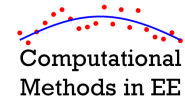




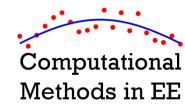
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Topic 5c – Curve Fitting to Polynomials & Interpolation/Extrapolation

EE 4386/5301 Computational Methods in EE

Outline



- Exact Fit Methods
 - Fitting Polynomials
- Interpolation & Extrapolation

Fitting Polynomials (Exact Fit, Clean Math)

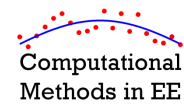
Statement of the Problem

Suppose we wish to fit the following N th-order polynomial to a set of points.

$$f(x) \approx a_0 + a_1x + a_2x^2 + \cdots + a_Nx^N$$

We have $N+1$ unknown coefficients so we need $N+1$ points to calculate the coefficients exactly.

Calculating the Unknown Polynomial Coefficients



First, we write the polynomial at $N+1$ points.

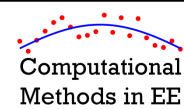
$$\begin{aligned} f(x_1) &= a_0 + a_1x_1 + a_2x_1^2 + \cdots + a_Nx_1^N \\ f(x_2) &= a_0 + a_1x_2 + a_2x_2^2 + \cdots + a_Nx_2^N \\ &\vdots \\ f(x_{N+1}) &= a_0 + a_1x_{N+1} + a_2x_{N+1}^2 + \cdots + a_Nx_{N+1}^N \end{aligned}$$

Since we are doing an exact fit, there are no residual terms.

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Calculating the Unknown Polynomial Coefficients



Second, we put the equations into matrix form.

$$\begin{aligned} f(x_1) &= a_0 + a_1x_1 + a_2x_1^2 + \cdots + a_Nx_1^N \\ f(x_2) &= a_0 + a_1x_2 + a_2x_2^2 + \cdots + a_Nx_2^N \\ &\vdots \\ f(x_{N+1}) &= a_0 + a_1x_{N+1} + a_2x_{N+1}^2 + \cdots + a_Nx_{N+1}^N \end{aligned} \quad \Rightarrow \quad \begin{bmatrix} f(x_1) \\ f(x_2) \\ f(x_3) \\ \vdots \\ f(x_{N+1}) \end{bmatrix} = \begin{bmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^N \\ 1 & x_2 & x_2^2 & \cdots & x_2^N \\ 1 & x_3 & x_3^2 & \cdots & x_3^N \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_{N+1} & x_{N+1}^2 & \cdots & x_{N+1}^N \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_N \end{bmatrix}$$

This can be written more compactly as

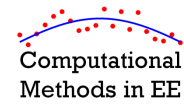
$$[f] = [X][a]$$

Aside:
This is a Vandermonde matrix and is usually ill-conditioned for large matrices. See Lagrange interpolation.

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Calculating the Unknown Polynomial Coefficients



Last, the matrix equation is solved for $[a]$ to find the polynomial coefficients.

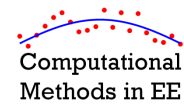
$$[a] = [X]^{-1} [f]$$

$$[a] = \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_N \end{bmatrix}$$

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Example 1 (1 of 2)

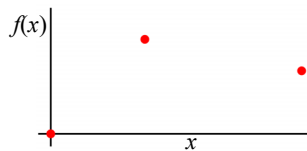


Fit the following points to a polynomial..

$$f(0) = 0$$

$$f(1.5) = 1.5$$

$$f(4.0) = 1.0$$



Step 1 – Determine order of polynomial

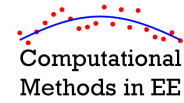
Since we have three points, we can fit a quadratic polynomial.

$$f(x) \approx a_0 + a_1x + a_2x^2$$

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Example 1 (2 of 2)



Step 2 – Our matrix equation is

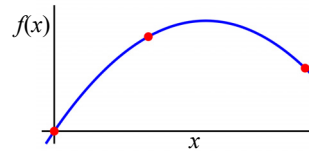
$$\begin{aligned} f_1 &= a_0 + a_1x_1 + a_2x_1^2 \\ f_2 &= a_0 + a_1x_2 + a_2x_2^2 \\ f_3 &= a_0 + a_1x_3 + a_2x_3^2 \end{aligned} \rightarrow \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix} = \begin{bmatrix} 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \\ 1 & x_3 & x_3^2 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} \rightarrow \begin{bmatrix} 0 \\ 1.5 \\ 1.0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1.5 & 2.25 \\ 1 & 4 & 16 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix}$$

Step 3 – Calculate polynomial coefficients.

$$\begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1.5 & 2.25 \\ 1 & 4 & 16 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1.5 \\ 1.0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1.45 \\ -0.3 \end{bmatrix}$$

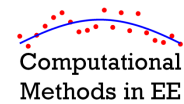
Step 4 – Write the final polynomial.

$$f(x) \approx 1.45x - 0.3x^2$$



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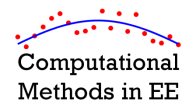


Interpolation & Extrapolation

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What is Interpolation & Extrapolation?



Sometimes we have some several precisely known values and want to know an intermediate value (interpolation) or a value outside of the range of the known values (extrapolation).

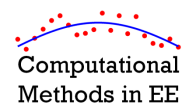
Measuring the new value may be difficult, expensive, or impossible.

Interpolation and extrapolation can be thought of as two steps:

Step 1 – Fit data to a curve.

Step 2 – Use the curve fit to calculate the new value(s).

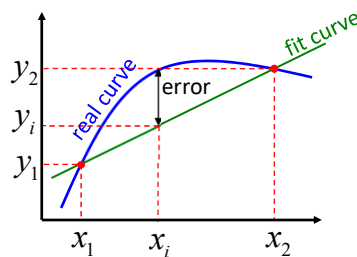
Linear Interpolation



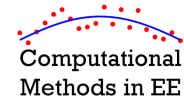
Given two data points, we can fit a line from which we can interpolate or extrapolate anything else.

Given two data points (x_1, y_1) and (x_2, y_2) , the equation for the line is

$$y_i = y_1 + \underbrace{\frac{y_2 - y_1}{x_2 - x_1}}_{\text{slope}} (x_i - x_1)$$

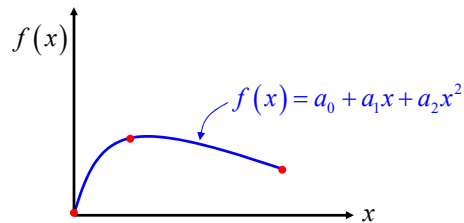


Polynomial Interpolation (1 of 3)



Interpolation using a quadratic polynomial is likely the most common method of interpolation.

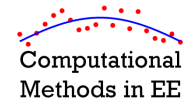
This requires three points.



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Polynomial Interpolation (2 of 3)



Write the matrix equation.

$$\begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix} = \begin{bmatrix} 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \\ 1 & x_3 & x_3^2 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix}$$

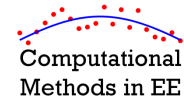
Solve for polynomial coefficients.

$$\begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \\ 1 & x_3 & x_3^2 \end{bmatrix}^{-1} \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix} = \begin{bmatrix} \frac{x_2x_3}{(x_1-x_2)(x_1-x_3)} & -\frac{x_1x_3}{(x_1-x_2)(x_2-x_3)} & \frac{x_1x_2}{(x_1-x_3)(x_2-x_3)} \\ -\frac{x_2+x_3}{(x_1-x_2)(x_1-x_3)} & \frac{x_1+x_3}{(x_1-x_2)(x_2-x_3)} & -\frac{x_1+x_2}{(x_1-x_3)(x_2-x_3)} \\ \frac{1}{(x_1-x_2)(x_1-x_3)} & -\frac{1}{(x_1-x_2)(x_2-x_3)} & \frac{1}{(x_1-x_3)(x_2-x_3)} \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix}$$

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Polynomial Interpolation (3 of 3)



The final equations are

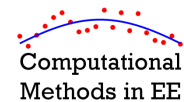
$$a_0 = \frac{x_2 x_3}{(x_1 - x_2)(x_1 - x_3)} f_1 - \frac{x_1 x_3}{(x_1 - x_2)(x_2 - x_3)} f_2 + \frac{x_1 x_2}{(x_1 - x_3)(x_2 - x_3)} f_3$$

$$a_1 = -\frac{x_2 + x_3}{(x_1 - x_2)(x_1 - x_3)} f_1 + \frac{x_1 + x_3}{(x_1 - x_2)(x_2 - x_3)} f_2 - \frac{x_1 + x_2}{(x_1 - x_3)(x_2 - x_3)} f_3$$

$$a_2 = \frac{1}{(x_1 - x_2)(x_1 - x_3)} f_1 - \frac{1}{(x_1 - x_2)(x_2 - x_3)} f_2 + \frac{1}{(x_1 - x_3)(x_2 - x_3)} f_3$$

Typically, these are not calculated using these equations. Instead, the matrix inversion is performed numerically.

Example 1 – Quadratic Interpolation



Given the following points, interpolate the value at $x = 3$.

$$f(0) = 0$$

$$f(1.5) = 1.5$$

$$f(4.0) = 1.0$$

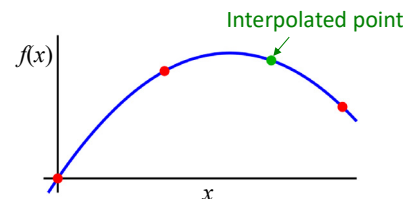
Solution

Step 1 – We already fit these points to a quadratic polynomial. The answer was

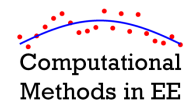
$$f(x) = 1.45x - 0.3x^2$$

Step 2 – We evaluate this at $x = 3$.

$$f(3) = 1.45(3) - 0.3(3)^2 = \boxed{1.65}$$



Example 2 – Quadratic Extrapolation



Given the following points, extrapolate the value at $x = -0.5$.

$$f(0) = 0$$

$$f(1.5) = 1.5$$

$$f(4.0) = 1.0$$

Solution

Step 1 – We already fit these points to a quadratic polynomial. The answer was

$$f(x) = 1.45x - 0.3x^2$$

Step 2 – We evaluate this at $x = -0.5$.

$$f(-0.5) = 1.45(-0.5) - 0.3(-0.5)^2 = \boxed{-0.8}$$

