Advanced Computation: Computational Electromagnetics

Maxwell’s Equations

Outline

• Maxwell’s equations
• Physical Boundary conditions
•Preparing Maxwell’s equations for CEM
• Scaling properties of Maxwell’s equations
Maxwell’s Equations

James Clerk Maxwell

Born  June 13, 1831
Edinburgh, Scotland

Died  November 5, 1879
Cambridge, England

Sign Conventions for Waves

To describe a wave propagating the positive $z$ direction, we have two choices:

$E(z,t) = A \cos(\omega t - kz)$  Most common in engineering

$E(z,t) = A \cos(-\omega t + kz)$  Most common science and physics

Both are correct, but we must choose a convention and be consistent with it. For time-harmonic signals, this becomes

$E(z) = A \exp(-jkz)$  Negative sign convention

$E(z) = A \exp(+jkz)$  Positive sign convention
**Time-Harmonic Maxwell’s Equations**

**Time-Domain**
\[
\nabla \cdot \vec{D} = \rho_v \\
\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \\
\nabla \cdot \vec{B} = 0 \\
\n\nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}
\]

**Frequency-Domain**
\[
\text{(}e^{jkz}\text{ convention)} \\
\nabla \cdot \vec{D} = \rho_v \\
\nabla \times \vec{E} = j\omega \vec{B} \\
\nabla \cdot \vec{B} = 0 \\
\n\nabla \times \vec{H} = \vec{J} - j\omega \vec{D}
\]

**Frequency-Domain**
\[
\text{(}e^{-jkz}\text{ convention)} \\
\nabla \cdot \vec{D} = \rho_v \\
\nabla \times \vec{E} = -j\omega \vec{B} \\
\nabla \cdot \vec{B} = 0 \\
\n\nabla \times \vec{H} = \vec{J} + j\omega \vec{D}
\]
Lorentz Force Law

One additional equation is needed to completely describe classical electromagnetism...

\[ \vec{F} = q\vec{E} + q\vec{v} \times \vec{B} \]

Electric Force  Magnetic Force

Gauss’s Law

\[ \nabla \cdot \vec{D} = \rho \]

Electric fields diverge from positive charges and converge on negative charges.

If there are no charges, electric fields must form loops.
**Gauss’s Law for Magnetism**

\[ \nabla \cdot \vec{B} = 0 \]

Magnetic fields always form loops.

\[ \nabla \cdot \vec{B} = \frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z} \]

**Consequence of Zero Divergence**

The divergence theorems force the \( \vec{D} \) and \( \vec{B} \) fields to be perpendicular to the propagation direction \( \vec{k} \) of a plane wave.

\[ \nabla \cdot \vec{D} = 0 \]

\[ \nabla \cdot (\vec{d}e^{-jkr}) = 0 \]

\[ \vec{k} \perp \vec{D} \]

\[ \vec{k} \cdot \vec{d} = 0 \]

\[ jk \cdot \vec{d} = 0 \]

\[ \nabla \cdot \vec{B} = 0 \]

\[ \nabla \cdot (\vec{b}e^{-jkr}) = 0 \]

\[ \vec{k} \perp \vec{B} \]

\[ \vec{k} \cdot \vec{b} = 0 \]

\[ jk \cdot \vec{b} = 0 \]
Ampere’s Law with Maxwell’sCorrection

\[ \nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t} \]

Circulating magnetic fields induce currents and/or time varying electric fields. 
Currents and/or time varying electric fields induce circulating magnetic fields.

Faraday’s Law of Induction

\[ \nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \]

Circulating electric fields induce time varying magnetic fields. 
Time varying magnetic fields induce circulating electric fields.
The curl equations predict electromagnetic waves!!

**Electric Field**
- Electric field intensity (V/m)
- Initial electric “push.”
- Induced electric field.
- Electric energy in vacuum.
- Permittivity (F/m)
  - Measure of how well a material stores electric energy.
- Electric flux density (C/m²)
  - Pretends as if all electric energy is displaced charge.
  - Includes electric energy in vacuum and matter.

**Magnetic Field**
- Magnetic field intensity (A/m)
- Initial magnetic “push.”
- Induced magnetic field.
- Magnetic energy in vacuum.
- Permeability (H/m)
  - Measure of how well a material stores magnetic energy.
- Magnetic flux density (Wb/m²)
  - Pretends as if all magnetic energy is tilted magnetic dipoles.
  - Includes magnetic energy in vacuum and matter.
Material Classifications

Linear, isotropic and non-dispersive materials:
\[ \tilde{D}(t) = \varepsilon(t) \tilde{E}(t) \]

Dispersive materials:
\[ \tilde{D}(t) = \varepsilon(t) * \tilde{E}(t) \]

Anisotropic materials:
\[ \tilde{D}(t) = [\varepsilon] \tilde{E}(t) \]

Nonlinear materials:
\[ D(t) = \varepsilon_0 \chi_e^{(1)} E(t) + \varepsilon_0 \chi_e^{(2)} E^2(t) + \varepsilon_0 \chi_e^{(3)} E^3(t) + \cdots \]

A key point is that you can wrap all of the complexities associated with modeling strange materials into this single equation. This will make your code more modular and easier to modify. It may not be as efficient as it could be though.

All Together Now...

Divergence Equations
\[ \nabla \cdot \tilde{B} = 0 \]
\[ \nabla \cdot \tilde{D} = \rho_v \]

Curl Equations
\[ \nabla \times \tilde{H} = \tilde{J} + \frac{\partial \tilde{D}}{\partial t} \]
\[ \nabla \times \tilde{E} = - \frac{\partial \tilde{B}}{\partial t} \]

Constitutive Relations
\[ \tilde{D}(t) = [\varepsilon(t)] \ast \tilde{E}(t) \quad * \text{ means convolution} \]
\[ \tilde{B}(t) = [\mu(t)] \ast \tilde{H}(t) \quad [ ] \text{ means tensor} \]

What produces fields

How fields interact with materials
Maxwell’s Equations in Cartesian Coordinates (1 of 4)

Vector Terms
\[ \vec{E} = E_x \hat{x} + E_y \hat{y} + E_z \hat{z} \]
\[ \vec{H} = H_x \hat{x} + H_y \hat{y} + H_z \hat{z} \]
\[ \vec{D} = D_x \hat{x} + D_y \hat{y} + D_z \hat{z} \]
\[ \vec{B} = B_x \hat{x} + B_y \hat{y} + B_z \hat{z} \]

\[ \vec{J} = J_x \hat{x} + J_y \hat{y} + J_z \hat{z} \]

Divergence Equations
\[ \nabla \cdot \vec{D} = 0 \]
\[ \nabla \cdot \vec{B} = 0 \]

\[ \frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z} = 0 \]
\[ \frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z} = 0 \]

Maxwell’s Equations in Cartesian Coordinates (2 of 4)

Constitutive Relations
\[ \vec{D} = \varepsilon [ \vec{E} ] \]
\[ D_x \hat{x} + D_y \hat{y} + D_z \hat{z} = (\varepsilon_x E_x + \varepsilon_y E_y + \varepsilon_z E_z) \hat{x} + (\varepsilon_x E_x + \varepsilon_y E_y + \varepsilon_z E_z) \hat{y} + (\varepsilon_x E_x + \varepsilon_y E_y + \varepsilon_z E_z) \hat{z} \]
\[ \vec{B} = \mu [ \vec{H} ] \]
\[ B_x \hat{x} + B_y \hat{y} + B_z \hat{z} = (\mu_x H_x + \mu_y H_y + \mu_z H_z) \hat{x} + (\mu_x H_x + \mu_y H_y + \mu_z H_z) \hat{y} + (\mu_x H_x + \mu_y H_y + \mu_z H_z) \hat{z} \]
Maxwell’s Equations in Cartesian Coordinates (3 of 4)

Curl Equations
\[ \nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \]
\[
\left( \frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} \right) \hat{a}_x + \left( \frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} \right) \hat{a}_y + \left( \frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} \right) \hat{a}_z = -\frac{\partial}{\partial t} \left( B_y \hat{a}_x + B_z \hat{a}_y + B_x \hat{a}_z \right)
\]
\[
\left( \frac{\partial E_x}{\partial y} - \frac{\partial E_y}{\partial x} \right) \hat{a}_z + \left( \frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} \right) \hat{a}_x + \left( \frac{\partial E_z}{\partial x} - \frac{\partial E_x}{\partial z} \right) \hat{a}_y = -\frac{\partial B_x}{\partial t} \hat{a}_x - \frac{\partial B_y}{\partial t} \hat{a}_y - \frac{\partial B_z}{\partial t} \hat{a}_z
\]

Maxwell’s Equations in Cartesian Coordinates (4 of 4)

Curl Equations
\[ \nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t} \]
\[
\left( \frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} \right) \hat{a}_x + \left( \frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} \right) \hat{a}_y + \left( \frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} \right) \hat{a}_z = \left( J_x \hat{a}_x + J_y \hat{a}_y + J_z \hat{a}_z \right) + \frac{\partial}{\partial t} \left( D_x \hat{a}_x + D_y \hat{a}_y + D_z \hat{a}_z \right)
\]
\[
\left( \frac{\partial H_x}{\partial y} - \frac{\partial H_y}{\partial x} \right) \hat{a}_z + \left( \frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} \right) \hat{a}_x + \left( \frac{\partial H_z}{\partial x} - \frac{\partial H_x}{\partial z} \right) \hat{a}_y = \left( J_x + \frac{\partial D_x}{\partial t} \right) \hat{a}_x + \left( J_y + \frac{\partial D_y}{\partial t} \right) \hat{a}_y + \left( J_z + \frac{\partial D_z}{\partial t} \right) \hat{a}_z
\]
\[
\frac{\partial H_x}{\partial y} - \frac{\partial H_y}{\partial x} = J_x + \frac{\partial D_x}{\partial t}
\]
\[
\frac{\partial H_y}{\partial z} - \frac{\partial H_z}{\partial y} = J_y + \frac{\partial D_y}{\partial t}
\]
\[
\frac{\partial H_z}{\partial x} - \frac{\partial H_x}{\partial z} = J_z + \frac{\partial D_z}{\partial t}
\]
Alternative Form of Maxwell’s Equations in Cartesian Coordinates (1 of 2)

Alternate Curl Equations

\[ \nabla \times \vec{H} = \epsilon \frac{\partial \vec{E}}{\partial t} \]

\[
\begin{align*}
\left( \frac{\partial H_y}{\partial z} - \frac{\partial H_z}{\partial y} \right) \hat{a}_x &+ \left( \frac{\partial H_z}{\partial x} - \frac{\partial H_x}{\partial z} \right) \hat{a}_y &+ \left( \frac{\partial H_x}{\partial y} - \frac{\partial H_y}{\partial x} \right) \hat{a}_z = & \left( \epsilon_x \frac{\partial E_x}{\partial t} + \epsilon_y \frac{\partial E_y}{\partial t} + \epsilon_z \frac{\partial E_z}{\partial t} \right) \hat{a}_x \\
\left( \frac{\partial H_x}{\partial y} - \frac{\partial H_y}{\partial x} \right) &+ \left( \frac{\partial H_y}{\partial z} - \frac{\partial H_z}{\partial y} \right) &+ \left( \frac{\partial H_z}{\partial x} - \frac{\partial H_x}{\partial z} \right) \hat{a}_y & \left( \epsilon_x \frac{\partial E_x}{\partial t} + \epsilon_y \frac{\partial E_y}{\partial t} + \epsilon_z \frac{\partial E_z}{\partial t} \right) \hat{a}_y \\
\left( \frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} \right) &+ \left( \frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} \right) &+ \left( \frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} \right) \hat{a}_z & \left( \epsilon_x \frac{\partial E_x}{\partial t} + \epsilon_y \frac{\partial E_y}{\partial t} + \epsilon_z \frac{\partial E_z}{\partial t} \right) \hat{a}_z
\end{align*}
\]

Alternative Form of Maxwell’s Equations in Cartesian Coordinates (2 of 2)

Alternate Curl Equations

\[ \nabla \times \vec{E} = -\mu \frac{\partial \vec{H}}{\partial t} \]

\[
\begin{align*}
\left( \frac{\partial E_y}{\partial z} - \frac{\partial E_z}{\partial y} \right) \hat{a}_x &+ \left( \frac{\partial E_z}{\partial x} - \frac{\partial E_x}{\partial z} \right) \hat{a}_y &+ \left( \frac{\partial E_x}{\partial y} - \frac{\partial E_y}{\partial x} \right) \hat{a}_z = & \left( \mu_x \frac{\partial H_x}{\partial t} + \mu_y \frac{\partial H_y}{\partial t} + \mu_z \frac{\partial H_z}{\partial t} \right) \hat{a}_x \\
\left( \frac{\partial E_x}{\partial y} - \frac{\partial E_y}{\partial x} \right) &+ \left( \frac{\partial E_y}{\partial z} - \frac{\partial E_z}{\partial y} \right) &+ \left( \frac{\partial E_z}{\partial x} - \frac{\partial E_x}{\partial z} \right) \hat{a}_y & \left( \mu_x \frac{\partial H_x}{\partial t} + \mu_y \frac{\partial H_y}{\partial t} + \mu_z \frac{\partial H_z}{\partial t} \right) \hat{a}_y \\
\left( \frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} \right) &+ \left( \frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} \right) &+ \left( \frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} \right) \hat{a}_z & \left( \mu_x \frac{\partial H_x}{\partial t} + \mu_y \frac{\partial H_y}{\partial t} + \mu_z \frac{\partial H_z}{\partial t} \right) \hat{a}_z
\end{align*}
\]
Physical Boundary Conditions

\[\mu_1 \text{ and } \varepsilon_1\]

- Tangential components of \(E\) and \(H\) are continuous across an interface.
- \(E\) and \(H\) fields normal to the interface are discontinuous across an interface.
- Note: Normal components of \(D\) and \(B\) are continuous across the interface.
- Tangential components of the wave vector are continuous across an interface.

\[\mu_2 \text{ and } \varepsilon_2\]

- These are more complicated boundary conditions, physically and analytically.

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Slide 24
Preparing Maxwell’s Equations for CEM

Simplifying Maxwell’s Equations

1. Assume no charges or current sources: \( \rho = 0, \ J = 0 \)

\[
\begin{align*}
\nabla \cdot \vec{B} &= 0 \\
\nabla \times \vec{H} &= \epsilon \frac{\partial \vec{D}}{\partial t} \\
\n\vec{D}(t) &= \left[ \epsilon(t) \right] * \vec{E}(t) \\
\n\nabla \cdot \vec{D} &= 0 \\
\n\nabla \times \vec{E} &= -\epsilon \frac{\partial \vec{B}}{\partial t} \\
\n\vec{B}(t) &= \left[ \mu(t) \right] * \vec{H}(t)
\end{align*}
\]

2. Transform Maxwell’s equations to frequency-domain:

\[
\begin{align*}
\nabla \cdot \vec{B} &= 0 \\
\n\nabla \times \vec{H} &= j \omega \vec{D} \\
\n\vec{D} &= \left[ \epsilon \right] \vec{E} \\
\n\nabla \cdot \vec{D} &= 0 \\
\n\nabla \times \vec{E} &= -j \omega \vec{B} \\
\n\vec{B} &= \left[ \mu \right] \vec{H}
\end{align*}
\]

Convolution becomes simple multiplication

Note: We have chosen to proceed with the negative sign convention.

3. Substitute constitutive relations into Maxwell’s equations:

\[
\begin{align*}
\nabla \cdot \left[ \mu \right] \vec{H} &= 0 \\
\n\nabla \times \vec{H} &= j \omega \left[ \epsilon \right] \vec{E} \\
\n\nabla \cdot \left[ \epsilon \right] \vec{E} &= 0 \\
\n\nabla \times \vec{E} &= -j \omega \left[ \mu \right] \vec{H}
\end{align*}
\]

Note: It is useful to retain \( \mu \) and \( \epsilon \) and not replace them with refractive index \( n \).
Isotropic Materials

For anisotropic materials, the permittivity and permeability terms are tensor quantities.

$$
[\varepsilon] = \begin{bmatrix}
\varepsilon_{xx} & \varepsilon_{xy} & \varepsilon_{xz} \\
\varepsilon_{yx} & \varepsilon_{yy} & \varepsilon_{yz} \\
\varepsilon_{zx} & \varepsilon_{zy} & \varepsilon_{zz}
\end{bmatrix}
$$

$$
[\mu] = \begin{bmatrix}
\mu_{xx} & \mu_{xy} & \mu_{xz} \\
\mu_{yx} & \mu_{yy} & \mu_{yz} \\
\mu_{zx} & \mu_{zy} & \mu_{zz}
\end{bmatrix}
$$

For isotropic materials, the tensors reduce to a single scalar quantity.

$$
[\varepsilon] = \begin{bmatrix}
\varepsilon & 0 & 0 \\
0 & \varepsilon & 0 \\
0 & 0 & \varepsilon
\end{bmatrix} = \varepsilon
$$

$$
[\mu] = \begin{bmatrix}
\mu & 0 & 0 \\
0 & \mu & 0 \\
0 & 0 & \mu
\end{bmatrix} = \mu
$$

Maxwell’s equations can then be written as

$$
\nabla \cdot (\varepsilon \varepsilon E) = 0 \quad \nabla \cdot (\mu \mu H) = 0
$$

$$
\nabla \times H = j \omega \varepsilon \varepsilon E
$$

$$
\nabla \times E = -j \omega \mu \mu H
$$

Expand Maxwell’s Equations

Divergence Equations

$$
\nabla \cdot (\varepsilon \varepsilon E) = 0
$$

$$
\frac{\partial (\varepsilon E_x)}{\partial x} + \frac{\partial (\varepsilon E_y)}{\partial y} + \frac{\partial (\varepsilon E_z)}{\partial z} = 0
$$

Curl Equations

$$
\nabla \times H = j \omega \varepsilon \varepsilon E
$$

$$
\frac{\partial H_x}{\partial y} - \frac{\partial H_y}{\partial x} = j \omega \varepsilon \varepsilon E_x
$$

$$
\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} = j \omega \varepsilon \varepsilon E_y
$$

$$
\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} = j \omega \varepsilon \varepsilon E_z
$$

$$
\nabla \times E = -j \omega \mu \mu H
$$

$$
\frac{\partial E_x}{\partial y} - \frac{\partial E_y}{\partial x} = -j \omega \mu \mu H_x
$$

$$
\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} = -j \omega \mu \mu H_y
$$

$$
\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} = -j \omega \mu \mu H_z
$$
Normalize the Magnetic Field

Standard form of “Maxwell’s Curl Equations”

\[ \nabla \times \vec{E} = -j \omega \mu_0 \mu_r \vec{H} \]

\[ \nabla \times \vec{H} = j \omega \varepsilon_0 \varepsilon_r \vec{E} \]

Normalized Magnetic Field

\[ \frac{|\vec{E}|}{|\vec{H}|} \approx \frac{377}{n} \]

\[ \vec{H} = -j \sqrt{\frac{\mu_0}{\varepsilon_0}} \vec{H} \]

Note:

\[ k_0 = \omega \sqrt{\mu_0 \varepsilon_0} \]

- Equalizes \( E \) and \( H \) amplitudes
- No sign inconsistency
- Just have \( k_0 \)

Normalized Maxwell’s Equations

\[ \nabla \times \vec{E} = k_0 \mu_r \vec{H} \]

\[ \nabla \times \vec{H} = k_0 \varepsilon_r \vec{E} \]

Starting Point for Most CEM

We arrive at the following set of equations that are the same regardless of the sign convention used.

\[ \frac{\partial E_x}{\partial y} - \frac{\partial E_y}{\partial z} = k_0 \mu_{xx} \vec{H}_x \]

\[ \frac{\partial E_y}{\partial z} - \frac{\partial E_z}{\partial x} = k_0 \mu_{yy} \vec{H}_y \]

\[ \frac{\partial E_z}{\partial x} - \frac{\partial E_x}{\partial y} = k_0 \mu_{zz} \vec{H}_z \]

\[ \frac{\partial \vec{H}_x}{\partial y} - \frac{\partial \vec{H}_y}{\partial z} = k_0 \varepsilon_{xx} \vec{E}_x \]

\[ \frac{\partial \vec{H}_y}{\partial z} - \frac{\partial \vec{H}_z}{\partial x} = k_0 \varepsilon_{yy} \vec{E}_y \]

\[ \frac{\partial \vec{H}_z}{\partial x} - \frac{\partial \vec{H}_x}{\partial y} = k_0 \varepsilon_{zz} \vec{E}_z \]

The manner in which the magnetic field is normalized does depend on the sign convention chosen.

\[ \vec{H} = \begin{cases} 
- j \eta_0 \vec{H} & \text{negative sign convention} \\
+ j \eta_0 \vec{H} & \text{positive sign convention} 
\end{cases} \]
Scaling Properties of Maxwell’s Equations

There is no fundamental length scale in Maxwell’s equations.

Devices may be scaled to operate at different frequencies just by scaling the mechanical dimensions or material properties in proportion to the change in frequency.

This assumes it is physically possible to scale systems in this manner. In practice, building larger or smaller features may not be practical. Further, the properties of the materials may be different at the new operating frequency.
Scaling Dimensions

We start with the wave equation and write the parameters dependence on position explicitly.

\[ \nabla \times \frac{1}{\mu} \nabla \times \vec{E}(\vec{r}) = \omega^2 \mu_0 \varepsilon_0 \cdot \varepsilon_r(\vec{r}) \cdot \vec{E}(\vec{r}) \]

Next, we scale the dimensions by a factor \( a \).

\[ (a \nabla) \times \frac{1}{\mu_0} (a \nabla) \times \vec{E}(\vec{r}/a) = \omega^2 \mu_0 \varepsilon_0 \cdot \varepsilon_r(\vec{r}/a) \cdot \vec{E}(\vec{r}/a) \]

The scale factors multiplying the \( \nabla \) operators are moved to multiply the frequency term.

\[ (a \nabla) \times \frac{1}{\mu_0} \nabla \times \vec{E}(\vec{r}') = \left( \frac{\omega}{a} \right)^2 \mu_0 \varepsilon_0 \cdot \varepsilon_r(\vec{r}') \cdot \vec{E}(\vec{r}') \quad \vec{r}' = \frac{\vec{r}}{a} \]

The effect of scaling the dimensions is just a shift in frequency.

Visualization of Size Scaling

\( a = 1.0 \)

\( f_c = 500 \text{ MHz} \)

\( a = 0.5 \)

\( f_c = 1000 \text{ MHz} \)
Scaling $\mu$ and $\varepsilon$

We apply separate scaling factors to $\mu$ and $\varepsilon$.

$$\nabla \times \frac{1}{(a_\mu \mu_r)} \nabla \times \vec{E} = \omega^2 \mu_0 \varepsilon_0 \cdot (a_\varepsilon \varepsilon_t) \cdot \vec{E}$$

The scale factors are moved to multiply the frequency term.

$$\nabla \times \frac{1}{\mu_t} \nabla \times \vec{E} = \left(\omega \sqrt{a_\mu a_\varepsilon}\right)^2 \mu_0 \varepsilon_0 \cdot \varepsilon_t \cdot \vec{E}$$

The effect of scaling the material properties is just a factor in frequency.