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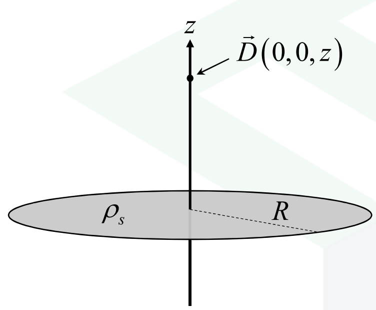
Electromagnetics:

Electromagnetic Field Theory

Example: Uniform Circular Plate Charge

1

Total Charge



What is the total charge Q_{Total}?

1. Draw the problem.
2. Choose a coordinate system.
   Cylindrical
3. Write general equation.
   $Q_{\text{Total}} = \iint_s \rho_s ds$
4. Write expressions for each term.
   $\rho_s = \rho_s \qquad ds = \rho d\rho d\phi$
5. Choose limits of integration.
   $Q_{\text{Total}} = \int_{\phi=0}^{2\pi} \int_{\rho=0}^{R} \rho_s \rho d\rho d\phi$

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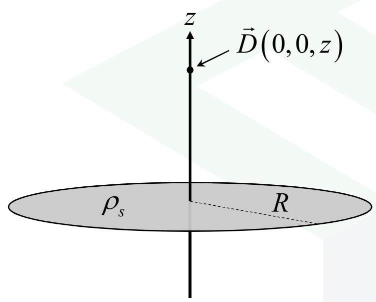
Total Charge



What is the total charge Q_{Total}?

6. Solve the integral.

$$Q_{\text{Total}} = \int_{\phi=0}^{2\pi}\int_{\rho=0}^{R} \rho_s \rho d\rho d\phi$$

$$= \rho_s \int_{\phi=0}^{2\pi}\left(\int_{\rho=0}^{R} \rho d\rho\right) d\phi$$

$$= \rho_s \int_{\phi=0}^{2\pi}\left(\left.\frac{\rho^2}{2}\right|_{\rho=0}^{R}\right) d\phi$$

$$= \frac{\rho_s R^2}{2}\int_{\phi=0}^{2\pi} d\phi$$

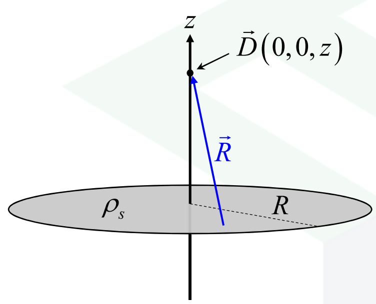
$$= \frac{\rho_s R^2}{2}\cdot 2\pi$$

$$\boxed{Q_{\text{Total}} = \pi R^2 \rho_s}$$

7. Interpret the result.

$Q_{\text{Total}} = \rho_s S$ *for uniform charge density*

Total Field



What is the total field $\vec{D}$

1. Draw the problem.
2. Choose a coordinate system.

   Cylindrical

3. Write general equation.

$$\vec{D}_{\text{Total}} = \iint_s \frac{\rho_s ds}{4\pi R^2}\hat{a}_R = \frac{\rho_s}{4\pi}\iint_s \frac{\vec{R}}{|\vec{R}|^3} ds$$

4. Write expressions for each term.

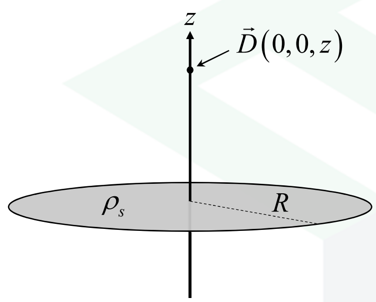
$\rho_s = \rho_s$ $\quad \vec{R} = (0,\phi,z) - (\rho,\phi,0) = (-\rho,0,z)$

$ds = \rho d\rho d\phi$ $\quad |\vec{R}| = \sqrt{\rho^2 + z^2}$

5. Choose limits of integration.

$$\vec{D}_{\text{Total}} = \frac{\rho_s}{4\pi}\int_{\phi=0}^{2\pi}\int_{\rho=0}^{R} \frac{-\rho\hat{a}_\rho + z\hat{a}_z}{(\rho^2+z^2)^{3/2}} \rho d\rho d\phi$$

Total Field



What is the total field $\vec{D}$

6. Solve the integral.

$$\vec{D}_{\text{Total}} = \frac{\rho_s}{4\pi}\int_{\phi=0}^{2\pi}\int_{\rho=0}^{R}\frac{-\rho\hat{a}_\rho + z\hat{a}_z}{\left(\rho^2+z^2\right)^{3/2}}\rho d\rho d\phi$$

Due to symmetry, we can ignore the ρ component.

$$\vec{D}_{\text{Total}} = \frac{\rho_s}{4\pi}\int_{\phi=0}^{2\pi}\int_{\rho=0}^{R}\frac{\cancel{-\rho\hat{a}_\rho} + z\hat{a}_z}{\left(\rho^2+z^2\right)^{3/2}}\rho d\rho d\phi$$

$$= \frac{\rho_s z}{4\pi}\hat{a}_z\int_{\phi=0}^{2\pi}\int_{\rho=0}^{R}\left(\rho^2+z^2\right)^{-3/2}\rho d\rho d\phi$$

$$= \frac{\rho_s z}{4\pi}\hat{a}_z\int_{\rho=0}^{R}\left(\int_{\phi=0}^{2\pi}d\phi\right)\left(\rho^2+z^2\right)^{-3/2}\rho d\rho$$

$$= \frac{\rho_s z}{4\pi}\hat{a}_z\int_{\rho=0}^{R}(2\pi)\left(\rho^2+z^2\right)^{-3/2}\rho d\rho$$

$$= \frac{\rho_s z}{2}\hat{a}_z\int_{\rho=0}^{R}\left(\rho^2+z^2\right)^{-3/2}\rho d\rho$$

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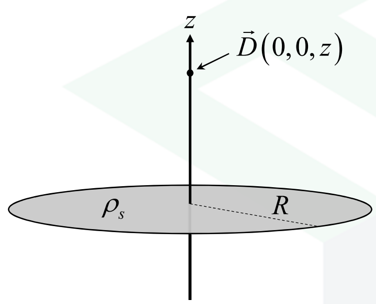
Total Field



What is the total field $\vec{D}$

6. Solve the integral.

$$\vec{D}_{\text{Total}} = \frac{\rho_s z}{2}\hat{a}_z\int_{\rho=0}^{R}\left(\rho^2+z^2\right)^{-3/2}\rho d\rho$$

Let $v = \rho^2 + z^2$

Then $dv = 2\rho d\rho + \cancel{z^2}$

$$d\rho = \frac{dv}{2\rho}$$

$$\rho_1 = 0 \quad\rightarrow\quad v_1 = 0^2 + z^2 = z^2$$

$$\rho_2 = R \quad\rightarrow\quad v_2 = R^2 + z^2$$

Our integral in terms of v is then

$$= \frac{\rho_s z}{2}\hat{a}_z\int_{v=z^2}^{R^2+z^2}v^{-3/2}\rho\frac{dv}{2\rho}$$

$$= \frac{\rho_s z}{4}\hat{a}_z\int_{v=z^2}^{R^2+z^2}v^{-3/2}dv$$

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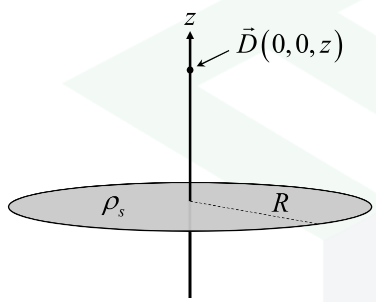
Total Field



What is the total field $\vec{D}$

6. Solve the integral.

$$\vec{D}_{\text{Total}} = \frac{\rho_s z}{4}\hat{a}_z \int_{v=z^2}^{R^2+z^2} v^{-3/2}\,dv$$

$$= \frac{\rho_s z}{4}\hat{a}_z \left.\frac{v^{-1/2}}{-1/2}\right|_{z^2}^{R^2+z^2}$$

$$= -\frac{\rho_s z}{2}\hat{a}_z \left.\frac{1}{\sqrt{v}}\right|_{z^2}^{R^2+z^2}$$

$$= -\frac{\rho_s z}{2}\hat{a}_z \left(\frac{1}{\sqrt{R^2+z^2}} - \frac{1}{\sqrt{z^2}}\right)$$

$$= \frac{\rho_s}{2}\left(\frac{z}{\sqrt{z^2}} - \frac{z}{\sqrt{R^2+z^2}}\right)\hat{a}_z$$

$$\boxed{\vec{D}_{\text{Total}} = \frac{\rho_s}{2}\left(1 - \frac{z}{\sqrt{R^2+z^2}}\right)\hat{a}_z}$$

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