods in EE

- Does not require bounds.
- Requires a “good” initial guess.
- Requires $f(x)$ and $f'(x)$ to be analytical.
- Converges extremely fast for functions that are near linear.
- Algorithm vulnerable to instability
- Can converge to the wrong root if multiple roots exist.
- Newton-Raphson’s Method ≠ Newton’s Method
  - NRM is for root finding, whereas NM is for optimization.

Example #1

Computational Methods in EE

Let $f(x) = \sin x$. What is the root of $f(x)$ in the proximity of $x = 4$?

Step 1 -- Derive analytical expression for $f(x)/f'(x)$

$$f(x) = \sin x$$

$$\Delta x = \frac{f(x)}{f'(x)} = \frac{\sin x}{\frac{d}{dx}\sin x} = \frac{\sin x}{\cos x} = \tan x$$

This means our update equation is

$$x_{i+1} = x_i - \Delta x_i = x_i - \tan x_i$$

Step 2 -- MATLAB code

```
xr  = 4;
tol = 1e-6;
dx  = inf;
while abs(dx) > tol
    dx = tan(xr);
    xr = xr - dx;
end
```

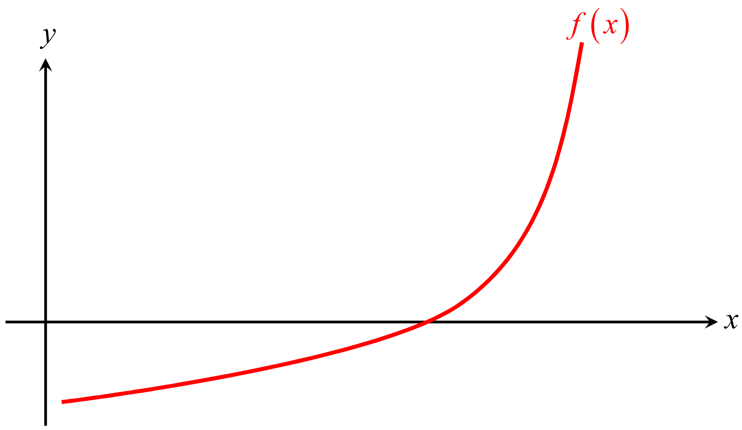
Converges to 3.1416 after 4 iterations.

Computational Methods in EE

Secant Method

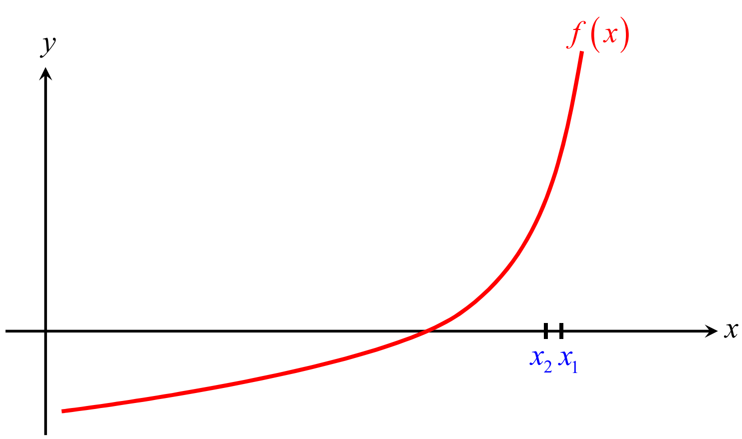
We Start with Some Function

Computational Methods in EE



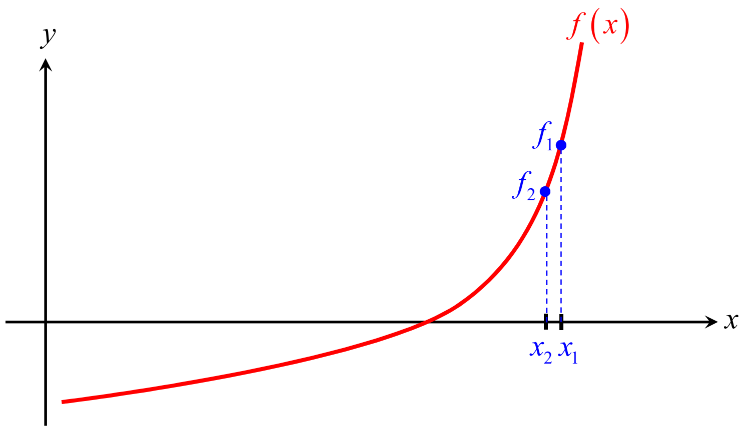
We Make Two Initial Guesses for the Position of the Root, x_1 and x_2

Computational Methods in EE



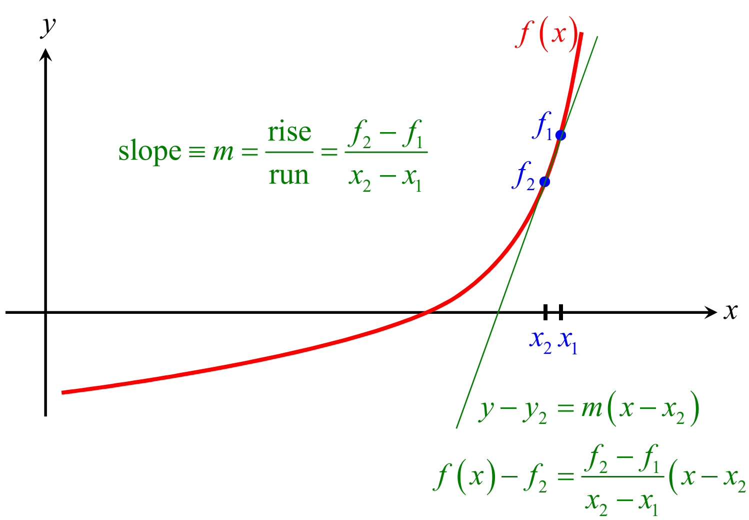
Evaluate the Function at x_1 and x_2

Computational Methods in EE



Calculate the Equation of the Line Connecting (x_1, f_1) and (x_2, f_2)

Computational Methods in EE



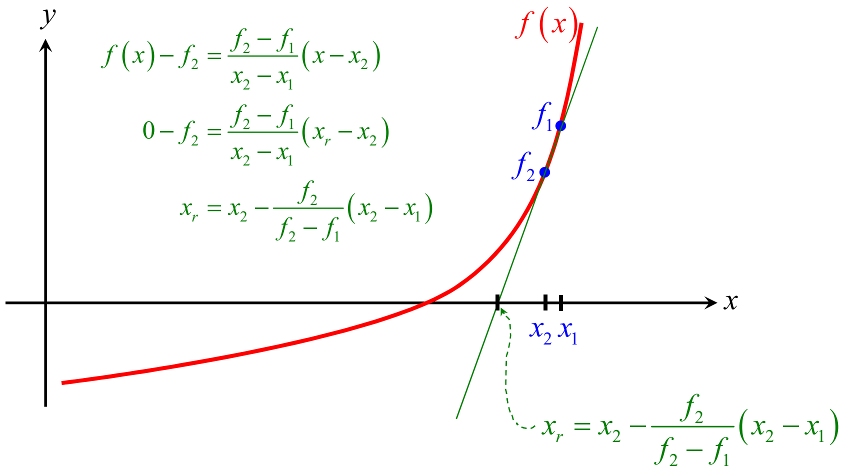
$$\text{slope} \equiv m = \frac{\text{rise}}{\text{run}} = \frac{f_2 - f_1}{x_2 - x_1}$$

$$y - y_2 = m(x - x_2)$$

$$f(x) - f_2 = \frac{f_2 - f_1}{x_2 - x_1}(x - x_2)$$

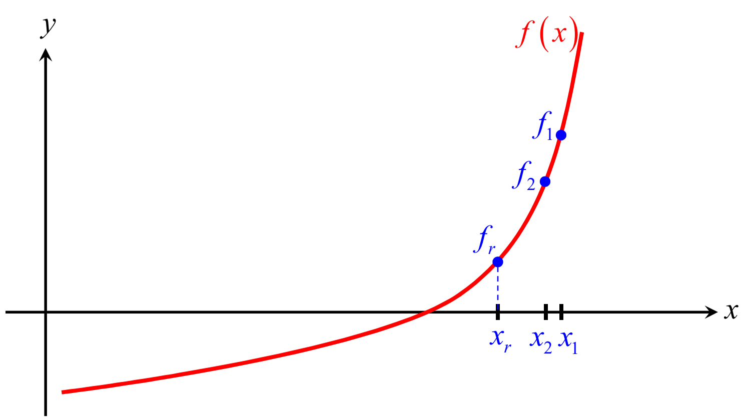
Calculate Where the Line Crosses the x-Axis

Computational Methods in EE



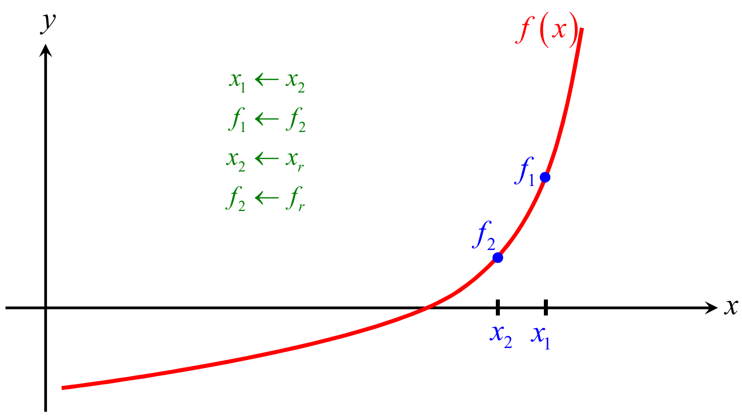
Evaluate the Function at the New Point x_r

Computational Methods in EE



Adjust Points to x_1 and x_2

Computational Methods in EE



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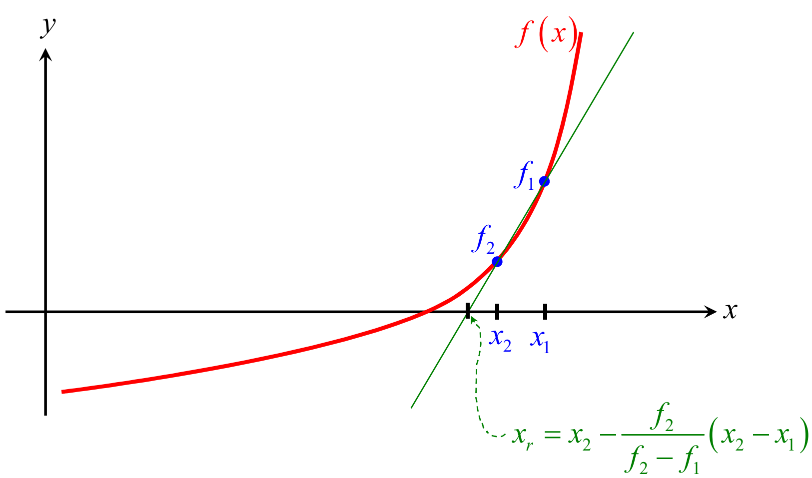
Calculate Where the New Line Crosses the x-Axis

Computational Methods in EE



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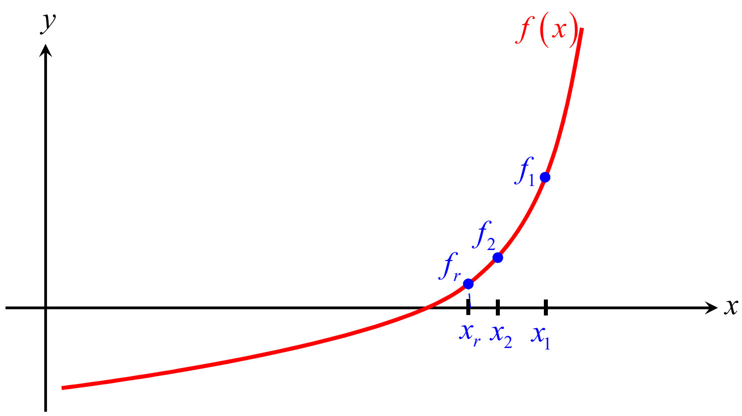
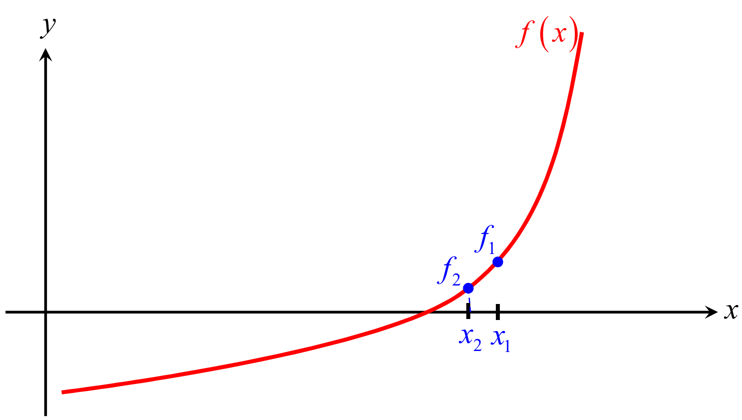
Evaluate the Function at the New Point x_r

Computational Methods in EE



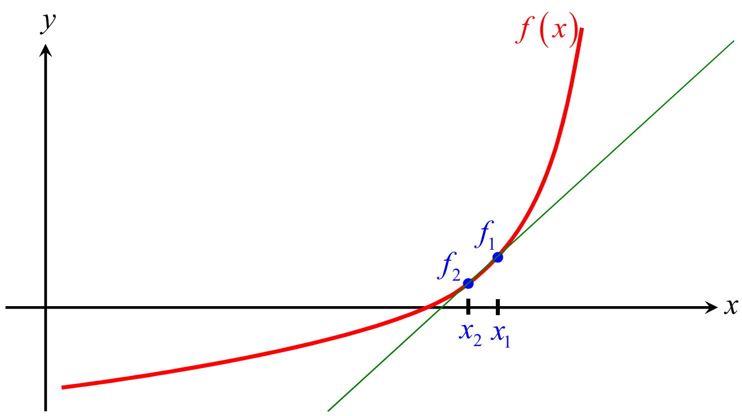
Adjust Points to x_1 and x_2

Computational Methods in EE



And so on…

Computational Methods in EE



Algorithm

Computational Methods in EE

1. Determine two good initial guesses: x_1 and x_2.
2. Evaluate the function at x_1.

   $$f_1 = f(x_1)$$

3. Iterate until converged
   1. Evaluate function at x_2.

      $$f_2 = f(x_2)$$

   2. Calculate Δx.

      $$\Delta x = \frac{f_2}{f_2 - f_1}(x_2 - x_1)$$

   3. Make new first point the old second point.

      $$x_1 = x_2 \quad \text{and} \quad f_1 = f_2$$

   4. Calculate new x_2.

      $$x_2 = x_2 - \Delta x$$

   5. If $|\Delta x| <$ tolerance, Done!

Notes on Secant Method

Computational Methods in EE

- Does not require bounds.
- Does not require $f(x)$ or $f'(x)$ to be analytical
- Requires two "good" initial guesses
- Fully numerical version of Newton's method
- Same weaknesses as Newton-Raphson method
  - Algorithm vulnerable to instability
  - Can converge to the wrong root

Example #1

Computational Methods in EE

Let $f(x) = \sin x$. What is the root of $f(x)$ in the proximity of $x = 4$?

1. Determine two good initial guesses: $x_1=4.0$ and $x_2=3.9$.
2. Evaluate the function at x_1.
   $$f_1 = f(x_1)$$
3. Iterate until converged
   a. Evaluate function at x_2.
   $$f_2 = f(x_2)$$
   b. Calculate Δx.
   $$\Delta x = \frac{f_2}{f_2 - f_1}(x_2 - x_1)$$
   c. Make new first point the old second point.
   $$x_1 = x_2 \quad \text{and} \quad f_1 = f_2$$
   d. Calculate new x_2.
   $$x_2 = x_2 - \Delta x$$
   e. If $|\Delta x| < \text{tolerance}$, Done!

```
% DASHBOARD
func = @sin;
x1   = 4.0;
x2   = 3.9;
tol  = 1e-6;

% IMPLEMENT SECANT METHOD
f1 = func(x1);
dx  = inf;
while abs(dx) > tol
    f2 = func(x2);
    dx = (x2 - x1)*f2/(f2 - f1);
    x1 = x2;
    f1 = f2;
    x2 = x2 - dx;
end
xr = x2;
```

Converges to 3.1416 after 5 iterations.

Computational Methods in EE

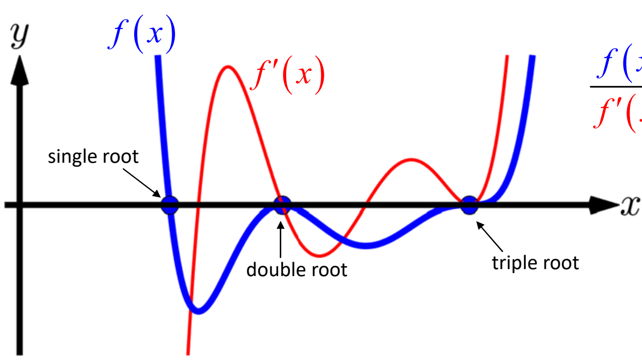
Multiple Roots

Problem with Multiple Roots

Computational Methods in EE

Our update equation for both Newton-Raphson method and secant method involve $f(x)/f'(x)$.

When we have multiple roots, both $f(x)$ and $f'(x)$ can go to zero at the root. This causes a divide by zero problem.

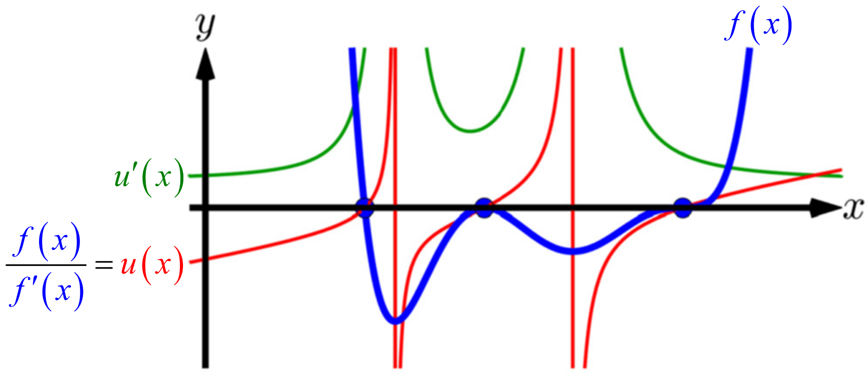


$$\frac{f(x)}{f'(x)} = \frac{0}{0} = \text{undefined}$$

A Useful Property

Computational Methods in EE

We can define an auxiliary function $u(x)$ that will have the same roots as $f(x)$ but who's derivative $u'(x)$ will not go to zero at the roots.



This is the same auxiliary function we used for bracketing methods.

The Fix

Computational Methods in EE

If $f(x)$ has multiple roots, perform root finding on the auxiliary function $u(x)$ instead.

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)} \quad \rightarrow \quad x_{i+1} = x_i - \frac{u(x_i)}{u'(x_i)}$$

We can write our new update equation completely in terms of $f(x)$ and its derivatives as follows.

$$\boxed{x_{i+1} = x_i - \frac{f(x_i) f'(x_i)}{\left[f'(x_i)\right]^2 - f(x_i) f''(x_i)}}$$