Electromagnetics:
Electromagnetic Field Theory

Capacitor Examples

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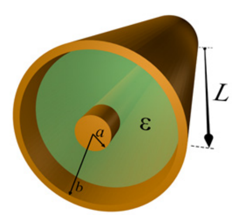
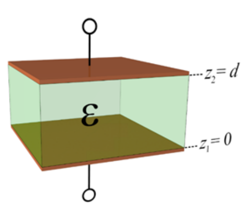
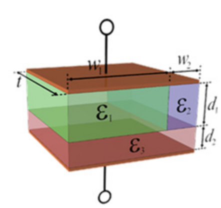
Outline

- Parallel plate capacitor
- How big is a Farad?
- Coaxial capacitor
- RG-59 coax
- Inhomogeneous capacitor







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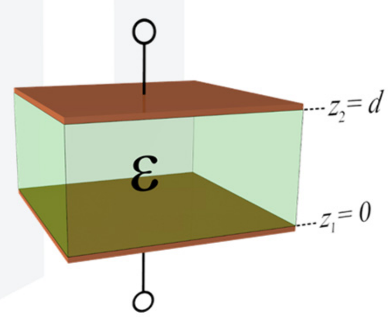
Example #1:
Parallel Plate Capacitor

Example 1

Step 1 – Choose a convenient coordinate system.

Cartesian

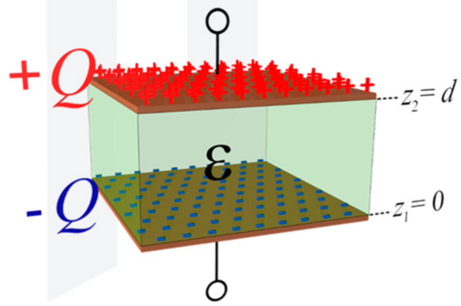


Example 1

Step 1 – Choose a convenient coordinate system.

Cartesian

Step 2 – Let the plates carry charges $+Q$ and $-Q$.



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Example 1

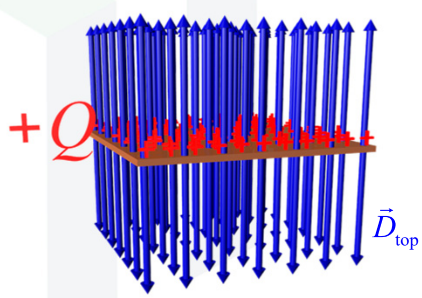
Step 3 – Calculate $\vec{D}$ using Gauss' law.

Recall the field around an infinite plate.

$$\vec{D} = \frac{\rho_S}{2}\hat{a}_n$$

Field below the top plate,

$$\vec{D}_{\text{top}} = -\frac{\rho_S}{2}\hat{a}_z$$



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Example 1

Step 3 – Calculate $\vec{D}$ using Gauss' law.

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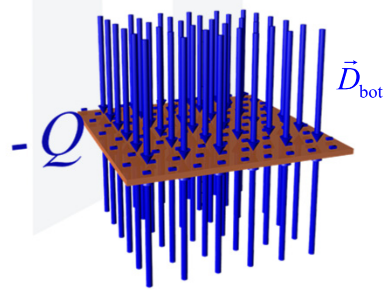
$$\vec{D} = \frac{\rho_S}{2}\hat{a}_n$$

Field below the top plate,

$$\vec{D}_{\text{top}} = -\frac{\rho_S}{2}\hat{a}_n$$

Field above the bottom plate,

$$\vec{D}_{\text{bot}} = -\frac{\rho_S}{2}\hat{a}_z$$



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Example 1

Step 3 – Calculate $\vec{D}$ using Gauss' law.

Recall the field around an infinite plate.

$$\vec{D} = \frac{\rho_S}{2}\hat{a}_n$$

Field below the top plate,

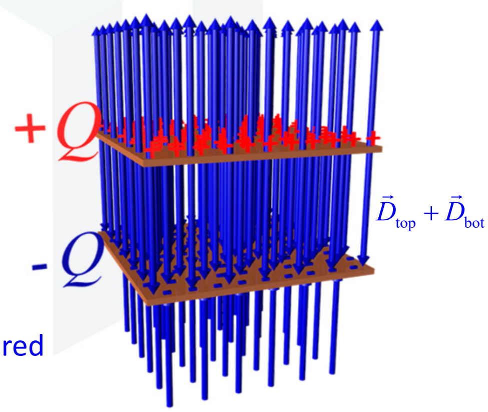
$$\vec{D}_{\text{top}} = -\frac{\rho_S}{2}\hat{a}_n$$

Field above the bottom plate,

$$\vec{D}_{\text{bot}} = -\frac{\rho_S}{2}\hat{a}_n$$

When both plates are considered

$$\vec{D} = \vec{D}_{\text{top}} + \vec{D}_{\text{bot}} = -\rho_S\hat{a}_z$$



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Example 1

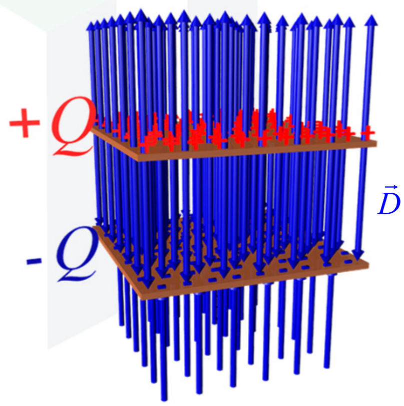
Step 3 – Calculate $\vec{D}$ using Gauss' law.

The surface charge density is

$$\rho_S = \frac{Q}{S}$$

The final express for $\vec{D}$ is

$$\vec{D} = -\rho_S \hat{a}_z = -\frac{Q}{S}\hat{a}_z$$



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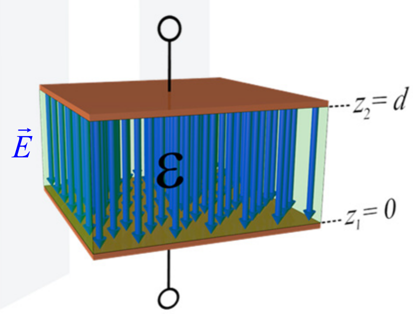
Example 1

Step 4 – Calculate $\vec{E}$.

Calculate $\vec{E}$ from the constitutive relation.

$$\vec{E} = \frac{\vec{D}}{\varepsilon} = -\frac{Q}{\varepsilon S}\hat{a}_z$$



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Example 1

Step 5 – Calculate V_0.

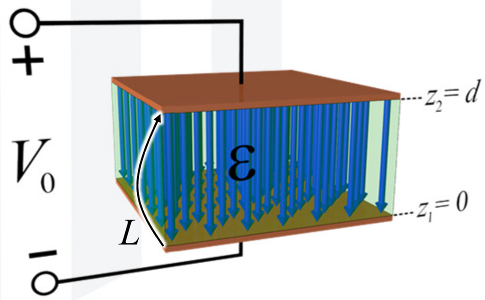
Given $\vec{E}$, calculate V_0 by integrating from the bottom plate to the top plate.

$$V_0 = -\int_L \vec{E} \bullet d\vec{\ell}$$

$$V_0 = -\int_0^d \left(-\frac{Q}{\varepsilon S} \hat{a}_z \right) \bullet dz\hat{a}_z$$

$$V_0 = \frac{Q}{\varepsilon S} \int_0^d dz$$

$$V_0 = \frac{Qd}{\varepsilon S}$$



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Example 1

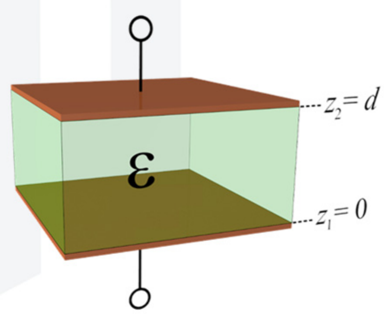
Step 6 – Calculate capacitance C.

$$C = \frac{|Q|}{|V_0|} = \frac{|Q|}{\left|\frac{Qd}{\varepsilon S}\right|} = \frac{Q}{\frac{Qd}{\varepsilon S}} = \frac{\varepsilon S}{d}$$

The final answer is

$$\boxed{C = \frac{\varepsilon S}{d}}$$

Self-check – C should not be a function of Q or V_0.



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Example #2:

How Big is a Farad?

Example #2

Suppose the plates of a capacitor are 10 m by 20 m and the gap between the plates is 1 mm.

$$C = \frac{\varepsilon S}{d} = \frac{\varepsilon_0 \varepsilon_r LW}{d} = \frac{\left(8.854\times10^{-12}\ \text{F/m}\right)(1.0)(10\ \text{m})(20\ \text{m})}{(0.001\ \text{m})}$$

$$= 1.78\times10^{-6}\ \text{F}$$

$$= 1.78\ \mu\text{F}$$

The capacitor is physically very large, yet the capacitance is very small.

The Farad is a HUGE unit!!!

Example #3:

Coaxial Capacitor

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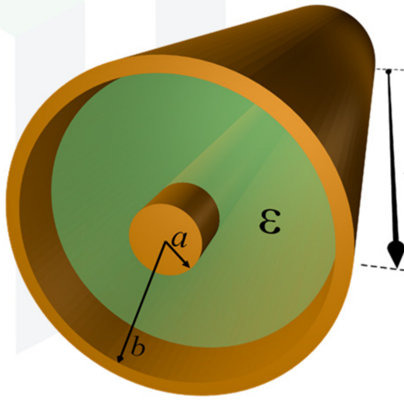
Example #3 – Coaxial Capacitor

Step 1 – Choose a convenient coordinate system.

Cylindrical (ρ,ϕ,z)



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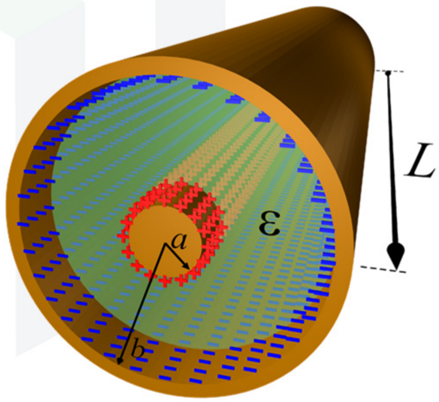
Example #3 – Coaxial Capacitor

Step 1 – Choose a convenient coordinate system.

Cylindrical (ρ,ϕ,z)

Step 2 – Let the plates carry charges $+Q$ and $-Q$.



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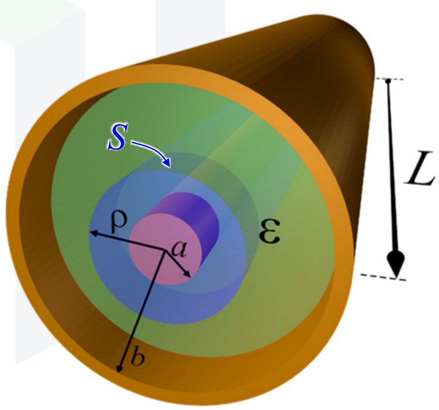
Example #3 – Coaxial Capacitor

Step 3 – Calculate $\vec{D}$ using Gauss' law.

Define a Gaussian surface with radius ρ to be inside of the dielectric.

$$Q = \oiint_S \vec{D} \bullet d\vec{s}$$

From the boundary conditions, it is known that the electric field will be normal at the interfaces to the metal.



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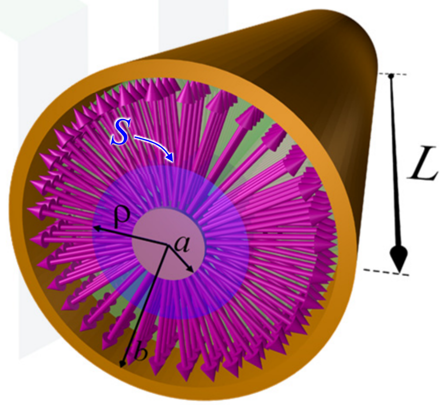
Example #3 – Coaxial Capacitor

Step 3 – Calculate $\vec{D}$ using Gauss' law.

The only field configuration that makes sense considering the boundary conditions is when the field is purely radially directed.

$$\vec{D} = D_\rho\left(\rho,\phi,z\right)\hat{a}_\rho$$



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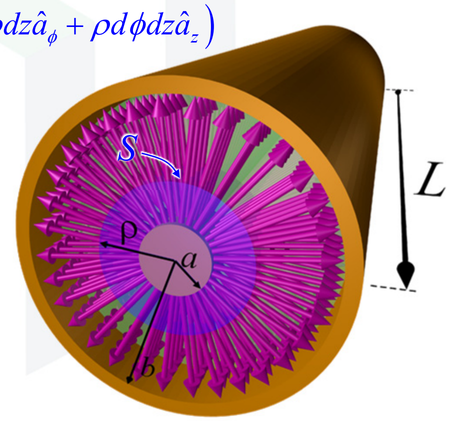
Example #3 – Coaxial Capacitor

Step 3 – Calculate $\vec{D}$ using Gauss' law.

Gauss' law becomes

$$\begin{aligned}
Q &= \int_0^L \int_0^{2\pi} \left(D_\rho \hat{a}_\rho\right) \bullet \left(\rho d\phi dz \hat{a}_\rho + d\rho dz \hat{a}_\phi + \rho d\phi dz \hat{a}_z\right) \\
&= \int_0^L \int_0^{2\pi} D_\rho \rho d\phi dz \\
&= D_\rho \rho \int_0^L \left(\int_0^{2\pi} d\phi\right) dz \\
&= D_\rho \rho \int_0^L \left(2\pi\right) dz \\
&= 2\pi D_\rho \rho \left(\int_0^L dz\right) = 2\pi D_\rho \rho L
\end{aligned}$$



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Example #3 – Coaxial Capacitor

Step 3 – Calculate $\vec{D}$ using Gauss' law.

Solving for $\vec{D}$ gives

$$D_\rho(\rho,\phi,z) = \frac{Q}{2\pi\rho L}$$

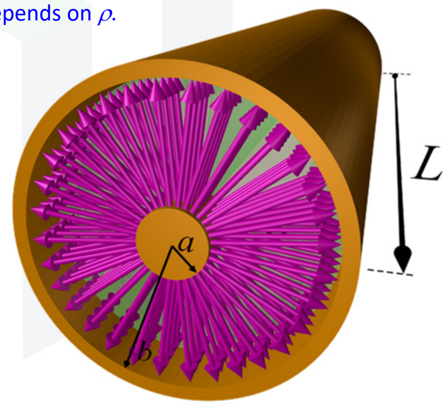
$\vec{D}$ only depends on ρ.

$$\vec{D}(\rho) = \frac{Q}{2\pi\rho L}\hat{a}_\rho$$

Step 4 – Calculate $\vec{E}$.

Calculate $\vec{E}$ from the constitutive relation.

$$\vec{E}(\rho) = \frac{\vec{D}}{\varepsilon} = \frac{Q}{2\pi\varepsilon\rho L}\hat{a}_\rho$$



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Example #3 – Coaxial Capacitor

Step 5 – Calculate V_0.

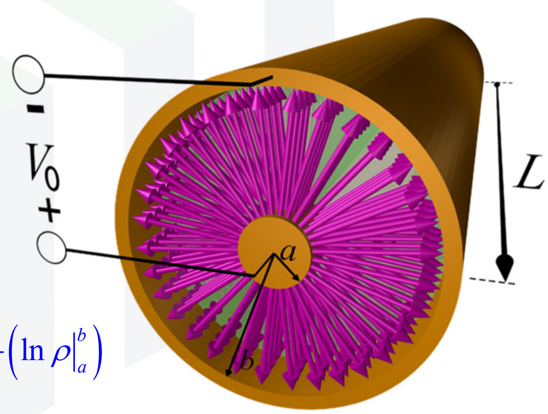
Given $\vec{E}$, calculate V_0 by integrating from the inner conductor to the outer conductor.

$$V_0 = -\int_L \vec{E} \bullet d\vec{\ell}$$

$$= -\int_a^b \left(\frac{Q}{2\pi\varepsilon\rho L}\hat{a}_\rho\right) \bullet \left(d\rho\hat{a}_\rho\right)$$

$$= -\int_a^b \frac{Q}{2\pi\varepsilon\rho L} d\rho$$

$$= -\frac{Q}{2\pi\varepsilon L}\int_a^b \frac{1}{\rho} d\rho = -\frac{Q}{2\pi\varepsilon L}\left(\ln \rho\big|_a^b\right)$$



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Example #3 – Coaxial Capacitor

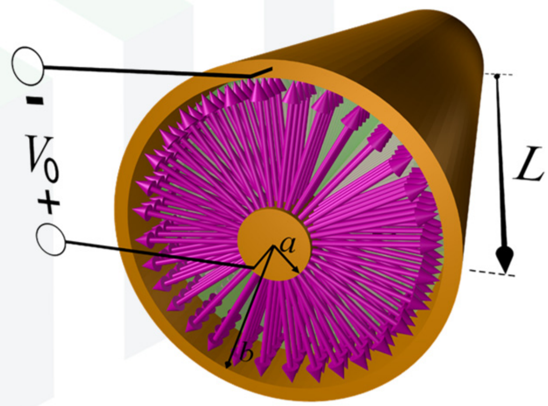
Step 5 – Calculate V_0.

Continued…

$$V_0 = -\frac{Q}{2\pi\varepsilon L}\left(\ln \rho\Big|_a^b\right)$$

$$= -\frac{Q}{2\pi\varepsilon L}\left(\ln b - \ln a\right)$$

$$= -\frac{Q}{2\pi\varepsilon L}\ln\left(\frac{b}{a}\right)$$



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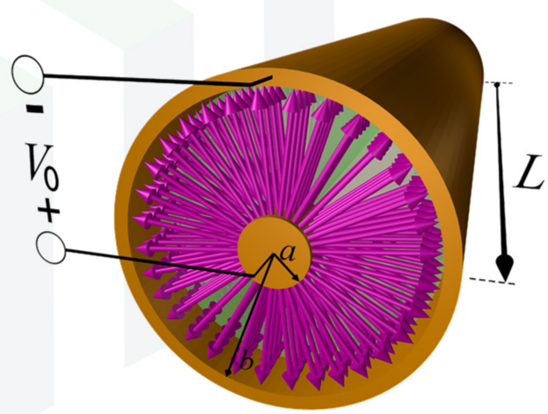
Example #3 – Coaxial Capacitor

Step 6 – Calculate capacitance C.

$$C = \frac{|Q|}{|V_0|}$$

$$= \frac{Q}{\frac{Q}{2\pi\varepsilon L}\ln\left(\frac{b}{a}\right)}$$

$$\boxed{C = \frac{2\pi\varepsilon L}{\ln\left(\frac{b}{a}\right)}}$$



* Self Check – C is not a function of Q or V_0.

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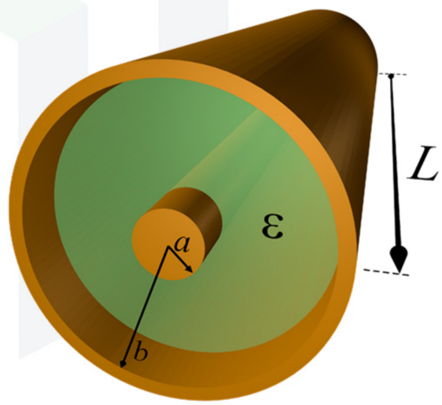
Example #3 – Coaxial Capacitor

Distributed Capacitance

Sometimes it is desired to specify the capacitance without knowledge of L.

This is done using the distributed capacitance, which is defined as capacitance per unit length.

$$\frac{C}{L} = \frac{2\pi\varepsilon}{\ln\left(\frac{b}{a}\right)}$$



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Example #4:

RG-59 Coax

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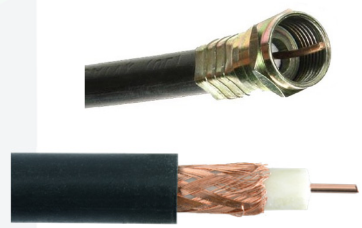
Example #4 – RG-59 Coax

A standard RG-59 coax has

Inner conductor diameter:	0.81 mm (20 AWG)
Outer conductor diameter:	3.66 mm
Dielectric constant:	2.1
Specified capacitance:	86.9 pF/m



$$\frac{C}{L} = \frac{2\pi\varepsilon_0\varepsilon_r}{\ln(b/a)}$$

$$= \frac{2\pi(8.854\times10^{-12}\ \text{F/m})(2.1)}{\ln(3.66\ \text{mm}/0.81\ \text{mm})}$$

$$= 7.746\times10^{-11}\ \text{F/m} = \boxed{77.46\ \text{pF/m}}$$

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Example #5:

Inhomogeneous Capacitor

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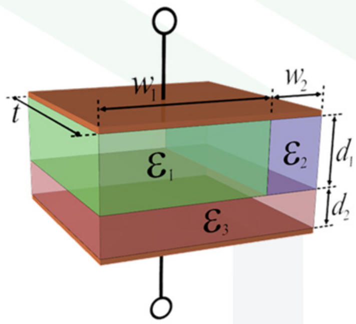
Example #5 – Inhomogeneous Capacitor

Suppose there exists an inhomogeneous capacitor.



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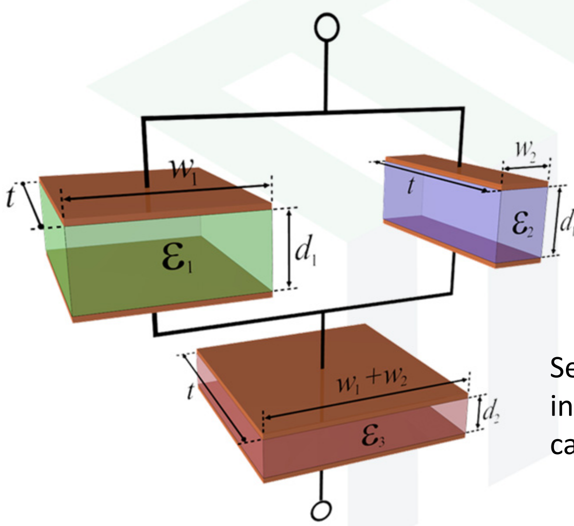
Example #5 – Inhomogeneous Capacitor



Separate the inhomogeneous capacitor into a network of homogeneous capacitors.

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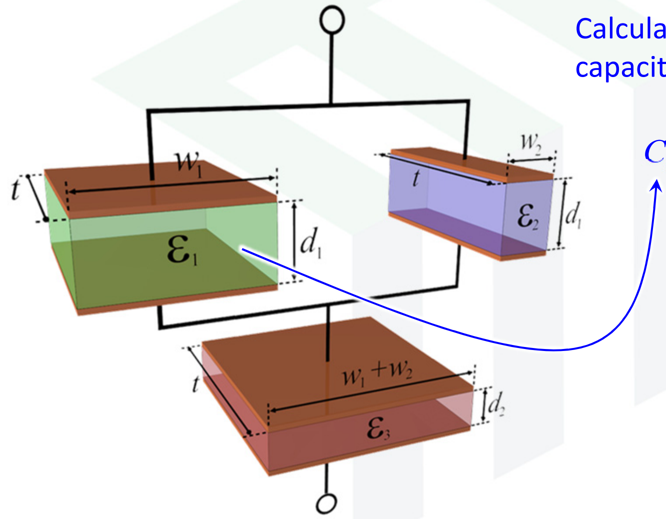
Example #5 – Inhomogeneous Capacitor



Calculate each homogeneous capacitor independently.

$$C_1 = \frac{\varepsilon_1 S_1}{d_1} = \frac{\varepsilon_1 t w_1}{d_1}$$

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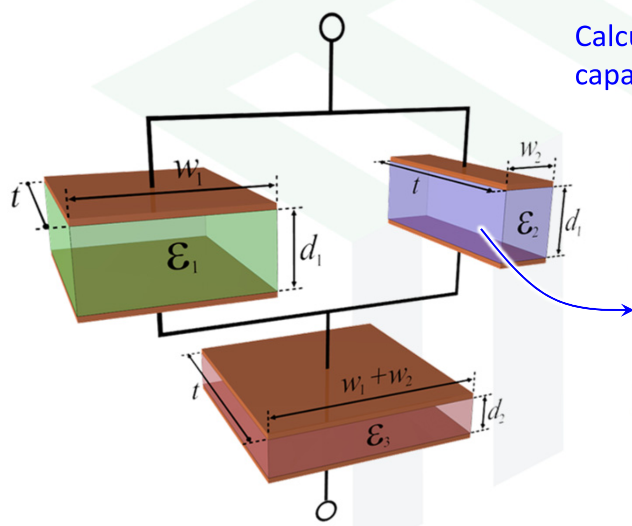
Example #5 – Inhomogeneous Capacitor



Calculate each homogeneous capacitor independently.

$$C_1 = \frac{\varepsilon_1 S_1}{d_1} = \frac{\varepsilon_1 t w_1}{d_1}$$

$$C_2 = \frac{\varepsilon_2 S_2}{d_1} = \frac{\varepsilon_2 t w_2}{d_1}$$

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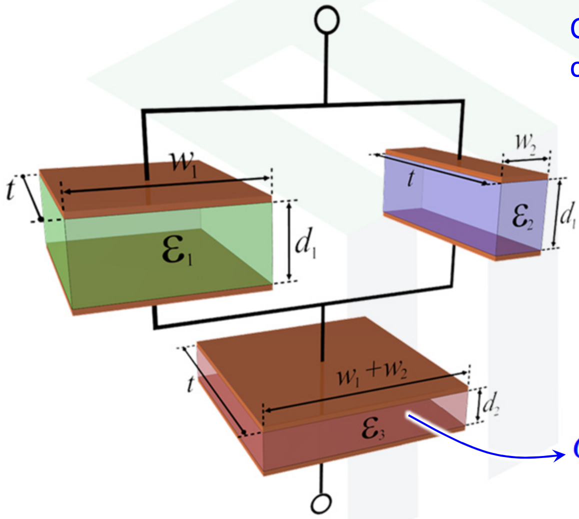
Example #5 – Inhomogeneous Capacitor



Calculate each homogeneous capacitor independently.

$$C_1 = \frac{\varepsilon_1 S_1}{d_1} = \frac{\varepsilon_1 t w_1}{d_1}$$

$$C_2 = \frac{\varepsilon_2 S_2}{d_1} = \frac{\varepsilon_2 t w_2}{d_1}$$

$$C_3 = \frac{\varepsilon_3 S_3}{d_2} = \frac{\varepsilon_3 t \left(w_1 + w_2\right)}{d_2}$$

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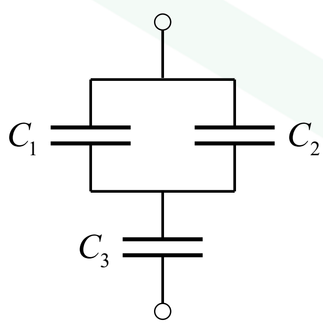
Example #5 – Inhomogeneous Capacitor



View the capacitor as a series/parallel network of capacitors.

The equivalent capacitance is

$$C_{\text{eq}} = \left(C_1 + C_2\right) \| C_3 = \left(\frac{\varepsilon_1 t w_1}{d_1} + \frac{\varepsilon_2 t w_2}{d_1}\right) \Bigg\| \frac{\varepsilon_3 t \left(w_1 + w_2\right)}{d_2}$$

$$\boxed{C_{\text{eq}} = \frac{t \varepsilon_3 \left(\varepsilon_1 w_1 + \varepsilon_2 w_2\right)\left(w_1 + w_2\right)}{d_2 \left(\varepsilon_1 w_1 + \varepsilon_2 w_2\right) + \varepsilon_3 d_1 \left(w_1 + w_2\right)}}$$

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