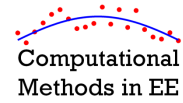




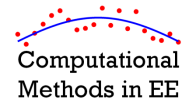
**Course Instructor**  
Dr. Raymond C. Rumpf  
Office: A-337  
Phone: (915) 747-6958  
E-Mail: rcrumpf@utep.edu



# Topic 7b – Numerical Analysis of Slab Waveguides

*EE 4386/5301 Computational Methods in EE*

## Outline



- Slab Waveguides
- Formulation
- Solution
- Implementation in MATLAB

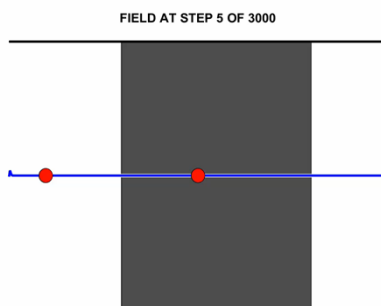
# Slab Waveguides

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## Refractive Index $n$

Light travels at different speeds when it is inside different materials.



Frequency is constant.  
Speed changes.  
Wavelength changes.

The factor by which light slows down is called the *refractive index*.

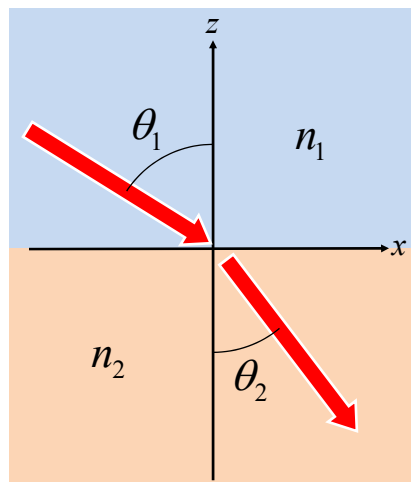
$$n = \frac{c}{v}$$

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## Snell's Law

Computational  
Methods in EE



Snell's law quantifies the angles of light rays at an interface.

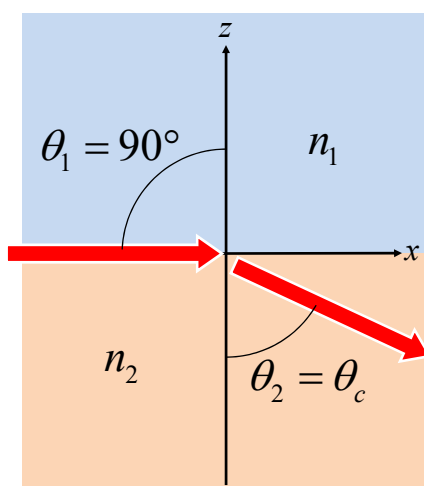
$$n_1 \sin \theta_1 = n_2 \sin \theta_2$$

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## Critical Angle

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There exists a special angle, the *critical angle*, where the ray in the low-index medium is at  $90^\circ$ .

$$n_1 \sin \theta_1 = n_2 \sin \theta_2$$

$$n_1 \sin 90^\circ = n_2 \sin \theta_c$$

$$n_1 = n_2 \sin \theta_c$$

$$\sin \theta_c = n_1/n_2$$

$$\theta_c = \sin^{-1}(n_1/n_2)$$

$$\theta_c = \sin^{-1}(n_1/n_2)$$

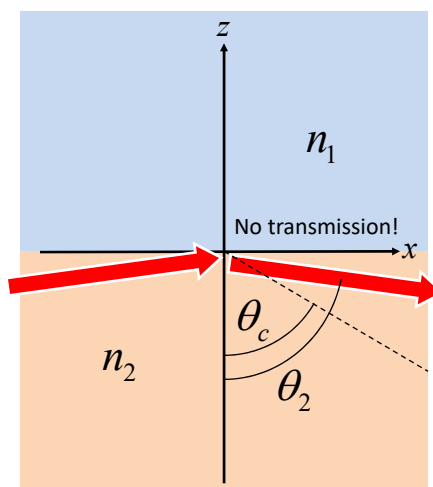
where  $n_2 > n_1$

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## Total Internal Reflection (TIR)

Computational  
Methods in EE



When a light ray is incident onto an interface at an angle greater than the critical angle, the light completely reflects and no light is transmitted.

This is called *total internal reflection* (TIR).

$$\theta_2 > \theta_c$$

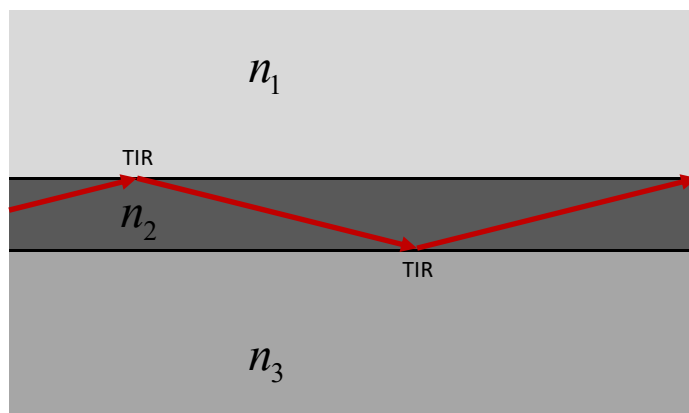
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## The Slab Waveguide

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If we “sandwich” a slab of high-index material between two materials with lower refractive index, we form a slab waveguide.

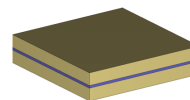


### Conditions

$$n_2 > n_1$$

and

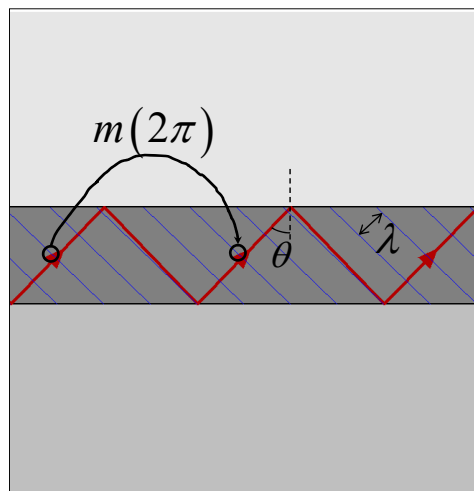
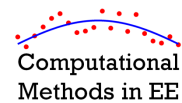
$$n_2 > n_3$$



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## Ray Tracing Picture



The round trip phase of a ray must be an integer multiple of  $2\pi$ . Otherwise the wave will interfere with itself and escape from the slab.

Because of this, only certain angles are allowed to propagate in the waveguide.

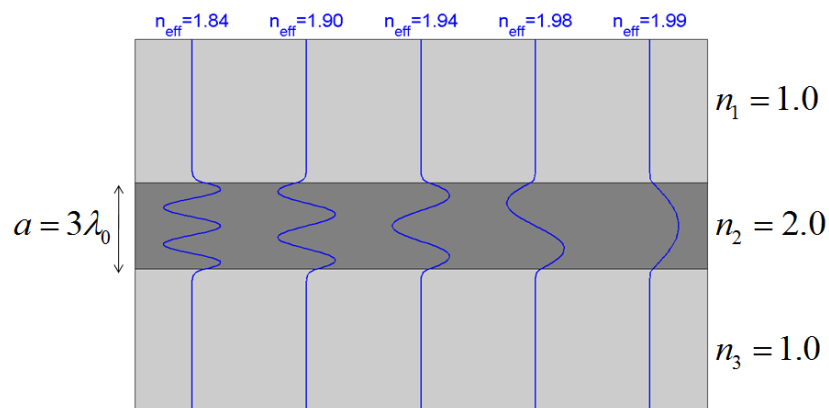
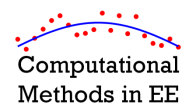
This is the origin of discrete modes in a waveguide.

$$\beta = k_0 n_{\text{eff}} = k_0 n \sin \theta$$

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## Rigorous Analysis

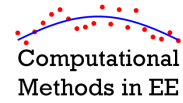


$$\beta = k_0 n_{\text{eff}} = k_0 n \sin \theta$$

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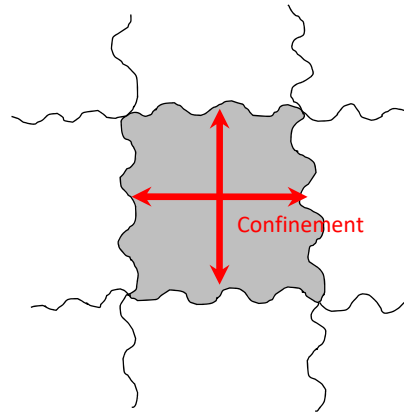
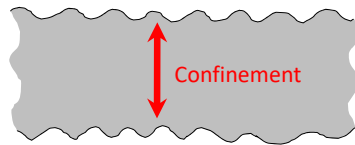
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# Slab Vs. Channel Waveguides



Slab waveguides confine energy in only one transverse direction.

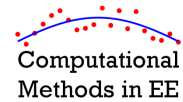
Channel waveguides confine energy in both transverse directions.



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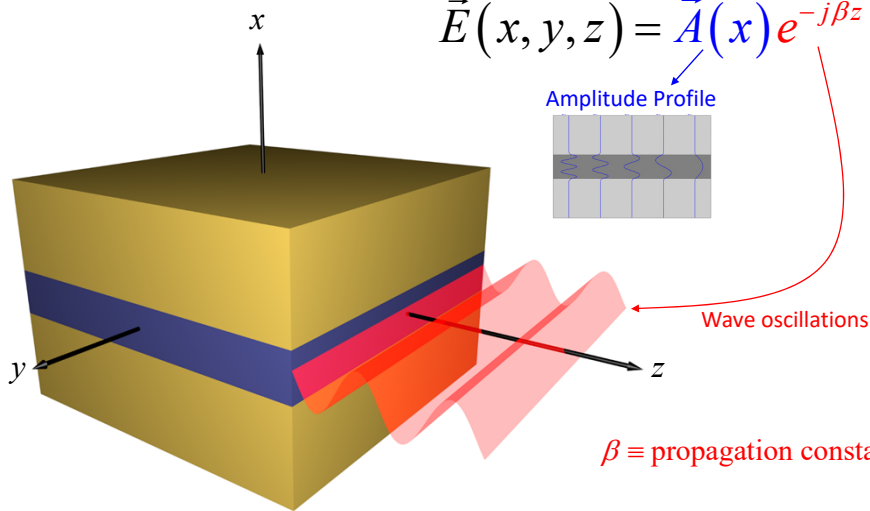
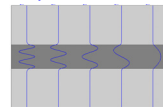
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# Mathematical Form of Solution



$$\vec{E}(x, y, z) = \vec{A}(x) e^{-j\beta z}$$

Amplitude Profile



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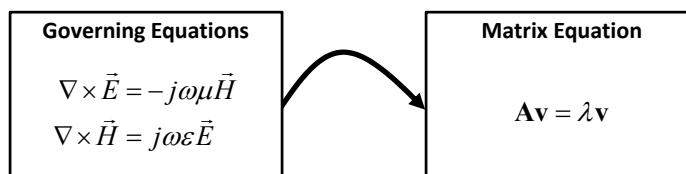
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# Formulation

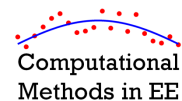
## What is Formulation?

Formulation is the initial analytical work we do before implementing a computer code.

Usually we start with the governing equation(s) and end with the matrix equation to be solved.



## Governing Equations



Since this is an electromagnetic problem, we start with Maxwell's curl equations.

$$\nabla \times \vec{E} = -j\omega\mu\vec{H}$$

$$\nabla \times \vec{H} = j\omega\varepsilon\vec{E}$$

### Vector Curl

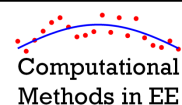
The curl of a vector is a measure of the vector field's tendency to circulate about an axis. The curl quantity is directly along this axis and the magnitude measures the strength of the circulation.

$$\nabla \times \vec{A} = \left( \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} \right) \hat{a}_x + \left( \frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} \right) \hat{a}_y + \left( \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right) \hat{a}_z$$

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## Expand Governing Equations (1 of 2)



If we expand the first equation into its vector components, we get

$$\nabla \times \vec{E} = -j\omega\mu\vec{H}$$

$$\left( \frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} \right) \hat{a}_x + \left( \frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} \right) \hat{a}_y + \left( \frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} \right) \hat{a}_z = -j\omega\mu (H_x \hat{a}_x + H_y \hat{a}_y + H_z \hat{a}_z)$$

The vector components on each side must be equal.

$$x\text{-component: } \frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} = -j\omega\mu H_x$$

$$y\text{-component: } \frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} = -j\omega\mu H_y$$

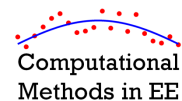
$$z\text{-component: } \frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = -j\omega\mu H_z$$

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## Expand Governing Equations (2 of 2)



We now have

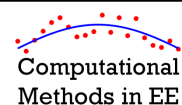
$$\nabla \times \vec{E} = -j\omega\mu\vec{H} \rightarrow \begin{aligned} \frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} &= -j\omega\mu H_x \\ \frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} &= -j\omega\mu H_y \\ \frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} &= -j\omega\mu H_z \end{aligned}$$

$$\nabla \times \vec{H} = j\omega\varepsilon\vec{E} \rightarrow \begin{aligned} \frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} &= j\omega\varepsilon E_x \\ \frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} &= j\omega\varepsilon E_y \\ \frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} &= j\omega\varepsilon E_z \end{aligned}$$

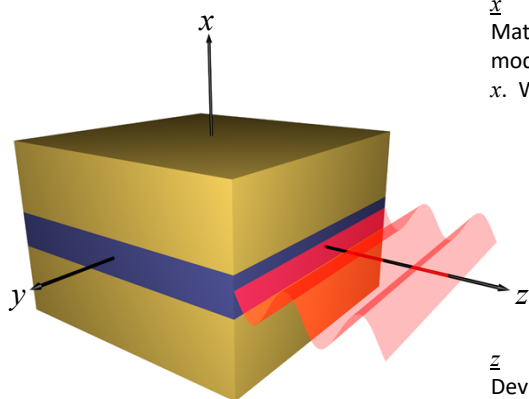
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## How to Reduce Dimensions



It is always good practice to minimize the number of dimensions utilized in a numerical analysis.



$\underline{x}$   
Material changes as a function of  $x$ . The mode profile will change as a function of  $x$ . We must retain this dimension.

$\underline{y}$   
Device is uniform. Wave does not propagate in this direction. Mode profile is uniform.

$$\frac{\partial}{\partial y} = 0$$

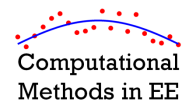
$\underline{z}$   
Device is uniform. Wave propagates in this direction so wave phase is increasing.

$$\frac{\partial}{\partial z} = -j\beta$$

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## Apply $\partial/\partial y = 0$



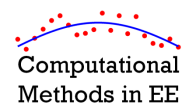
Since nothing is changing in the  $y$  direction, any derivative with respect to  $y$  must be zero.

$$\begin{array}{ccc}
 \cancel{\frac{\partial E_y}{\partial y}} - \frac{\partial E_y}{\partial z} = -j\omega\mu H_x & & -\frac{\partial E_y}{\partial z} = -j\omega\mu H_x \\
 \frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} = -j\omega\mu H_y & \longrightarrow & \frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} = -j\omega\mu H_y \\
 \frac{\partial E_y}{\partial x} - \cancel{\frac{\partial E_z}{\partial y}} = -j\omega\mu H_z & & \frac{\partial E_y}{\partial x} = -j\omega\mu H_z \\
 \\ 
 \cancel{\frac{\partial H_z}{\partial y}} - \frac{\partial H_y}{\partial z} = j\omega\varepsilon E_x & & -\frac{\partial H_y}{\partial z} = j\omega\varepsilon E_x \\
 \frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} = j\omega\varepsilon E_y & \longrightarrow & \frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} = j\omega\varepsilon E_y \\
 \frac{\partial H_y}{\partial x} - \cancel{\frac{\partial H_x}{\partial y}} = j\omega\varepsilon E_z & & \frac{\partial H_y}{\partial x} = j\omega\varepsilon E_z
 \end{array}$$

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## Two Distinct Mode Types



Our revised governing equations have separated into two distinct mode types.

We will analyze the  $E_y$  mode

### Mode Type 1 – $E_y$ Mode

$$\begin{array}{l}
 \frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} = j\omega\varepsilon E_y \\
 -\frac{\partial E_y}{\partial z} = -j\omega\mu H_x \\
 \frac{\partial E_y}{\partial x} = -j\omega\mu H_z
 \end{array}$$

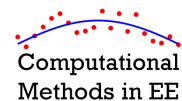
### Mode Type 2 – $H_y$ Mode

$$\begin{array}{l}
 -\frac{\partial E_y}{\partial z} = -j\omega\mu H_x \\
 \frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} = -j\omega\mu H_y \\
 \frac{\partial E_y}{\partial x} = -j\omega\mu H_z \\
 \\ 
 -\frac{\partial H_y}{\partial z} = j\omega\varepsilon E_x \\
 \frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} = j\omega\varepsilon E_y \\
 \frac{\partial H_y}{\partial x} = j\omega\varepsilon E_z
 \end{array}$$

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## What About $\partial/\partial z$ ?



Our guided mode has the following mathematical form

$$\vec{E}(x, y, z) = \vec{A}(x) e^{-j\beta z}$$

Let's calculate the partial derivative with respect to  $z$  and see what happens.

$$\begin{aligned} \frac{\partial}{\partial z} \vec{E}(x, y, z) &= \frac{\partial}{\partial z} [\vec{A}(x) e^{-j\beta z}] = \vec{A}(x) \frac{\partial}{\partial z} e^{-j\beta z} + \cancel{e^{-j\beta z} \frac{\partial}{\partial z} \vec{A}(x)} \\ &= -j\beta \underbrace{\vec{A}(x) e^{-j\beta z}}_{\vec{E}(x, y, z)} = -j\beta \vec{E}(x, y, z) \end{aligned}$$

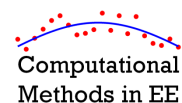
We conclude that

$$\frac{\partial}{\partial z} = -j\beta$$

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## 1D Governing Equations



The equations for the  $E_y$  mode were

$$\begin{aligned} \frac{\partial}{\partial z} H_x - \frac{\partial H_z}{\partial x} &= j\omega\epsilon E_y \\ -\frac{\partial}{\partial z} E_y &= -j\omega\mu H_x \\ \frac{\partial E_y}{\partial x} &= -j\omega\mu H_z \end{aligned}$$

We simply replace  $\partial/\partial z$  with  $-j\beta$ .

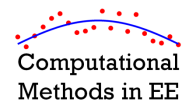
$$\begin{aligned} -j\beta H_x - \frac{dH_z}{dx} &= j\omega\epsilon E_y \\ j\beta E_y &= -j\omega\mu H_x \\ \frac{dE_y}{dx} &= -j\omega\mu H_z \end{aligned}$$

The partial derivative has become an ordinary derivative because there is only one independent variable remaining.

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## Normalize



Before converting our equations to matrix form, we should normalize the spatial coordinate  $x$  to put it in terms of wavelength in some manner.

$$\tilde{x} = \frac{x}{\lambda_0}$$

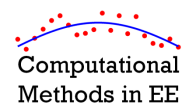
Additionally, it will be mathematically convenient to normalize by multiplying  $x$  by the free space wave number  $k_0$ .

$$\tilde{x} = k_0 x \quad k_0 = \frac{2\pi}{\lambda_0} = \frac{\omega\sqrt{\mu\varepsilon}}{n}$$

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## Normalizing Maxwell's Eqs.



We start with the following equation,

$$-j\beta H_x - \frac{dH_z}{dx} = j\omega\varepsilon E_y$$

and replace  $x$  with  $\tilde{x}/k_0$ .

$$-j\beta H_x - k_0 \frac{dH_z}{d\tilde{x}} = j\omega\varepsilon E_y$$

Next we divide both sides of the equation by  $k_0$ .

$$-j \frac{\beta}{k_0} H_x - \frac{dH_z}{d\tilde{x}} = \frac{j\omega\varepsilon}{k_0} E_y$$

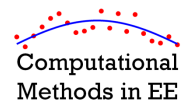
Recognizing that  $\beta = k_0 n_{\text{eff}}$  this equation becomes

$$\begin{aligned} -jn_{\text{eff}} H_x - \frac{dH_z}{d\tilde{x}} &= \frac{j\omega\varepsilon}{k_0} E_y \\ &= \frac{j\omega\varepsilon_0\varepsilon_r}{\omega\sqrt{\mu_0\varepsilon_0}} E_y = j\sqrt{\frac{\varepsilon_0}{\mu_0}} \varepsilon_r E_y \end{aligned}$$

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## Normalized Equations



Applying our normalizations to all equations, we get

$$\begin{aligned}
 -jn_{\text{eff}}H_x - \frac{dH_z}{d\tilde{x}} &= j\sqrt{\frac{\epsilon_0}{\mu_0}}\epsilon_r E_y \\
 jn_{\text{eff}}E_y &= -j\sqrt{\frac{\mu_0}{\epsilon_0}}\mu_r H_x \\
 \frac{dE_y}{d\tilde{x}} &= -j\sqrt{\frac{\mu_0}{\epsilon_0}}\mu_r H_z
 \end{aligned}$$

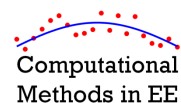
Last, at optical frequencies, the permeability is negligible so  $\mu_r = 1$ .

$$\begin{aligned}
 -jn_{\text{eff}}H_x - \frac{dH_z}{d\tilde{x}} &= j\sqrt{\frac{\epsilon_0}{\mu_0}}\epsilon_r E_y \\
 jn_{\text{eff}}E_y &= -j\sqrt{\frac{\mu_0}{\epsilon_0}}H_x \\
 \frac{dE_y}{d\tilde{x}} &= -j\sqrt{\frac{\mu_0}{\epsilon_0}}H_z
 \end{aligned}$$

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## Final Governing Equation



We solve the last two equations for  $H_x$  and  $H_z$ .

$$\begin{aligned}
 -jn_{\text{eff}}H_x - \frac{dH_z}{d\tilde{x}} &= j\sqrt{\frac{\epsilon_0}{\mu_0}}\epsilon_r E_y \\
 jn_{\text{eff}}E_y &= -j\sqrt{\frac{\mu_0}{\epsilon_0}}H_x \quad \rightarrow \quad H_x = -n_{\text{eff}}\sqrt{\frac{\epsilon_0}{\mu_0}}E_y \\
 \frac{dE_y}{d\tilde{x}} &= -j\sqrt{\frac{\mu_0}{\epsilon_0}}H_z \quad \rightarrow \quad H_z = j\sqrt{\frac{\epsilon_0}{\mu_0}}\frac{dE_y}{d\tilde{x}}
 \end{aligned}$$

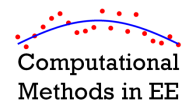
These are substituted into the first equation to get a single equation containing only  $E_y$ . This is why it was called the  $E_y$  mode.

$$\begin{aligned}
 -jn_{\text{eff}}H_x - \frac{dH_z}{d\tilde{x}} &= j\sqrt{\frac{\epsilon_0}{\mu_0}}\epsilon_r E_y \\
 -jn_{\text{eff}}\left(-n_{\text{eff}}\sqrt{\frac{\epsilon_0}{\mu_0}}E_y\right) - \frac{d}{d\tilde{x}}\left(j\sqrt{\frac{\epsilon_0}{\mu_0}}\frac{dE_y}{d\tilde{x}}\right) &= j\sqrt{\frac{\epsilon_0}{\mu_0}}\epsilon_r E_y \\
 n_{\text{eff}}^2 E_y - \frac{d^2 E_y}{d\tilde{x}^2} &= \epsilon_r E_y \quad \longrightarrow \quad \frac{d^2 E_y}{d\tilde{x}^2} + \epsilon_r E_y = n_{\text{eff}}^2 E_y
 \end{aligned}$$

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## Eigen-Value Problem



For optical problems, we like to put everything in terms of refractive index. We have  $\epsilon_r = n^2$ .

$$\frac{d^2 E_y}{d\tilde{x}^2} + n^2 E_y = n_{\text{eff}}^2 E_y$$

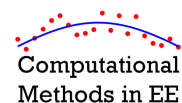
Our governing equation can be rearranged to the form of a standard eigen-value problem.

$$\left[ \frac{d^2}{d\tilde{x}^2} + n^2(x) \right] E_y(x) = n_{\text{eff}}^2 E_y(x)$$

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## Matrix Form



We go term-by-term to write the equation in matrix form.

$$\left[ \frac{d^2}{d\tilde{x}^2} + n^2(x) \right] E_y(x) = n_{\text{eff}}^2 E_y(x)$$

$$(\mathbf{D}_{\tilde{x}}^2 + \mathbf{n}^2) \mathbf{e}_y = n_{\text{eff}}^2 \mathbf{e}_y$$

or

$$(\mathbf{D}_{\tilde{x}}^2 + \boldsymbol{\epsilon}) \mathbf{e}_y = n_{\text{eff}}^2 \mathbf{e}_y$$

Eigen Matrix

Eigen Value

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# Solution

## Solving the Eigen-Value Problem

$$(\mathbf{D}_x^2 + \boldsymbol{\varepsilon})\mathbf{e}_y = n_{\text{eff}}^2 \mathbf{e}_y \rightarrow \begin{array}{l} \mathbf{V} \equiv \text{Eigen-vector matrix} \\ \boldsymbol{\lambda} \equiv \text{Eigen-value matrix} \end{array}$$

$$\mathbf{V} = \begin{bmatrix} \begin{bmatrix} e_y^{(1)}(1) \\ e_y^{(1)}(2) \\ e_y^{(1)}(3) \\ \vdots \\ e_y^{(1)}(N_x-1) \\ e_y^{(1)}(N_x) \end{bmatrix} & \begin{bmatrix} e_y^{(2)}(1) \\ e_y^{(2)}(2) \\ e_y^{(2)}(3) \\ \vdots \\ e_y^{(2)}(N_x-1) \\ e_y^{(2)}(N_x) \end{bmatrix} & \dots & \begin{bmatrix} e_y^{(M)}(1) \\ e_y^{(M)}(2) \\ e_y^{(M)}(3) \\ \vdots \\ e_y^{(M)}(N_x-1) \\ e_y^{(M)}(N_x) \end{bmatrix} \end{bmatrix}$$

$M = \# \text{ modes}$   
Usually  $M = N_x$

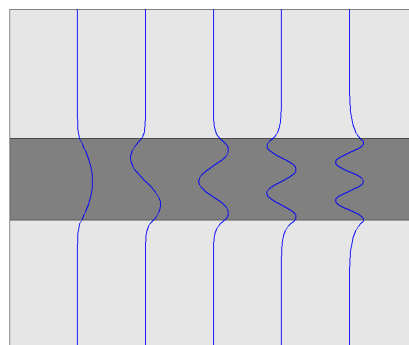
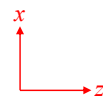
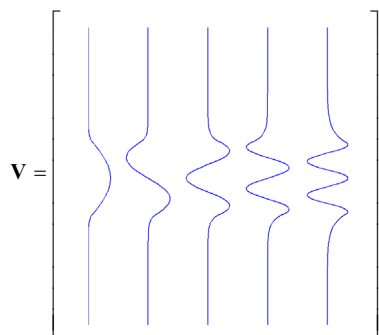
$$\mathbf{D} = \begin{bmatrix} (n_{\text{eff}}^{(1)})^2 & & & \\ & (n_{\text{eff}}^{(2)})^2 & & \\ & & \ddots & \\ & & & (n_{\text{eff}}^{(M)})^2 \end{bmatrix}$$

Eigen-vectors and eigen-values  
come in pairs.

Do not mix up their pairing!

## Visualizing the Solution

The columns of the eigen-vector matrix are pictures of the modes.

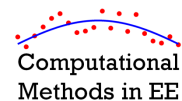


The eigen-values are the effective refractive indices of the modes.

## Implementation in MATLAB



## Implementation Outline

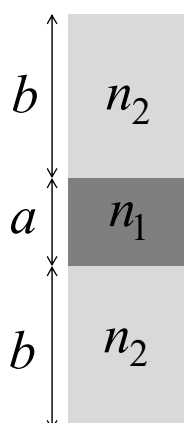
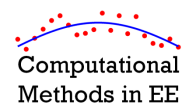


1. Initialize MATLAB
2. Dashboard (materials, dimensions, etc.)
3. Calculate Grid
4. Build Device on Grid
5. Perform Finite-Difference Analysis
6. Visualize the Results

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## Dashboard



How big should we make  $b$ ?  
 → Enough to allow the mode to decay to zero before reaching the boundary.

What grid resolution should we use?  
 → Convergence

```
% slabdemo.m

% INITIALIZE MATLAB
close all;
clc;
clear all;

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%% DASHBOARD
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

% FREE SPACE WAVELENGTH
lam0 = 1.0;

% SLAB PARAMETERS
n1 = 2.0;
n2 = 1.0;
a = 3*lam0;

% GRID
b = 5*lam0;
dx = lam0/20;

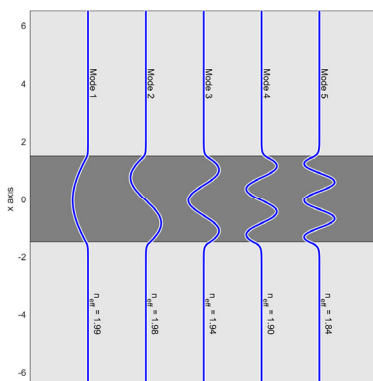
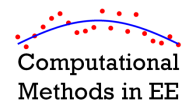
% NUMBER OF MODES TO CALCULATE
M = 5;
```

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# Visualize the Results



```

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%% VISUALIZE
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

% SORT MODES
[~,ind] = sort(real(NEFF),'descend');
V       = V(:,ind);
NEFF    = NEFF(ind);

% OPEN FIGURE WINDOW
figure('Color','w');
hold on;

% DRAW SLAB WAVEGUIDE
x = [0 2*(M+1) 2*(M+1) 0 0];
y = [ -b-a/2 -b-a/2 b+a/2 b+a/2 -b-a/2 ];
fill(x,y,0.9*[1 1 1]);
y = [ -a/2 -a/2 a/2 a/2 -a/2 ];
fill(x,y,0.5*[1 1 1]);

% DRAW AND LABEL MODES
for m = 1 : M
    x0 = 2*m;
    y0 = (a + b)/2;
    x = x0 + 3*(1:m);
    y = linspace(-b-a/2,b+a/2,Nx);
    line(x,y,'Color','w','LineWidth',4);
    h = line(x,y,'Color','b','LineWidth',2);
    text(x0,y0,['Mode ' num2str(m)],'Rotation',-90,...
        'HorizontalAlignment','center','VerticalAlignment','bottom');
    text(x0,-y0,['n_eff = ' num2str(NEFF(m),'%.2f')],'Rotation',-90,...
        'HorizontalAlignment','center','VerticalAlignment','bottom');
end

% SET GRAPHICS VIEW
hold off;
h2 = get(h,'Parent');
set(h2,'XTick',[]);
axis equal tight;
xlabel('x axis');

```

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