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Advanced Computation:
Computational Electromagnetics

TMM Extras

1

Outline

- Circuit/Wave Equivalence
- Calculating Internal Fields
- Calculating Slab Waveguide Modes
- TMM for General Bi-Anisotropic Media

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Slide 2

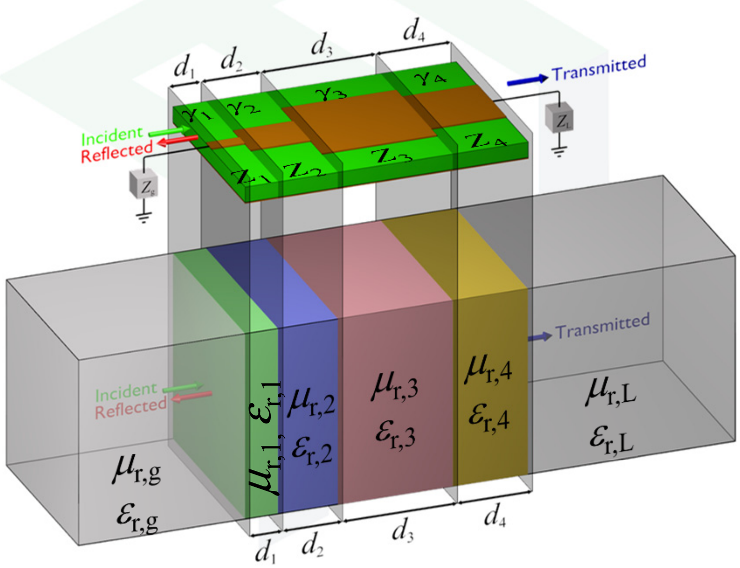
2

Circuit-Wave Equivalence

Slide 3

3

Circuit-Wave Equivalence



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Slide 4

4

Derivation (1 of 3)

We wish to derive equations that convert between the transmission line framework and the transfer matrix method framework.

$$\gamma, Z \quad \leftrightarrow \quad \mu_r, \varepsilon_r$$

Step 1 – Extract the complex refractive index n and complex impedance η from the complex propagation constant γ and characteristic impedance Z.

$$e^{-jk_0 nz} = e^{-\gamma z}$$

$$\downarrow$$

$$n = \frac{\gamma}{jk_0}$$

$$\eta = Z$$

Derivation (2 of 3)

Step 2 – Relate the complex refractive index n and complex impedance η to the complex permittivity ε_r and complex permeability μ_r.

$$n = \sqrt{\mu_r \varepsilon_r} \qquad \eta = \eta_0 \sqrt{\frac{\mu_r}{\varepsilon_r}}$$

Step 3 – Solve the above equations for the complex permittivity ε_r and complex permeability μ_r.

$$\varepsilon_r = n\frac{\eta_0}{\eta} \qquad \mu_r = n\frac{\eta}{\eta_0}$$

Step 4 – Replace the complex refractive index n and complex impedance η with the complex propagation constant γ and characteristic impedance Z from Step 1.

$$\boxed{\varepsilon_r = \frac{\gamma}{j\omega\varepsilon_0 Z}} \qquad \boxed{\mu_r = \frac{\gamma Z}{j\omega\mu_0}}$$

Derivation (3 of 3)

Step 5 – Solve the equations from Step 4 for the complex propagation constant γ and characteristic impedance Z.

$$\gamma = jk_0\sqrt{\mu_r\varepsilon_r}$$

$$Z = \eta_0\sqrt{\frac{\mu_r}{\varepsilon_r}}$$

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Slide 7

7

Calculation of Internal Fields

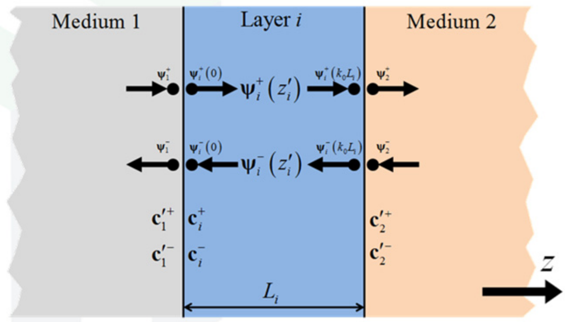
Slide 8

8

Information Needed

The field inside the i^{th} layer is calculated as:

$$\boldsymbol{\psi}_i(z_i') = \begin{bmatrix} E_{x,i}(z_i') \\ E_{y,i}(z_i') \\ \tilde{H}_{x,i}(z_i') \\ \tilde{H}_{y,i}(z_i') \end{bmatrix} = \begin{bmatrix} \mathbf{W}_i & \mathbf{W}_i \\ \mathbf{V}_i & -\mathbf{V}_i \end{bmatrix} \begin{bmatrix} e^{\lambda_i z_i'} & \mathbf{0} \\ \mathbf{0} & e^{-\lambda_i z_i'} \end{bmatrix} \begin{bmatrix} \mathbf{c}_i^+ \\ \mathbf{c}_i^- \end{bmatrix}$$



To do this, you must store the layer parameters and calculate the internal mode coefficients from the external mode coefficients to apply this equation.

For a detailed description along with implementation in MATLAB, see: https://empossible.thinkific.com/courses/tmmmatlab

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Slide 9

9

Calculating Slab Waveguide Modes

Slide 10

10

Using TMM to Calculate Slab Waveguide Modes

1. For the incident wave vector, set it to

$$\tilde{k}_{x,\text{inc}} = n_{\text{eff}}$$
$$\tilde{k}_{y,\text{inc}} = 0$$
$$\tilde{k}_{z,\text{inc}} = 0$$

2. Perform standard TMM to construct the global scattering matrix.

3. Guided modes are identified as values of n_{eff} that give

$$\det\left[\mathbf{S}^{(\text{global})}\right] = 0$$

4. Perform root finding to identify precise values of n_{eff}.

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Slide 11

11

TMM For General Bi-Anisotropic Media

Slide 12

12

Matrix Wave Equation

$$\frac{\partial}{\partial z'}\boldsymbol{\psi}_i(z) - \boldsymbol{\Omega}_i \boldsymbol{\psi}_i(z) = \mathbf{0} \qquad \boldsymbol{\psi}_i(z) = \begin{bmatrix} E_{i,x}(z) \\ E_{i,y}(z) \\ \tilde{H}_{i,x}(z) \\ \tilde{H}_{i,y}(z) \end{bmatrix}$$

$$\boldsymbol{\Omega}_i = \begin{bmatrix}
-j\left(\tilde{k}_y \frac{\mu_{i,yz}}{\mu_{i,zz}} + \tilde{k}_x \frac{\varepsilon_{i,zx}}{\varepsilon_{i,zz}}\right) & j\tilde{k}_x\left(\frac{\mu_{i,yz}}{\mu_{i,zz}} - \frac{\varepsilon_{i,zy}}{\varepsilon_{i,zz}}\right) & \left(\frac{\tilde{k}_x\tilde{k}_y}{\varepsilon_{i,zz}} + \mu_{i,yx} - \frac{\mu_{i,yz}\mu_{i,zx}}{\mu_{i,zz}}\right) & \left(-\frac{\hat{k}_x^2}{\varepsilon_{i,zz}} + \mu_{i,yy} - \frac{\mu_{i,yz}\mu_{i,zy}}{\mu_{i,zz}}\right) \\
j\tilde{k}_y\left(\frac{\mu_{i,xz}}{\mu_{i,zz}} - \frac{\varepsilon_{i,zx}}{\varepsilon_{i,zz}}\right) & -j\left(\tilde{k}_x \frac{\mu_{i,xz}}{\mu_{i,zz}} + \tilde{k}_y \frac{\varepsilon_{i,zy}}{\varepsilon_{i,zz}}\right) & \left(\frac{\tilde{k}_y^2}{\varepsilon_{i,zz}} - \mu_{i,xx} + \frac{\mu_{i,xz}\mu_{i,zx}}{\mu_{i,zz}}\right) & \left(-\frac{\tilde{k}_x\tilde{k}_y}{\varepsilon_{i,zz}} - \mu_{i,xy} + \frac{\mu_{i,xz}\mu_{i,zy}}{\mu_{i,zz}}\right) \\
\left(\frac{\tilde{k}_x\tilde{k}_y}{\mu_{i,zz}} + \varepsilon_{i,yx} - \frac{\varepsilon_{i,yz}\varepsilon_{i,zx}}{\varepsilon_{i,zz}}\right) & \left(-\frac{\tilde{k}_x^2}{\mu_{i,zz}} + \varepsilon_{i,yy} - \frac{\varepsilon_{i,yz}\varepsilon_{i,zy}}{\varepsilon_{i,zz}}\right) & -j\left(\tilde{k}_y \frac{\varepsilon_{i,yz}}{\varepsilon_{i,zz}} + \tilde{k}_x \frac{\mu_{i,zx}}{\mu_{i,zz}}\right) & j\tilde{k}_x\left(\frac{\varepsilon_{i,yz}}{\varepsilon_{i,zz}} - \frac{\mu_{i,zy}}{\mu_{i,zz}}\right) \\
\left(\frac{\tilde{k}_y^2}{\mu_{i,zz}} - \varepsilon_{i,xx} + \frac{\varepsilon_{i,xz}\varepsilon_{i,zx}}{\varepsilon_{i,zz}}\right) & \left(-\frac{\tilde{k}_x\tilde{k}_y}{\mu_{i,zz}} - \varepsilon_{i,xy} + \frac{\varepsilon_{i,xz}\varepsilon_{i,zy}}{\varepsilon_{i,zz}}\right) & j\tilde{k}_y\left(\frac{\varepsilon_{i,xz}}{\varepsilon_{i,zz}} - \frac{\mu_{i,zx}}{\mu_{i,zz}}\right) & -j\left(\tilde{k}_x \frac{\varepsilon_{i,xz}}{\varepsilon_{i,zz}} + \tilde{k}_y \frac{\mu_{i,zy}}{\mu_{i,zz}}\right)
\end{bmatrix}$$

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Slide 13

13

Calculate & Sort Eigen-Modes

$$\boldsymbol{\Omega}_i \rightarrow \begin{matrix} \mathbf{W}'_i & \text{Unsorted eigen-vector matrix} \\ \boldsymbol{\lambda}'_i & \text{Unsorted eigen-value matrix} \end{matrix}$$

After the eigen-modes are sorted, they will have the following general form.

$$\mathbf{W}'_i \quad \rightarrow \quad \mathbf{W}_i = \begin{bmatrix} \mathbf{W}^+_{i,E} & \mathbf{W}^-_{i,E} \\ \mathbf{W}^+_{i,H} & \mathbf{W}^-_{i,H} \end{bmatrix} \begin{matrix} E_{i,x} \\ E_{i,y} \\ \tilde{H}_{i,x} \\ \tilde{H}_{i,y} \end{matrix}$$

$$\boldsymbol{\lambda}' \quad \rightarrow \quad e^{\boldsymbol{\lambda}_i z'_i} = \begin{bmatrix} e^{\boldsymbol{\lambda}^+_i z'} & \mathbf{0} \\ \mathbf{0} & e^{-\boldsymbol{\lambda}^-_i z'} \end{bmatrix}$$

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Slide 14

14

Scattering Matrix Calculation

$$\begin{bmatrix} \mathbf{S}_{i,11} & \mathbf{S}_{i,12} \\ \mathbf{S}_{i,21} & \mathbf{S}_{i,22} \end{bmatrix} = \begin{bmatrix} -\mathbf{A}_{i,12} & \mathbf{B}_{i,11} \\ -\mathbf{A}_{22}^{i} & \mathbf{B}_{i,21} \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{A}_{i,11} & -\mathbf{B}_{i,12} \\ \mathbf{A}_{i,21} & -\mathbf{B}_{i,22} \end{bmatrix}$$

$$\mathbf{A}_i = \begin{bmatrix} \mathbf{X}_i & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \mathbf{W}_i^{-1} \mathbf{W}_1$$

$$\mathbf{B}_i = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{X}_i \end{bmatrix} \mathbf{W}_i^{-1} \mathbf{W}_2$$

$$\mathbf{X}_i = e^{k_0 \boldsymbol{\lambda}_i^+ L_i}$$

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Slide 15

15