Problem #1: Wave Equation

Starting with Maxwell's curl equations

$$\nabla \times \vec{E} = -j\omega\mu\vec{H} \,, \tag{1}$$

$$\nabla \times \vec{H} = j\omega\varepsilon\vec{E} \,, \tag{2}$$

derive the following wave equation for the electric field in a linear, homogeneous, and isotropic (LHI) medium.

$$\nabla^2 \vec{E} + \left(k_0 n\right)^2 \vec{E} = 0 \tag{3}$$

In this equation, n is refractive index and k_0 is the free space wave number. These are defined as

$$n = \sqrt{\mu_{\mathrm{r}}\varepsilon_{\mathrm{r}}} \tag{4}$$

$$k_0 = 2\pi/\lambda_0 \tag{5}$$

Problem #2: Dispersion Relation

Given that the solution to the wave equation is a plane wave of the form

$$\vec{E}\left(\vec{r}\right) = \vec{E}_0 e^{-j\vec{k}\cdot\vec{r}} \,, \tag{6}$$

substitute this solution back into the wave equation to derive the dispersion relation for LHI media given in Eq. (7).

$$k_x^2 + k_y^2 + k_z^2 = \left(k_0 n\right)^2 \tag{7}$$

Problem #3: Inverse of a Block 2×2 Matrix

Using proper matrix algebra rules, derive the following expression for the inverse of a 2×2 block matrix. Remember that **A**, **B**, **C**, and **D** are square matrices themselves.

$$\begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix}^{-1} = \begin{bmatrix} \mathbf{D}\mathbf{X}^{-1} & -\mathbf{B}\mathbf{X}^{-1} \\ -\mathbf{C}\mathbf{X}^{-1} & \mathbf{A}\mathbf{X}^{-1} \end{bmatrix} \qquad \mathbf{X} = \mathbf{AD} - \mathbf{BC} \tag{8}$$

Problem #4: Block Matrix Division

Using proper matrix algebra rules, simplify the following block matrix expression where $\mathbf{W}_i$, $\mathbf{W}_j$, $\mathbf{V}_i$ and $\mathbf{V}_j$ are themselves square matrices.

$$\begin{bmatrix} \mathbf{W}_i & \mathbf{W}_i \\ \mathbf{V}_i & -\mathbf{V}_i \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{W}_j & \mathbf{W}_j \\ \mathbf{V}_j & -\mathbf{V}_j \end{bmatrix}, \tag{9}$$

Your final answer should be

$$\frac{1}{2}\begin{bmatrix} \mathbf{A}_{ij} & \mathbf{B}_{ij} \\ \mathbf{B}_{ij} & \mathbf{A}_{ij} \end{bmatrix} \qquad \text{where} \qquad \begin{aligned} \mathbf{A}_{ij} &= \mathbf{W}_i^{-1}\mathbf{W}_j + \mathbf{V}_i^{-1}\mathbf{V}_j \\ \mathbf{B}_{ij} &= \mathbf{W}_i^{-1}\mathbf{W}_j - \mathbf{V}_i^{-1}\mathbf{V}_j \end{aligned} \,, \tag{10}$$