Advanced Computation: Computational Electromagnetics

Maxwell’s Equations on a Yee Grid

Outline

• Yee Grid
• Maxwell’s equations on a Yee grid
• Finite-difference approximations of Maxwell’s equations on a Yee grid
• Consequences of the Yee grid
• Alternative grids
Kane S. Yee was born in Canton, China on March 26, 1934. He received a B.S.E.E., M.S.E.E, and Ph.D. in Applied Mathematics from the University of California at Berkeley in 1957, 1958, and 1963, respectively.

He did research on electromagnetic diffraction while employed by Lockheed Missile and Space Co. (1959-1961). He has been associated with the Lawrence Livermore Laboratory since 1963. At present he is a professor in mathematics at Kansas State University. His main areas of interest are electromagnetics, hydrodynamics and numerical solution to partial differential equations.

A three-dimensional grid looks like this:

One cell from the grid looks like this:

\[
\begin{align*}
\Delta x, \Delta y, \Delta z &= \text{grid resolution parameters} \\
N_x &= 15 \\
N_y &= 10 \\
N_z &= 10
\end{align*}
\]

Collocated Grid

Within the grid cell, where should \( E_x, E_y, E_z, H_x, H_y \), and \( H_z \) be placed?

A straightforward approach would be to locate all of the field components at a common point within in a grid cell; perhaps at the origin.
Instead, the field components will be staggered within the grid cell.


Stereo Image of Yee Cell

To view the Yee cell if full 3D, look past the image above so that they appear double. When the double images overlap so that you see three Yee cells, the middle image will be three-dimensional.
Reasons to Use the Yee Grid

1. Divergence-free
\[ \nabla \cdot (\varepsilon \vec{E}) = 0 \]
\[ \nabla \cdot (\mu \vec{H}) = 0 \]

2. Physical boundary conditions are naturally satisfied

\[
\begin{align*}
\mu_1 & \parallel & \mu_2 & \parallel \\
\varepsilon_1 & \parallel & \varepsilon_2 & \parallel \\
\eta_1 & \parallel & \eta_2 & \parallel \\
\varepsilon_{E,1} & \parallel & \varepsilon_{E,2} & \parallel \\
\mu_{H,1} & \parallel & \mu_{H,2} & \parallel \\
\kappa_1 & \parallel & \kappa_2 & \parallel 
\end{align*}
\]

3. Elegant arrangement to approximate curl equations
Visualizing Extended Yee Grids

2x2x2 Grid

4x4 Grid (E Mode)

Finite-Difference Approximations of Maxwell’s Equations on a Yee Grid
**Starting Point**

Start with Maxwell’s equations in the following form.

\[
\begin{align*}
\frac{\partial E_y}{\partial y} - \frac{\partial E_z}{\partial z} &= k_0 \mu_y \tilde{H}_y, \\
\frac{\partial E_z}{\partial z} - \frac{\partial E_x}{\partial x} &= k_0 \mu_z \tilde{H}_z, \\
\frac{\partial E_x}{\partial x} - \frac{\partial E_y}{\partial y} &= k_0 \mu_x \tilde{H}_x,
\end{align*}
\]

Here only diagonally anisotropic material tensors were retained. This will be needed to incorporate a uniaxial perfectly matched layer (UPML) absorbing boundary condition.

These equations are valid independent of the chosen sign convention.

**Normalize the Grid Coordinates**

The grid is normalized according to

\[
\begin{align*}
x' &= k_0 x, & y' &= k_0 y, & z' &= k_0 z
\end{align*}
\]

This “absorbs” the \( k_0 \) term into the spatial derivatives and simplifies Maxwell’s equations to

\[
\begin{align*}
\frac{\partial E_y}{\partial y} - \frac{\partial E_z}{\partial z} &= \mu_y \tilde{H}_y, \\
\frac{\partial E_z}{\partial z} - \frac{\partial E_x}{\partial x} &= \mu_z \tilde{H}_z, \\
\frac{\partial E_x}{\partial x} - \frac{\partial E_y}{\partial y} &= \mu_x \tilde{H}_x,
\end{align*}
\]

\[
\begin{align*}
\frac{\partial \tilde{H}_y}{\partial y} - \frac{\partial \tilde{H}_z}{\partial z} &= \varepsilon_{xx} E_x, \\
\frac{\partial \tilde{H}_z}{\partial z} - \frac{\partial \tilde{H}_x}{\partial x} &= \varepsilon_{yy} E_y, \\
\frac{\partial \tilde{H}_x}{\partial x} - \frac{\partial \tilde{H}_y}{\partial y} &= \varepsilon_{zz} E_z.
\end{align*}
\]
Starting Point

\[ \frac{\partial E_z}{\partial y'} - \frac{\partial E_y}{\partial z'} = \mu_{xx} \frac{\partial \tilde{H}_x}{\partial y'} \]

\[ \frac{\partial E_z}{\partial z'} - \frac{\partial E_z}{\partial x'} = \mu_{yy} \frac{\partial \tilde{H}_y}{\partial x'} \]

\[ \frac{\partial E_y}{\partial x'} - \frac{\partial E_x}{\partial y'} = \mu_{zz} \frac{\partial \tilde{H}_z}{\partial x'} \]

\[ \frac{\partial \tilde{H}_x}{\partial y'} - \frac{\partial \tilde{H}_y}{\partial z'} = \varepsilon_{xx} E_x \]

\[ \frac{\partial \tilde{H}_x}{\partial z'} - \frac{\partial \tilde{H}_z}{\partial x'} = \varepsilon_{yy} E_y \]

\[ \frac{\partial \tilde{H}_y}{\partial x'} - \frac{\partial \tilde{H}_y}{\partial y'} = \varepsilon_{zz} E_z \]

Finite-Difference Equation for \( \tilde{H}_x \)

\[ \frac{\partial E_z}{\partial y'} - \frac{\partial E_y}{\partial z'} + \mu_{xx} \frac{\partial \tilde{H}_x}{\partial y'} = \mu_{xx} \tilde{H}_x \]

\[ = \mu_{xx}^{i,j,k} \tilde{H}_x^{i,j,k} \]
Finite-Difference Equation for $H_x$

$$\frac{\partial E_z}{\partial y'} - \frac{\partial E_y}{\partial z'} = \mu_{xx} \tilde{H}_x$$

$$\frac{E_{zi,j+1,k} - E_{zi,j,k}}{\Delta y'} = \mu_{xx} \frac{i,j,k}{i,j,k} \tilde{H}_x$$

Finite-Difference Equation for $H_x$

$$\frac{\partial E_z}{\partial y'} \left( \frac{\partial E_y}{\partial z'} \right) = \mu_{xx} \tilde{H}_x$$

$$\frac{E_{zi,j+1,k} - E_{zi,j,k}}{\Delta y'} - \frac{E_{yi,j+1,k} - E_{yi,j,k}}{\Delta z'} = \mu_{xx} \frac{i,j,k}{i,j,k} \tilde{H}_x$$
Finite-Difference Equation for $H_y$

\[ \frac{\partial E_x}{\partial z'} - \frac{\partial E_z}{\partial x'} = \mu_{yy} \tilde{H}_y \]

\[ = \mu_{yy}^{i,j,k} \tilde{H}_y^{i,j,k} \]

Finite-Difference Equation for $H_y$

\[ \frac{\partial E_x}{\partial z'} - \frac{\partial E_z}{\partial x'} = \mu_{yy} \tilde{H}_y \]

\[ = \frac{E_x^{i,j,k+1} - E_x^{i,j,k}}{\Delta z'} \]

\[ = \mu_{yy}^{i,j,k} \tilde{H}_y^{i,j,k} \]
Finite-Difference Equation for $H_y$

$$\frac{\partial E_x}{\partial z'} \frac{\partial E_z}{\partial x'} = \mu_{yy} \tilde{H}_y$$

$$\frac{E_{x}^{i,j,k+1} - E_{x}^{i,j,k}}{\Delta z'} - \frac{E_{z}^{i+1,j,k} - E_{z}^{i,j,k}}{\Delta x'} = \mu_{yy}^{i,j,k} \tilde{H}_y^{i,j,k}$$

Finite-Difference Equation for $H_z$

$$\frac{\partial E_y}{\partial x'} - \frac{\partial E_x}{\partial y'} = \mu_{zz} \tilde{H}_z$$

$$= \mu_{zz}^{i,j,k} \tilde{H}_z^{i,j,k}$$
Finite-Difference Equation for $H_z$

\[
\frac{\partial E_y}{\partial x'} \left( \frac{\partial E_x}{\partial y'} \right) = \mu_{zz} \tilde{H}_z
\]

\[
\frac{E_{y}^{i+1,j,k} - E_{y}^{i,j,k}}{\Delta x'} = \mu_{zz}^{i,j,k} \tilde{H}_z^{i,j,k}
\]

Finite-Difference Equation for $H_z$

\[
\frac{\partial E_y}{\partial x'} \left( \frac{\partial E_x}{\partial y'} \right) = \mu_{zz} \tilde{H}_z
\]

\[
\frac{E_{y}^{i+1,j,k} - E_{y}^{i,j,k}}{\Delta x'} - \frac{E_{x}^{i,j+1,k} - E_{x}^{i,j,k}}{\Delta y'} = \mu_{zz}^{i,j,k} \tilde{H}_z^{i,j,k}
\]
Finite-Difference Equation for $E_x$

$$\frac{\partial \tilde{H}_z}{\partial y'} - \frac{\partial \tilde{H}_y}{\partial z'} = \varepsilon_{xx} E_x$$

$$= \varepsilon_{xx}^{i,j,k} E_x^{i,j,k}$$

Finite-Difference Equation for $E_x$

$$\frac{\partial \tilde{H}_z}{\partial y'} - \frac{\partial \tilde{H}_y}{\partial z'} = \varepsilon_{xx} E_x$$

$$\frac{\tilde{H}_z^{i,j,k} - \tilde{H}_z^{i,j-1,k}}{\Delta y'} = \varepsilon_{xx}^{i,j,k} E_x^{i,j,k}$$
Finite-Difference Equation for $E_x$

$$\frac{\partial \tilde{H}_z}{\partial y'} \frac{\partial \tilde{H}_y}{\partial z'} = \varepsilon_{xx} E_x$$

$$\begin{align*}
\frac{\tilde{H}_z^{i,j,k} - \tilde{H}_z^{i,j,k-1}}{\Delta y'} - \frac{\tilde{H}_y^{i,j,k} - \tilde{H}_y^{i,j,k-1}}{\Delta z'} &= \varepsilon_{xx}^{i,j,k} E_x^{i,j,k}.
\end{align*}$$

Finite-Difference Equation for $E_y$

$$\frac{\partial \tilde{H}_x}{\partial z'} - \frac{\partial \tilde{H}_z}{\partial x'} = \varepsilon_{yy} E_y$$

$$\begin{align*}
\frac{\tilde{H}_x^{i,j,k} - \tilde{H}_x^{i,j,k-1}}{\Delta z'} - \frac{\tilde{H}_z^{i,j,k} - \tilde{H}_z^{i,j,k-1}}{\Delta x'} &= \varepsilon_{yy}^{i,j,k} E_y^{i,j,k}.
\end{align*}$$
Finite-Difference Equation for $E_y$

$$\frac{\partial \tilde{H}_x}{\partial z'} - \frac{\partial \tilde{H}_z}{\partial x'} = \varepsilon_{yy} E_y$$

$$\frac{\tilde{H}_x^{i,j,k} - \tilde{H}_x^{i,j,k-1}}{\Delta z'} = \varepsilon_{yy}^{i,j,k} E_y^{i,j,k}$$

Finite-Difference Equation for $E_y$

$$\frac{\partial \tilde{H}_x}{\partial z'} - \frac{\partial \tilde{H}_z}{\partial x'} = \varepsilon_{yy} E_y$$

$$\frac{\tilde{H}_x^{i,j,k} - \tilde{H}_x^{i,j,k-1}}{\Delta z'} - \frac{\tilde{H}_z^{i,j,k} - \tilde{H}_z^{i,j,k-1}}{\Delta x'} = \varepsilon_{yy}^{i,j,k} E_y^{i,j,k}$$
Finite-Difference Equation for $E_z$

$$\frac{\partial \tilde{H}_y}{\partial x'} - \frac{\partial \tilde{H}_x}{\partial y'} = \varepsilon_{zz} E_z$$

$$= \varepsilon_{zz} i, j, k E_{z}^{i, j, k}$$

Finite-Difference Equation for $E_z$

$$\frac{\partial \tilde{H}_y}{\partial x'} - \frac{\partial \tilde{H}_x}{\partial y'} = \varepsilon_{zz} E_z$$

$$\frac{\tilde{H}_y^{i, j, k} - \tilde{H}_y^{i-1, j, k}}{\Delta x'} = \varepsilon_{zz} i, j, k E_{z}^{i, j, k}$$
Finite-Difference Equation for $E_z$

\[
\frac{\partial \tilde{H}_y}{\partial x'} \frac{\partial \tilde{H}_x}{\partial y'} = \varepsilon_{zz} E_z
\]

\[
\frac{\tilde{H}_{y}^{i,j,k} - \tilde{H}_{y}^{i-1,j,k}}{\Delta x'} - \frac{\tilde{H}_{x}^{i,j,k} - \tilde{H}_{x}^{i-1,j,k}}{\Delta y'} = \varepsilon_{zz} E_z^{i,j,k}
\]

Summary of Finite-Difference Approximations of Maxwell’s Equations

\[
\frac{\partial E_x}{\partial y'} - \frac{\partial E_y}{\partial z'} = \mu_{xx} \tilde{H}_x
\]

\[
\frac{\partial E_x}{\partial z'} - \frac{\partial E_z}{\partial x'} = \mu_{yy} \tilde{H}_y
\]

\[
\frac{\partial E_y}{\partial x'} - \frac{\partial E_z}{\partial y'} = \mu_{zz} \tilde{H}_z
\]

\[
\frac{E_{x}^{i,j,k+1} - E_{x}^{i,j,k}}{\Delta z'} - \frac{E_{y}^{i,j,k+1} - E_{y}^{i,j,k}}{\Delta y'} = \mu_{xx} \tilde{H}_{x}^{i,j,k}
\]

\[
\frac{E_{x}^{i,j,k+1} - E_{x}^{i,j,k}}{\Delta z'} - \frac{E_{y}^{i,j,k+1} - E_{y}^{i,j,k}}{\Delta y'} = \mu_{yy} \tilde{H}_{y}^{i,j,k}
\]

\[
\frac{E_{x}^{i,j,k+1} - E_{x}^{i,j,k}}{\Delta z'} - \frac{E_{y}^{i,j,k+1} - E_{y}^{i,j,k}}{\Delta y'} = \mu_{zz} \tilde{H}_{z}^{i,j,k}
\]

\[
\frac{\tilde{H}_{x}^{i,j,k} - \tilde{H}_{x}^{i-1,j,k}}{\Delta x'} - \frac{\tilde{H}_{y}^{i,j,k} - \tilde{H}_{y}^{i,j,k-1}}{\Delta y'} = \varepsilon_{xx} E_{x}^{i,j,k}
\]

\[
\frac{\tilde{H}_{x}^{i,j,k} - \tilde{H}_{x}^{i-1,j,k}}{\Delta x'} - \frac{\tilde{H}_{y}^{i,j,k} - \tilde{H}_{y}^{i,j,k-1}}{\Delta y'} = \varepsilon_{yy} E_{y}^{i,j,k}
\]

\[
\frac{\tilde{H}_{x}^{i,j,k} - \tilde{H}_{x}^{i-1,j,k}}{\Delta x'} - \frac{\tilde{H}_{y}^{i,j,k} - \tilde{H}_{y}^{i,j,k-1}}{\Delta y'} = \varepsilon_{zz} E_{z}^{i,j,k}
\]
Consequences of the Yee Grid

Field Components are at Physically Different Positions

Even though the field components reside within the same Yee cell, they are out of phase.

Any time that a wave is injected, extracted, or analyzed within a simulation, this must be accounted for.
Field Components are at Physically Different Positions

Even though the field components reside within the same Yee cell, they may reside in different media. This happens when an interface slices through the middle of a Yee cell.

This is the main reason why each field component is assigned its own material properties.

\[ \varepsilon_{xx} \text{ assigned to } E_x. \]
\[ \varepsilon_{yy} \text{ assigned to } E_y. \]
\[ \varepsilon_{zz} \text{ assigned to } E_z. \]
\[ \mu_{xx} \text{ assigned to } H_x. \]
\[ \mu_{yy} \text{ assigned to } H_y. \]
\[ \mu_{zz} \text{ assigned to } H_z. \]

Dispersion on a Yee Grid

Recall the dispersion relation for an isotropic material with parameters \( \mu_t \) and \( \varepsilon_i \):

\[
\left( \frac{\omega}{c_0} \right)^2 \mu_t \varepsilon_i = k_x^2 + k_y^2 + k_z^2
\]

The analogous dispersion relation on a frequency-domain Yee grid filled with \( \mu_t \) and \( \varepsilon_i \) is

\[
\left( \frac{\omega}{\nu} \right)^2 \mu_t \varepsilon_i = \left[ \frac{2}{\Delta_x} \sin \left( \frac{k_x \Delta_x}{2} \right) \right]^2 + \left[ \frac{2}{\Delta_y} \sin \left( \frac{k_y \Delta_y}{2} \right) \right]^2 + \left[ \frac{2}{\Delta_z} \sin \left( \frac{k_z \Delta_z}{2} \right) \right]^2
\]

In this equation, the speed of light \( c_0 \) is written as \( \nu \) because the velocity is different due to the dispersion of the grid.
Drawback of Structured Grids

Structured grids exhibit highly anisotropic dispersion.

![Anisotropic Dispersion](image)

The numerical dispersion equation is solved for velocity $v$.

$$v = \omega \sqrt{\frac{\mu \varepsilon}{\Delta y}} \left[ \frac{2}{\Delta x} \sin \left( \frac{k_x \Delta_x}{2} \right) \right]^2 + \left[ \frac{2}{\Delta y} \sin \left( \frac{k_y \Delta_y}{2} \right) \right]^2 + \left[ \frac{2}{\Delta z} \sin \left( \frac{k_z \Delta_z}{2} \right) \right]^2 \right]^\frac{1}{2}$$

In the absence of grid dispersion, $v$ should be exactly the speed of light $c_0$. Due to the Yee grid, waves slow down by a factor $\gamma$.

$$v = \frac{c_0}{\gamma}$$

We can calculate this factor by combining the above equations.

$$\gamma = \frac{c_0}{\omega \sqrt{\mu \varepsilon}} \left[ \frac{2}{\Delta x} \sin \left( \frac{k_x \Delta_x}{2} \right) \right]^2 + \left[ \frac{2}{\Delta y} \sin \left( \frac{k_y \Delta_y}{2} \right) \right]^2 + \left[ \frac{2}{\Delta z} \sin \left( \frac{k_z \Delta_z}{2} \right) \right]^2$$
Compensation Factor $\gamma$ (2 of 2)

We can write a simpler and more useful expression for $\gamma$.

$$\gamma = \frac{1}{k_n n} \sqrt{\left[ \frac{2}{\Delta_x} \sin \left( \frac{k_x \Delta_x}{2} \right) \right]^2 + \left[ \frac{2}{\Delta_y} \sin \left( \frac{k_y \Delta_y}{2} \right) \right]^2 + \left[ \frac{2}{\Delta_z} \sin \left( \frac{k_z \Delta_z}{2} \right) \right]^2}$$

$$k_0 = \frac{2\pi}{\lambda_0}$$

$$n = \sqrt{\mu_r \varepsilon_r}$$

Compensating for Numerical Dispersion

Given that the wave slows down by factor $\gamma$ in the direction of $\vec{k}$, it follows that we can compensate for the dispersion by artificially “speeding up” the wave.

We do this by decreasing the values of $\mu_r$ and $\varepsilon_r$ across the entire grid by a factor of $\gamma$.

$$\mu'_r = \mu_r / \gamma \quad \varepsilon'_r = \varepsilon_r / \gamma$$

Notes:
1. We can only compensate for dispersion for one direction $k$.
2. We can only compensate for dispersion in one set of material values $\mu_r$ and $\varepsilon_r$.
3. It is best to choose average or dominant values for these parameters.
4. Choose $\theta = 22.5^\circ$ if nothing else is known.
Alternative Grids

Drawbacks of Uniform Grids

Uniform grids are the easiest to implement, but do not conform well to arbitrary structures and exhibit high anisotropic dispersion.

Anisotropic Dispersion (see Lecture 10)  Staircase Approximation (see Lecture 18)
Drawbacks of Structured Grids (1 of 2)

Structured grids are the easiest to implement, but do not conform well to arbitrary geometries.

Structured Grid  Unstructured Grid

Hexagonal Grids

Hexagonal grids are good for minimizing anisotropic dispersion suffered on Cartesian grids. This is very useful when extracting phase information.

(b) Staggered, uncirculated grid and its associated dual grid

See Text, pp. 101-103.
Nonuniform Orthogonal Grids (1 of 2)

Nonuniform orthogonal grids are still relatively simple to implement and provide some ability to refine the grid at localized regions.

See Text, pp. 464-471.

Nonuniform Orthogonal Grids (2 of 2)

Uniform Grid Simulation
- 80×110×16 cells
- 140,800 cells

Nonuniform Grid Simulation
- 64×76×16 cells
- 77,824 cells

Conclusion: Roughly 50% memory and time savings.
Curvilinear Coordinates

Maxwell’s equations can be transformed from curvilinear coordinates to Cartesian coordinates to conform to curved boundaries of a device.


See Text, pp. 484-492.

Structured Nonorthogonal Grids

This is a particularly powerful approach for simulating periodic structures with oblique symmetries.

Irregular Nonorthogonal Unstructured Grids

Unstructured grids are more tedious to implement, but can conform to highly complex shapes while maintaining good cell aspect ratios and global uniformity.

Comparison of convergence rates

Bodies of Revolution (Cylindrical Symmetry)

Three-dimensional devices with cylindrical symmetry can be very efficiently modeled using cylindrical coordinates.

Devices with cylindrical symmetry have fields that are periodic around their axis. Therefore, the fields can be expanded into a Fourier series in $\phi$.

$$E(r, \theta, \phi) = \sum_{m=0}^{\infty} \vec{E}_m(r, \theta) \cdot \cos(m\phi) + \vec{E}_m(r, \theta) \cdot \sin(m\phi)$$

$$\vec{H}(r, \theta, \phi) = \sum_{m=0}^{\infty} \vec{H}_m(r, \theta) \cdot \cos(m\phi) + \vec{H}_m(r, \theta) \cdot \sin(m\phi)$$

Due to a singularity at $r = 0$, update equations for fields on the $z$ axis are derived differently.
Some Devices with Cylindrical Symmetry

- Bent Waveguides
- Cylindrical Waveguides
- Dipole Antennas
- Conical Horn Antenna
- Diffractive Lenses
- Focusing Antennas