Module 2 – Spatially-Variant Planar Gratings

Course Website: http://emlab.utep.edu/scSVL.htm

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Lecture Outline

• The grating vector
• Grating phase
• Generating spatially-variant planar gratings
The Grating Vector $\vec{K}$

The Grating Vector in Two Dimensions

- The grating vector $\vec{K}$ is very similar to a wave vector $\vec{k}$.
- The direction of $\vec{K}$ is perpendicular to the grating planes.
- The magnitude of $\vec{K}$ is $2\pi$ divided by the period of the grating $\Lambda$.
  \[ |\vec{K}| = \frac{2\pi}{\Lambda} \]
- Given the slant angle $\theta$, it is calculated as
  \[ \vec{K} = \frac{2\pi}{\Lambda} (\hat{a}_x \cos \theta + \hat{a}_y \sin \theta) \]
- It allows a convenient calculation of the analog grating.
  \[ \cos (\vec{K} \cdot \vec{r}) \]
The Grating Vector in Three Dimensions

\[ |\vec{K}| = \frac{2\pi}{\Lambda} \]
\[ \vec{K} = K_x \hat{x} + K_y \hat{y} + K_z \hat{z} \]

The Analog Grating

For simplicity, calculation of the analog grating will be written as
\[ \varepsilon_a(\vec{r}) = \cos(\vec{K} \cdot \vec{r}) \]
or
\[ \varepsilon_a(\vec{r}) = \text{Re} \left[ \exp \left( j \vec{K} \cdot \vec{r} \right) \right] \]

If instead you wish to generate an actual analog grating, it will need to be scaled to convey physical permittivity values.
\[ \varepsilon(\vec{r}) = \varepsilon_{\text{avg}} + \Delta \varepsilon \cos(\vec{K} \cdot \vec{r}) \]
The Binary Grating

Generation of the grating using a cosine function gives us a smooth analog profile. This requires functionally grading the material properties to realize this. That is hard.

We can use a threshold to convert the analog profile to a binary grating. This is much easier to physically realize.

We can adjust the threshold value to control the duty cycle, or fill fraction, of the grating.

Grating Phase

\[ \Phi(\vec{r}) \]
**Concept of Grating Phase**

Here we make an analogy with a standard wave. A wave propagates in the direction of the wave vector \( \mathbf{k} \).

\[
\mathbf{E}(\mathbf{r}) = \mathbf{E}_0 \cos(\mathbf{k} \cdot \mathbf{r} - \omega t)
\]

As the wave propagates, it accumulates phase \( \phi \) which is a function that increases in the direction of \( \mathbf{k} \). The phase results from the wave propagating with wave vector \( \mathbf{k} \) and we could alternatively reconstruct the wave just from the phase.

\[
\mathbf{E}(\mathbf{r}) = \mathbf{E}_0 \cos(\phi(\mathbf{r})) \quad \quad \phi(\mathbf{r}) = \mathbf{k} \cdot \mathbf{r} - \omega t
\]

For gratings, the grating vector \( \mathbf{K} \) serves a similar purpose as the wave vector \( \mathbf{k} \) does for waves.

\[
\begin{align*}
\phi(\mathbf{r}) &\leftrightarrow \Phi(\mathbf{r}) \\
\mathbf{E}(\mathbf{r}) &\leftrightarrow \mathbf{e}_x(\mathbf{r}) \quad \quad \mathbf{k}(\mathbf{r}) \leftrightarrow \mathbf{K}(\mathbf{r})
\end{align*}
\]

**Definition of Grating Phase**

We can think of a grating as a wave because it has the same mathematical form. As a wave propagates it accumulates phase. The wave can be calculated from just the phase. Think of the grating phase \( \Phi \) this way...

\[
\begin{align*}
\mathbf{e}_x(\mathbf{r}) &= \cos(\mathbf{K} \cdot \mathbf{r}) = \cos[\Phi(\mathbf{r})] \\
\Phi(\mathbf{r}) &= \mathbf{K} \cdot \mathbf{r} = K_x \cdot x + K_y \cdot y
\end{align*}
\]

\[
\begin{align*}
\begin{array}{c}
K_x \quad \bullet \quad X \\
-5 & 0 & 5 \\
\end{array} &+ & \begin{array}{c}
K_y \quad \bullet \quad Y \\
-5 & 0 & 5 \\
\end{array}
\end{align*}
\]

\[
\Phi(\mathbf{r}) = \cos(\Phi(\mathbf{r}))
\]

Note: Unfortunately, this intuitive definition of grating phase only holds when \( \mathbf{K} \) is a constant and not a function of position.
Problem of a Chirped Grating

Suppose we wish to generate the following chirped grating with period \( \Lambda(z) \).

\[
\Lambda(z) = \Lambda_0 z \quad 1 \leq z \leq 2
\]

This 1D grating is described by the following grating vector function.

\[
K(z) = \frac{2\pi}{\Lambda(z)} = \frac{2\pi}{\Lambda_0 z}
\]

What happens when we try to calculate the grating using \( \cos\left(\vec{K} \cdot \vec{r}\right) \).

\[
\varepsilon_a(z) = \cos\left[K(z) \cdot z\right] = \cos\left[\frac{2\pi}{\Lambda_0 z} \cdot z\right] = \cos\left[\frac{2\pi}{\Lambda_0}\right]
\]

The argument in the cosine is a constant so we do not even generate a grating here.

Conclusion From the Chirped Grating

When the grating vector is a function of position, we can no longer calculate the grating directly from it.

\[
\varepsilon_a(\vec{r}) \neq \cos\left[\vec{K} \cdot \vec{r}\right] \quad \Phi(\vec{r}) \neq \vec{K}(\vec{r}) \cdot \vec{r}
\]

Instead, we must generate the grating through the intermediate parameter of the grating phase.

\[
\varepsilon_a(\vec{r}) = \cos\left[\Phi(\vec{r})\right]
\]

However, we must adopt a more rigorous definition of grating phase.

\[
\nabla \Phi(\vec{r}) = \vec{K}(\vec{r})
\]

Key equation
Now let’s construct the chirped grating using the grating phase.

\[
\nabla \Phi = K(\vec{r})
\]

\[
\frac{d\Phi}{dz} = \frac{2\pi}{\Lambda_0 z}
\]

\[
\Phi(z) = \int_{-\infty}^{z} \frac{2\pi}{\Lambda_0 z'} dz' = \int \frac{2\pi}{\Lambda_0 z'} dz' + \int \frac{2\pi}{\Lambda_0 z'} dz'
\]

\[
\Phi(z) = \frac{2\pi}{\Lambda_0} \ln z = \frac{2\pi}{\Lambda_0} \left( \ln z - \ln z_0 \right)
\]

The analog grating is then

\[
\varepsilon_z(z) = \cos \left[ \Phi(z) \right] = \cos \left[ \frac{2\pi}{\Lambda_0} \ln z \right]
\]

This term is just a constant. Since it is phase, we are free to choose whatever is convenient. Here we choose zero.
Procedure for Generating Spatially-Variant Planar Gratings

1. Define the grating vector function $\vec{K}(\vec{r})$, or $K$-function.

$$\vec{K}(\vec{r}) = \frac{2\pi}{\Lambda(\vec{r})} \left[ \hat{a}_\iota \cos[\theta(\vec{r})] + \hat{a}_\iota \sin[\theta(\vec{r})] \right]$$

\[ \forall \Phi(\vec{r}) = \vec{K}(\vec{r}) \]

2. Calculate the grating phase $\Phi(\vec{r})$ from $\vec{K}(\vec{r})$.

3. Calculate the analog grating from the grating phase.

$$\epsilon_a(\vec{r}) = \cos[\Phi(\vec{r})]$$

4. Calculate the binary grating from the analog grating.

$$\epsilon_b(\vec{r}) = \begin{cases} \epsilon_1 & \text{if } \epsilon_a(\vec{r}) \leq \gamma(\vec{r}) \\ \epsilon_2 & \text{if } \epsilon_a(\vec{r}) > \gamma(\vec{r}) \end{cases}$$
Build Input Functions

We need two things to build the $K$-function:

1. Orientation of the grating as a function of position, $\theta(\bar{r})$.
2. Period of the grating as a function of position, $\Lambda(\bar{r})$.

### % GRATING PARAMETERS

- $a = 1$;
- $er1 = 2.5$;
- $er2 = 1.0$;
- $gth = 0$;

### % GRID PARAMETERS

- $Sx = 5*a$;
- $Sy = Sx$;
- $Nz = 100$;
- $Ny = \text{round}(Nz*Sy/Sx)$;

### % CALCULATE GRID

- $dx = Sx/Nz$;
- $dy = Sy/Ny$;
- $xa = [1:Nz]*dx$;
- $ya = [1:Ny]*dy$;

- $[Y,X] = \text{meshgrid}(ya,xa)$;

### % SPATIALLY-VARIANT PARAMETERS

- $\text{PER} = a*\text{ones}(Nz,Ny)$;
- $\text{THETA} = \text{atan2}(Y,X)$;

Calculate the $K$-Function

The $K$-function is calculated directly from the input functions.

$$K_x(\bar{r}) = \frac{2\pi}{\Lambda(\bar{r})} \cos[\theta(\bar{r})]$$

$$K_y(\bar{r}) = \frac{2\pi}{\Lambda(\bar{r})} \sin[\theta(\bar{r})]$$

### % CALCULATE K-FUNCTION

- $Kx = 2*\pi/\text{PER}.*\text{cos}(\text{THETA})$;
- $Ky = 2*\pi/\text{PER}.*\text{sin}(\text{THETA})$;
Calculate Grating Phase

We calculate the grating phase from the $K$-function as a best fit.

$\nabla \Phi(\vec{r}) = \vec{K}$

\[
\begin{bmatrix}
  D_x \\
  D_y
\end{bmatrix}
\begin{bmatrix}
  k_x \\
  k_y
\end{bmatrix}
\]

\% COMPUTE GRATING PHASE
PHI = svlsolve(Kx,Ky,dx,dy);

A More Efficient Grid Strategy

Observe how smooth the grating phase function is compared to the final grating.

We can get away with a much coarser grid up to the point where the planar grating is calculated.
### Calculate Analog Grating

The analog grating is calculated directly from the grating phase.

\[ E_a(\bar{r}) = \cos[\Phi(\bar{r})] \]

\[ \text{ERA} = \cos(\text{PHI}); \]

### Calculate Binary Grating

The binary grating is calculated directly from the analog grating using the threshold technique.

\[ \begin{align*}
E_b(\bar{r}) &= \begin{cases}
E_1, & E_a(\bar{r}) \leq \gamma(\bar{r}) \\
E_2, & E_a(\bar{r}) > \gamma(\bar{r})
\end{cases} \\
\text{ERB} &= \text{er}^1(\text{ERA} \leq \text{gth}) + \text{er}^2(\text{ERA} > \text{gth});
\end{align*} \]
Controlling Fill Fraction Through the Threshold Function

Analog Grating $\epsilon_x(\vec{r})$

$$\epsilon_x(\vec{r}) = \begin{cases} 
\epsilon_1 & \epsilon_x(\vec{r}) \leq \gamma(\vec{r}) \\
\epsilon_2 & \epsilon_x(\vec{r}) > \gamma(\vec{r}) 
\end{cases}$$

We can estimate the threshold value in order to realize a given fill fraction $f$ of $\epsilon_1$.

$$\gamma(\vec{r}) \approx \cos[\pi f(\vec{r})]$$

- $f = 0$; $\gamma = 1.0$
- $f = 0.2$; $\gamma = 0.8$
- $f = 0.4$; $\gamma = 0.3$
- $f = 0.6$; $\gamma = -0.3$
- $f = 0.8$; $\gamma = -0.8$
- $f = 1.0$; $\gamma = -1.0$