Advanced Computation: Computational Electromagnetics

Diffraction Gratings & the Plane Wave Spectrum

Outline

- Fourier Series
- Diffraction from gratings
- The plane wave spectrum
- Plane wave spectrum for crossed gratings
- Power flow from gratings
Fourier Series

Jean Baptiste Joseph Fourier

Born: March 21, 1768 in Yonne, France.
Died: May 16, 1830 in Paris, France.

1D Complex Fourier Series

If a function $f(x)$ is periodic with period $\Lambda_x$, it can be expanded into a complex Fourier series.

$$f(x) = \sum_{p=-\infty}^{\infty} a_p e^{-\frac{2\pi px}{\Lambda_x}}$$

$$a_p = \frac{1}{\Lambda_x} \int_{-\Lambda_x/2}^{\Lambda_x/2} f(x) e^{-\frac{2\pi px}{\Lambda_x}} dx$$

Typically, only a finite number of terms are retained in the expansion.

$$f(x) = \sum_{p=-P}^{P} a_p e^{-\frac{2\pi px}{\Lambda_x}}$$
For 2D periodic functions, the complex Fourier series generalizes to

\[
f(x, y) = \sum_{p=-\infty}^{\infty} \sum_{q=-\infty}^{\infty} a_{p,q} e^{\frac{2\pi i (px + qy)}{\Lambda_x \Lambda_y}} \quad a_{p,q} = \frac{1}{\Lambda_x \Lambda_y} \int_{-\Lambda_x}^{\Lambda_x} \int_{-\Lambda_y}^{\Lambda_y} f(x, y) e^{-\frac{2\pi i (px + qy)}{\Lambda_x \Lambda_y}} \, dx \, dy
\]

For 3D periodic functions, the complex Fourier series generalizes to

\[
f(x, y, z) = \sum_{p=-\infty}^{\infty} \sum_{q=-\infty}^{\infty} \sum_{r=-\infty}^{\infty} a_{p,q,r} e^{\frac{2\pi i (px + qy + rz)}{\Lambda_x \Lambda_y \Lambda_z}} \quad a_{p,q,r} = \frac{1}{\Lambda_x \Lambda_y \Lambda_z} \iiint_{-\Lambda_x}^{\Lambda_x} \int_{-\Lambda_y}^{\Lambda_y} \int_{-\Lambda_z}^{\Lambda_z} f(x, y, z) e^{-\frac{2\pi i (px + qy + rz)}{\Lambda_x \Lambda_y \Lambda_z}} \, dx \, dy \, dz
\]
Diffraction from Gratings

Fields are Perturbed by Objects

A portion of the wave front is delayed after travelling through the dielectric object.
Fields in Periodic Structures

Waves in periodic structures take on the same periodicity as their host.

Diffraction from Gratings

The field is no longer a pure plane wave. The grating “chops” the wave front and sends the power into multiple discrete directions that are called *diffraction orders*.
Grating Lobes

If we were to plot the power exiting a periodic structure as a function of angle, we would get the following. The power peaks are called *grating lobes* or sometimes *side lobes*. The power minimums are called *nulls*.

---

**Diffraction Configurations**

<table>
<thead>
<tr>
<th>Planar Diffraction from a Ruled Grating</th>
<th>Conical Diffraction from a Ruled Grating</th>
</tr>
</thead>
<tbody>
<tr>
<td>- Diffraction is confined within a plane</td>
<td>- Diffraction is no longer confined to a plane</td>
</tr>
<tr>
<td>- Numerically much simpler than other cases</td>
<td>- Almost same analytical complexity as crossed grating case, but simpler numerically</td>
</tr>
<tr>
<td>- E and H modes are independent</td>
<td>- E and H modes are coupled</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Conical Diffraction from a Crossed Grating with Planar Incidence</th>
<th>Conical Diffraction from a Crossed Grating</th>
</tr>
</thead>
<tbody>
<tr>
<td>- Diffraction occurs in all directions</td>
<td>- Diffraction occurs in all directions</td>
</tr>
<tr>
<td>- Almost same numerical complexity as nest case</td>
<td>- Most complicated case numerically</td>
</tr>
<tr>
<td>- E and H modes are coupled</td>
<td>- E and H modes are coupled</td>
</tr>
<tr>
<td></td>
<td>- Essentially the same as previous case</td>
</tr>
</tbody>
</table>
Grating Equation for Planar Diffraction

The angles of the diffracted modes are related to the wavelength and grating period through the grating equation.

The grating equation only predicts the directions of the modes, not how much power is in them.

**Reflection Region**

\[ n_{\text{ref}} \sin \theta_m = n_{\text{inc}} \sin \theta_{\text{inc}} - m \frac{\lambda_0}{\Lambda_x} \]

**Transmission Region**

\[ n_{\text{trn}} \sin \theta_m = n_{\text{inc}} \sin \theta_{\text{inc}} - m \frac{\lambda_0}{\Lambda_x} \]

Effect of Grating Periodicity

- **Subwavelength Grating**
  \[ \Lambda_x < \frac{\lambda_0}{n_{\text{avg}}} \]

- **“Subwavelength” Grating**
  \[ \frac{\lambda_0}{n_{\text{avg}}} < \Lambda_x < \frac{\lambda_0}{n_{\text{inc}}} \]

- **Low Order Grating**
  \[ \Lambda_x > \frac{\lambda_0}{n_{\text{inc}}} \]

- **High Order Grating**
  \[ \Lambda_x >> \frac{\lambda_0}{n_{\text{inc}}} \]
Animation of Grating Diffraction (1 of 3)

\[ \frac{\Delta}{\lambda_1} = 0.30 \]
\[ \frac{\Delta}{\lambda_2} = 0.60 \]

\[ n_1 = 1.0 \]
\[ n_2 = 2.0 \]

Animation of Grating Diffraction (2 of 3)

\[ \frac{\Delta}{\lambda_1} = 0.42 \]
\[ \frac{\Delta}{\lambda_2} = 0.84 \]

\[ n_1 = 1.0 \]
\[ n_2 = 2.0 \]
The Plane Wave Spectrum
Periodic Functions Can Be Expanded into a Fourier Series

Waves in periodic structures obey Bloch's equation

\[ E(x, y) = A(x) e^{i\beta \cdot r} \]

The envelope \( A(x) \) is periodic along \( x \) with period \( \Lambda_x \) so it can be expanded into a Fourier series.

\[ A(x) = \sum_{m=-\infty}^{\infty} S(m) e^{-\frac{2\pi mx}{\Lambda_x}} \]

\[ S(m) = \int_{\Lambda} A(x) e^{\frac{2\pi mx}{\Lambda_x}} \, dx \]

Rearrange the Fourier Series (1 of 2)

A periodic field can be expanded into a Fourier series.

\[ E(x, y) = A(x) e^{i\beta \cdot r} \]

\[ = \left[ \sum_{m=-\infty}^{\infty} S(m) e^{-\frac{2\pi mx}{\Lambda_x}} \right] e^{i\beta \cdot r} \]

\[ = \sum_{m=-\infty}^{\infty} S(m) e^{i\beta \cdot \frac{2\pi x}{\Lambda_x}} \]

Here the plane wave term \( e^{i\beta \cdot r} \) is brought inside of the summation.
Rearrange the Fourier Series (2 of 2)

\[ E(x, y) = \sum_{m=-\infty}^{\infty} S(m) e^{i\beta_x x} e^{i\beta_y y} e^{i\beta_z z} \]

\[ = \sum_{m=-\infty}^{\infty} S(m) e^{i\beta_x x} e^{i\beta_y y} e^{i\left(\beta_z - \frac{2\pi m}{\Lambda_x}\right) x} \]

Now let \( k_{x,m} = \beta_x - \frac{2\pi m}{\Lambda_x} \) and \( k_{y,n} = \beta_y, k_{z,n} = \beta_z \)

\[ E(x, y) = \sum_{m=-\infty}^{\infty} S(m) e^{i\bar{k}(m) \cdot \mathbf{r}} \]

\[ \bar{k}(m) = \left( \beta_x - \frac{2\pi m}{\Lambda_x} \right) \hat{a}_x + \beta_y \hat{a}_y + \beta_z \hat{a}_z \]

The Plane Wave Spectrum

Terms were rearranged to show that a periodic field can also be thought of as an infinite sum of plane waves at different angles. This is the “plane wave spectrum” of a periodic field.
Longitudinal Wave Vector Components of the Plane Wave Spectrum

The wave incident on a grating can be written as

\[
E_{\text{inc}}(x, y) = E_0 e^{i(k_{x,\text{inc}}x + k_{z,\text{inc}}z)}
\]

\[
k_{x,\text{inc}} = k_0 n_{\text{inc}} \sin \theta_{\text{inc}}
\]

\[
k_{y,\text{inc}} = 0
\]

\[
k_{z,\text{inc}} = k_0 n_{\text{inc}} \cos \theta_{\text{inc}}
\]

Phase matching into the grating leads to

\[
k_x(m) = k_{x,\text{inc}} - m \frac{2\pi}{\Lambda_x} \quad m = \cdots, -2, -1, 0, 1, 2, \cdots
\]

Note: \( k_x \) is always real.

There are two possible solutions here.
1. Purely real \( k_z \)
2. Purely imaginary \( k_z \)

Each wave must satisfy the dispersion relation.

\[
k_x^2(m) + k_z^2(m) = (k_{n_{\text{grat}}}^2)^2
\]

\[
k_z(m) = \sqrt{(k_{n_{\text{grat}}}^2)^2 - k_x^2(m)}
\]

Visualizing Phase Matching into the Grating

The wave vector expansion for the first 11 modes can be visualized as...

Each of these is phase matched into material 2. The longitudinal component of the wave vector is calculated using the dispersion relation in material 2.

Note: The "evanescent" fields in material 2 are not completely evanescent. They have a purely real \( k_z \) so power flows in the transverse direction.
Conclusions About the Plane Wave Spectrum

- Fields in periodic media take on the same periodicity as the media they are in.
- Periodic fields can be expanded into a Fourier series.
- Each term of the Fourier series represents a spatial harmonic (plane wave).
- Since there are an infinite number of terms in the Fourier series, there are an infinite number of spatial harmonics.
- Only a few of the spatial harmonics are actually propagating waves. Only these can carry power away from a device. Tunneling is an exception.

Plane Wave Spectrum from Crossed Gratings
Diffraction from Crossed Gratings

Doubly-periodic gratings, also called crossed gratings, can diffract waves into many directions. They are described by two grating vectors, $\mathbf{K}_x$ and $\mathbf{K}_y$.

Two boundary conditions are necessary here.

\[
\begin{align*}
k_x(m) &= k_{x,\text{inc}} - mK_x & m = \ldots, -2, -1, 0, 1, 2, \ldots \\
k_y(n) &= k_{y,\text{inc}} - nK_y & n = \ldots, -2, -1, 0, 1, 2, \ldots \\
\end{align*}
\]

\[
\begin{align*}
\hat{K}_x &= \frac{2\pi}{\Lambda_x} \hat{x} \\
\hat{K}_y &= \frac{2\pi}{\Lambda_y} \hat{y}
\end{align*}
\]
Transverse Wave Vector Expansion (1 of 2)

Crossed gratings diffraction in two dimensions, $x$ and $y$.

To quantify diffraction for crossed gratings, we must calculate an expansion for both $k_x$ and $k_y$.

\[
k_x(m) = k_{x,\text{inc}} - \frac{2\pi m}{\Lambda_x}, \quad m = -\infty, \cdots, -2, 0, 1, 2, \cdots \infty
\]

\[
k_y(n) = k_{y,\text{inc}} - \frac{2\pi n}{\Lambda_y}, \quad n = -\infty, \cdots, -2, 0, 1, 2, \cdots \infty
\]

\[
k_{i}(m,n) = k_x(m) \hat{x} + k_y(n) \hat{y}
\]

% TRANSVERSE WAVE VECTOR EXPANSION
M = [-floor(Nx/2):floor(Nx/2)]';
N = [-floor(Ny/2):floor(Ny/2)]';
kx = kxinc - 2*pi*M/Lx;
ky = kyinc - 2*pi*N/Ly;
[ky,kx] = meshgrid(ky,kx);  

We will use this code for 2D PWEM, 3D RCWA, 3D FOTD, 3D FDFD, 3D MoL, and more.

Transverse Wave Vector Expansion (2 of 2)

The vector expansions can be visualized this way...

\[
k_x(m)
\]

\[
k_y(n)
\]

\[
k_{i}(m,n) = k_x(m) \hat{x} + k_y(n) \hat{y}
\]
Longitudinal Wave Vector Expansion (1 of 2)

The longitudinal components of the wave vectors are computed as

\[ k_{z,\text{ref}}(m,n) = \frac{1}{k_0 n_{\text{ref}}} \left( k_0^2 n_{\text{ref}}^2 - k_x^2(m) - k_y^2(m) \right) \]

\[ k_{z,\text{trn}}(m,n) = \sqrt{(k_0 n_{\text{trn}})^2 - k_x^2(m) - k_y^2(m)} \]

The center few modes will have real \( k_z \)'s. These correspond to propagating waves. The others will have imaginary \( k_z \)'s and correspond to evanescent waves that do not transport power.

Longitudinal Wave Vector Expansion (2 of 2)

The overall wave vector expansion can be visualized this way...

\[ \vec{k}_i(m,n) \rightarrow k_{z,\text{ref}}(m,n) \]

\[ k_{z,\text{ref}}(m,n) \rightarrow k_{z,\text{im}}(m,n) \]
Power Flow from Gratings

Electromagnetic Power Flow

The instantaneous direction and intensity of power flow at any point is given by the Poynting vector.

\[ \vec{\Phi}(\vec{r},t) = \vec{E}(\vec{r},t) \times \vec{H}(\vec{r},t) \]

The RMS power flow is then

\[ \tilde{\Phi}(\vec{r}, \omega) = \frac{1}{2} \text{Re}[\vec{E}(\vec{r}, \omega) \times \vec{H}^*(\vec{r}, \omega)] \]

This is typically just written in the frequency-domain as

\[ \tilde{\Phi} = \frac{1}{2} \text{Re}[\vec{E} \times \vec{H}^*] \]
Wave Expressions

To obtain an expression for the RMS Poynting vector, start by write expression for the electric and magnetic fields.

\[ \vec{E}(\vec{r}, \omega) = \vec{E}_0 e^{jkr} \]  
\[ \vec{H}(\vec{r}, \omega) = \frac{\vec{k} \times \vec{E}_0}{\omega \mu} e^{jkr} \]

Substituting these into the definition of RMS Poynting vector gives

\[ \vec{\phi} = \frac{1}{2} \text{Re} \left[ \left( \vec{E}_0 e^{-jkr} \right) \times \left( \frac{\vec{k} \times \vec{E}_0}{\omega \mu} e^{-jkr} \right) \right] = \frac{1}{2} \text{Re} \left[ \left( \vec{E}_0 e^{jkr} \right) \times \left( \frac{\vec{k}^* \times \vec{E}_0^*}{\omega \mu} e^{jkr} \right) \right] \]

\[ = \frac{1}{2} \text{Re} \left[ \frac{e^{jkr} e^{-jkr}}{\omega \mu} \left( \vec{E}_0 \times \vec{k}^* + \vec{E}_0^* \times \vec{k} \right) \right] = \frac{1}{2} \text{Re} \left[ \frac{e^{j(k-k')r}}{\omega \mu} \left| \vec{E}_0 \right|^2 \vec{k}^* \right] \]

\[ = \frac{\left| \vec{E}_0 \right|^2}{2} e^{-2\text{Im}[k'] \omega} \text{Re} \left[ \frac{\vec{k}^*}{2k_0\eta_0} \right] = \frac{\left| \vec{E}_0 \right|^2}{2k_0\eta_0} e^{-2\text{Im}[k'] \omega} \text{Re} \left[ \frac{k}{\mu_t} \right] \]

Power Flow Away From Grating

To calculate the power flow away from the grating, it is only the \( z \)-component of the Poynting vector that is of interest.

\[ \vec{\phi} = \frac{\left| \vec{E}_0 \right|^2}{2k_0\eta_0} e^{-2\text{Im}[k'] \omega} \text{Re} \left[ \frac{k}{\mu_t} \right] \]

\[ \varphi_z(z) = \frac{\left| \vec{E}_0 \right|^2}{2k_0\eta_0} e^{-2\text{Im}[k'] \omega} \text{Re} \left[ \frac{k_z}{\mu_t} \right] \]

\[ \text{Note: power travelling in the} \ x \ \text{direction does not transport power into or away from a device that is infinitely periodic along} \ x. \ \text{This simplifying statement is not true for devices of finite extent.} \]
RMS Power of the Diffracted Modes

Recall that the field scattered from a periodic structure can be decomposed into a Fourier series.

\[
\vec{E}(x, z) = \sum_{m=-\infty}^{\infty} \vec{S}(m) e^{-j k_x(m)x} e^{-j k_z(m)z}
\]

where

\[
k_x(m) = \frac{2\pi m}{\Lambda_x} \quad k_z(m) = \sqrt{(k_n)^2 - k_x(m)^2}
\]

The term \(\vec{S}(m)\) is the amplitude and polarization of the \(m\)th diffracted mode. Therefore, power flow away from the grating due to the \(m\)th diffraction order is

\[
\varphi_z(z) = \frac{|\vec{E}_0|^2}{2k_0\eta_0} e^{-2\text{Im}[k_z(m)] z} \text{Re}\left[ \frac{k_z}{\mu_z} \right]
\]

Z Directed Power

From the previous equation, the power flow of the incident, reflected, and transmitted waves into the grating are written as

\[
\varphi_{z,\text{inc}}(z) = \frac{|\vec{S}_{\text{inc}}|^2}{2k_0\eta_0} e^{-2\text{Im}[k_z\text{inc}] z} \text{Re}\left[ \frac{k_z\text{inc}}{\mu_z\text{inc}} \right]
\]

\[
\varphi_{z,\text{ref}}(z, m) = \frac{|\vec{S}_{\text{ref}}(m)|^2}{2k_0\eta_0} e^{-2\text{Im}[k_z\text{ref}(m)] z} \text{Re}\left[ \frac{-k_z\text{ref}(m)}{\mu_z\text{ref}} \right]
\]

\[
\varphi_{z,\text{trn}}(z, m) = \frac{|\vec{S}_{\text{trn}}(m)|^2}{2k_0\eta_0} e^{-2\text{Im}[k_z\text{trn}(m)] z} \text{Re}\left[ \frac{k_z\text{trn}(m)}{\mu_z\text{trn}} \right]
\]
Definition of Diffraction Efficiency

Diffraction efficiency is defined as the power in a specific diffraction order divided by the applied incident power.

\[ \text{DE} (m) = \frac{\phi_z^m (z)}{\phi_z^{\text{inc}}} \]

Despite the title “efficiency,” we don’t always want this quantity to be large. We often want to adjust how much power gets diffracted into each mode. So it is neither good nor bad to have high or low diffraction efficiency. It depends on how the grating is being used.

Putting it All Together

So far, expressions were derived for the incident power and power in the diffraction orders. Away from the edges of the grating, these are

\[ \phi_z^{\text{inc}} (z) = \frac{|S_{\text{inc}} (m)|}{2k\gamma_0} e^{-2ikz\sin(\theta)} \left| \frac{k_z^{\text{inc}}}{\mu_{z,\text{inc}}} \right| \]

\[ \phi_z^m (z,m) = \frac{|S_m (m)|}{2k\gamma_0} e^{-2ikz\sin(\theta)} \left| \frac{k_z^m (m)}{\mu_{z,m}} \right| \]

The diffraction efficiency of the \( m \)th diffraction order was also defined as

\[ \text{DE} (m) = \frac{\phi_z^m (z)}{\phi_z^{\text{inc}}} \]

It is now possible to derive expressions for the diffraction efficiencies of the diffraction orders by combining these expressions.

\[ R_{\text{inc}} (m) = \frac{\phi_z^{\text{inc}} (m)}{\phi_z^{\text{inc}}} = \frac{|S_{\text{inc}} (m)|}{|S_{\text{inc}}|} e^{-2ikz\sin(\theta)} \left| \frac{k_z^{\text{inc}}}{\mu_{z,\text{inc}}} \right| \]

\[ T_{\text{inc}} (m) = \frac{\phi_z^m (m)}{\phi_z^{\text{inc}}} = \frac{|S_m (m)|}{|S_{\text{inc}}|} e^{-2ikz\sin(\theta)} \left| \frac{k_z^m (m)}{\mu_{z,m}} \right| \]

Note that: \( \mu_{z,\text{inc}} = \mu_{z,\text{ref}} \)
Diffraction Efficiency for Magnetic Fields

Diffraction efficiency equations were just derived from electric field quantities.

\[
\bar{S}(m) = \text{Electric field amplitude of the } m^{th} \text{ diffraction order}
\]

\[
R_{ee}(m) = \frac{\varphi_{ee}^m(m)}{\varphi_{ee}^m} = \left| \frac{\bar{S}_{ee}(m)}{S_{ee}} \right| e^{\text{in}k_{ee}(m) + \text{out}(m)} \frac{\text{Re}[\bar{k}_{ee}^m(m)/\mu_{sce}]}{\text{Re}[k_{ee}^m/\mu_{sce}]} \]

\[
T_{ee}(m) = \frac{\varphi_{ee}^m(m)}{\varphi_{ee}^m} = \left| \frac{\bar{S}_{ee}(m)}{S_{ee}} \right| e^{\text{in}k_{ee}(m) + \text{out}(m)} \frac{\text{Re}[\bar{k}_{ee}^m(m)/\mu_{sce}]}{\text{Re}[k_{ee}^m/\mu_{sce}]} \]

Sometimes Maxwell’s equations in terms of the magnetic fields. In this case, the diffraction efficiency equations are

\[
\bar{U}(m) = \text{Magnetic field amplitude of the } m^{th} \text{ diffraction order}
\]

\[
R_{em}(m) = \frac{\varphi_{em}^m(m)}{\varphi_{em}^m} = \left| \frac{\bar{U}_{em}(m)}{U_{em}} \right| e^{\text{in}k_{em}(m) + \text{out}(m)} \frac{\text{Re}[\bar{k}_{em}^m(m)/\mu_{sce}]}{\text{Re}[k_{em}^m/\mu_{sce}]} \]

\[
T_{em}(m) = \frac{\varphi_{em}^m(m)}{\varphi_{em}^m} = \left| \frac{\bar{U}_{em}(m)}{U_{em}} \right| e^{\text{in}k_{em}(m) + \text{out}(m)} \frac{\text{Re}[\bar{k}_{em}^m(m)/\mu_{sce}]}{\text{Re}[k_{em}^m/\mu_{sce}]} \]

Reflectance, Transmittance, and Absorptance

The reflectance \( R \) is the sum of the diffraction efficiencies of the reflected modes.

\[
R = \sum_{m=-\infty}^{\infty} R_{DE}(m)
\]

The transmittance \( T \) is the sum of the diffraction efficiencies of the transmitted modes.

\[
T = \sum_{m=-\infty}^{\infty} T_{DE}(m)
\]

The absorptance \( A \) is the fraction of power absorbed by the device.

\[
A + R + T = 1
\]

\( A > 0 \) materials have loss

\( A = 0 \) materials have no loss

\( A < 0 \) materials have gain
Simplification for TMM

For TMM, the layers are homogeneous so there is no diffraction. Therefore, only the $m = 0$ diffraction order exists.

\[ S(0) = \text{Electric field amplitude of the } 0^{th} \text{ diffraction order} \]

\[ R_{\text{inc}}(0) = \frac{\phi_{\text{inc}}^{r}(0)}{\phi_{\text{inc}}^{s}} = \frac{\hat{S}_{\text{inc}}(0)}{\hat{S}_{\text{inc}}} \left[ \frac{e^{j2\pi(z_{0}+z_{m})}}{\text{Re} \left[ k_{z}^{\text{inc}}/\mu_{\text{inc}} \right]} \right] \]

\[ T_{\text{inc}}(0) = \frac{\phi_{\text{inc}}^{t}(0)}{\phi_{\text{inc}}^{s}} = \frac{\tilde{S}_{\text{inc}}(0)}{\hat{S}_{\text{inc}}} \left[ e^{-j2\pi(z_{0}+z_{m})} \frac{\text{Re} \left[ k_{z}^{\text{inc}}/\mu_{\text{inc}} \right]}{\text{Re} \left[ k_{z}^{\text{inc}}/\mu_{\text{inc}} \right]} \right] \]

Since there is only one diffraction order,

\[ \hat{E}_{\text{inc}} = \tilde{S}_{\text{inc}} \]
\[ \hat{E}_{\text{ref}} = \tilde{S}_{\text{inc}}(0) \]
\[ \hat{E}_{\text{trn}} = \tilde{S}_{\text{inc}}(0) \]
\[ R_{\text{inc}} = R_{\text{inc}}(0) \]
\[ T_{\text{inc}} = T_{\text{inc}}(0) \]

\[ k_{z,\text{ref}} = -k_{z,\text{inc}} \]

**Diffraction Efficiency Equations in Lossy LHI Materials Right Against Device (i.e. $z = 0$)**

Equations for Multiple Diffraction Orders

\[ R_{\text{DE}}(m) = \frac{\hat{E}_{\text{ref}}(m)}{\hat{E}_{\text{inc}}} \left[ \frac{\text{Re} \left[ -k_{z}^{\text{ref}}(m)/\mu_{\text{inc}} \right]}{\text{Re} \left[ k_{z}^{\text{inc}}/\mu_{\text{inc}} \right]} \right] \]

\[ T_{\text{DE}}(m) = \frac{\hat{E}_{\text{inc}}(m)}{\hat{E}_{\text{inc}}} \left[ \frac{\text{Re} \left[ k_{z}^{\text{inc}}(m)/\mu_{\text{inc}} \right]}{\text{Re} \left[ k_{z}^{\text{inc}}/\mu_{\text{inc}} \right]} \right] \]

\[ R_{\text{DE}}(m) = \frac{\hat{E}_{\text{inc}}(m)}{\hat{E}_{\text{inc}}} \left[ \frac{\text{Re} \left[ -k_{z}^{\text{ref}}(m)/\epsilon_{\text{inc}} \right]}{\text{Re} \left[ k_{z}^{\text{inc}}/\epsilon_{\text{inc}} \right]} \right] \]

\[ T_{\text{DE}}(m) = \frac{\hat{E}_{\text{inc}}(m)}{\hat{E}_{\text{inc}}} \left[ \frac{\text{Re} \left[ k_{z}^{\text{inc}}(m)/\epsilon_{\text{inc}} \right]}{\text{Re} \left[ k_{z}^{\text{inc}}/\epsilon_{\text{inc}} \right]} \right] \]

\[ R = \sum_{m} R_{\text{DE}}(m) \quad T = \sum_{m} T_{\text{DE}}(m) \]

TMM / Single Diffraction Order

\[ R = R_{\text{DE}} = \frac{\hat{E}_{\text{ref}}}{\hat{E}_{\text{inc}}} \]  \[ T = T_{\text{DE}} = \frac{\hat{E}_{\text{inc}}}{\hat{E}_{\text{inc}}} \left[ \frac{\text{Re} \left[ k_{z}^{\text{inc}}/\mu_{\text{inc}} \right]}{\text{Re} \left[ k_{z}^{\text{inc}}/\mu_{\text{inc}} \right]} \right] \]

\[ R = R_{\text{DE}} = \frac{\hat{E}_{\text{ref}}}{\hat{E}_{\text{inc}}} \]  \[ T = T_{\text{DE}} = \frac{\hat{E}_{\text{inc}}}{\hat{E}_{\text{inc}}} \left[ \frac{\text{Re} \left[ k_{z}^{\text{inc}}/\epsilon_{\text{inc}} \right]}{\text{Re} \left[ k_{z}^{\text{inc}}/\epsilon_{\text{inc}} \right]} \right] \]
Diffraction Efficiency Equations in Lossless LHI Materials Right Against Device (i.e. $z = 0$)

Equations for Multiple Diffraction Orders

$$\begin{align*}
R_{DE} (m) &= \frac{\tilde{S}_{ref} (m)}{S_{inc}} \frac{\text{Re}[-k^\text{ref}_z (m)]}{\text{Re}[k^\text{inc}_z]} \\
T_{DE} (m) &= \frac{\mu_{trn}}{\mu_{inc}} \frac{\tilde{S}_{inc} (m)}{S_{inc}} \frac{\text{Re}[k^\text{inc}_z (m)]}{\text{Re}[k^\text{inc}_z]} \\
R &= \sum_m R_{DE} (m) \\
T &= \sum_m T_{DE} (m)
\end{align*}$$

TMM / Single Diffraction Order

$$\begin{align*}
R_{DE} &= \frac{\tilde{E}_{ref}}{E_{inc}} \\
T_{DE} &= \frac{\mu_{inc}}{\mu_{inc}} \frac{\tilde{E}_{inc}}{E_{inc}} \frac{\text{Re}[k^\text{inc}_z]}{\text{Re}[k^\text{inc}_z]} \\
R &= R_{DE} \\
T &= T_{DE}
\end{align*}$$