Advanced Computation:
Computational Electromagnetics

Maxwell's Equations on a Yee Grid

1

Outline

- Electromagnetic waveguides
- Formulation of rigorous full-vectorial waveguide analysis
- Formulation of quasi-vectorial analysis
- Formulation of slab waveguide analysis
- Implementation in MATLAB
- Transmission Line Analysis
- Bent Waveguides

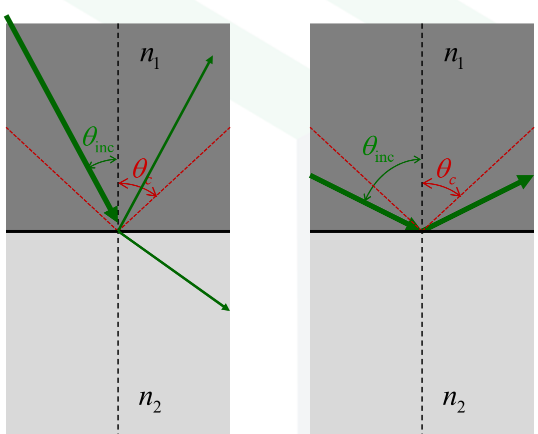
EMPossible Slide 2

2

Electromagnetic Waveguides

The Critical Angle and Total Internal Reflection

When an electromagnetic wave is incident on a material with a lower refractive index, it is totally reflected when the angle of incidence is greater than the critical angle.



$$\theta_c = \sin^{-1}\left(\frac{n_2}{n_1}\right)$$

Example

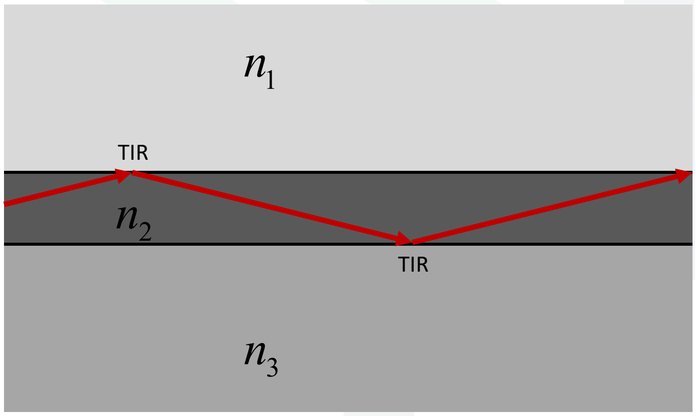
What is the critical angle for fused silica (glass).

The refractive index at optical frequencies is around 1.5.

$$\theta_c = \sin^{-1}\left(\frac{1.0}{1.5}\right) = 41.81^\circ$$

The Slab Waveguide

If we "sandwich" a slab of material between two materials with lower refractive index, we form a slab waveguide.

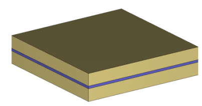


<u>Conditions</u>

$n_2 > n_1$

and

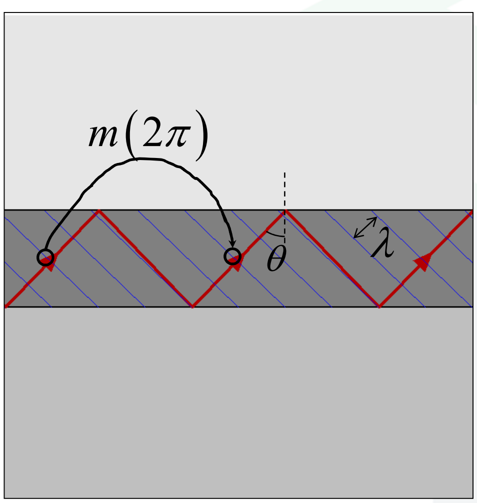
$n_2 > n_3$



EMPossible Slide 5

5

Ray Tracing Analysis



The roundtrip phase of a ray must be an integer multiple of 2π.

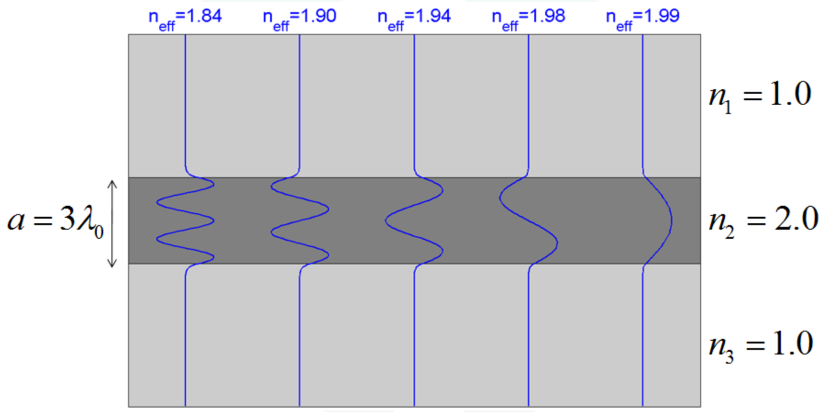
Because of this, only certain angles are allowed to propagate in the waveguide.

$$\beta = k_0 n_{\text{eff}} = k_0 n \sin\theta$$

EMPossible Slide 6

6

Exact Modal Analysis



$$\beta = k_0 n_{\text{eff}} = k_0 n \sin\theta$$

EMPossible

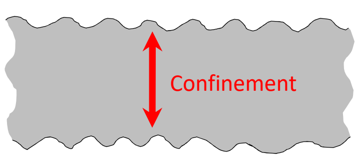
Slide 7

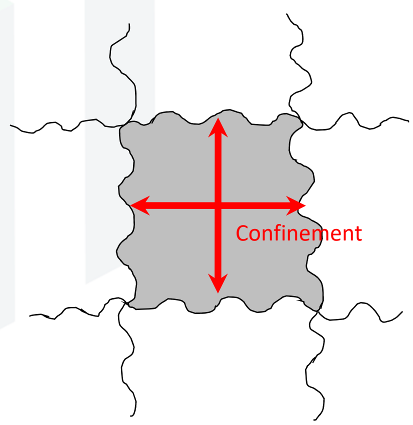
7

Slab Vs. Channel Waveguides

Slab waveguides confine energy in only one transverse direction.

Channel waveguides confine energy in both transverse directions.



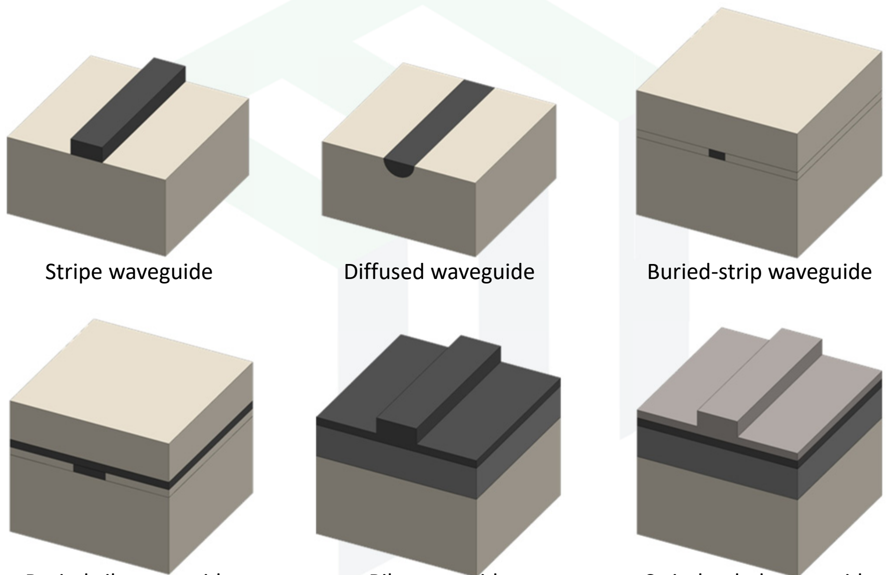


EMPossible

Slide 8

8

Channel Waveguides for Integrated Optics



Stripe waveguide — Diffused waveguide — Buried-strip waveguide

Buried-rib waveguide — Rib waveguide — Strip-loaded waveguide

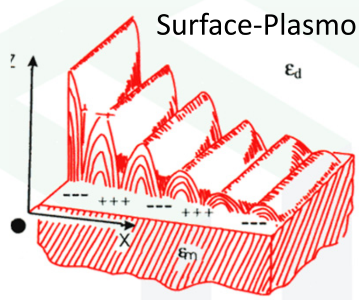
EMPossible

Slide 9

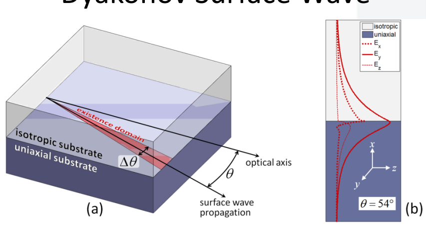
9

Structures Supporting Surface Waves

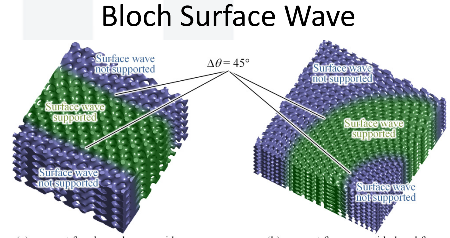
Surface-Plasmon Polariton (SPP)



Dyakonov Surface Wave



Bloch Surface Wave



(a) concept for channel waveguide for Bloch surface waves

(b) concept for waveguide bend for Bloch surface waves

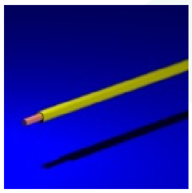
EMPossible

Slide 10

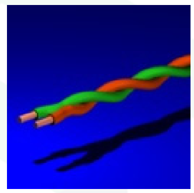
10

Channel Waveguides for Radio Frequencies

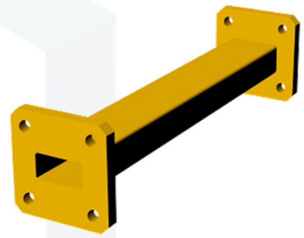
Isolated Wire



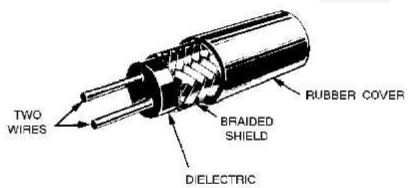
Twisted Pair Transmission Line



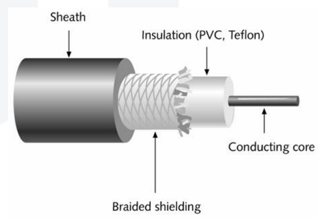
Rectangular Waveguide



Shielded-Pair Transmission Line



Coaxial Cable



EMPossible

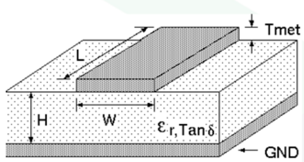
Slide 11

11

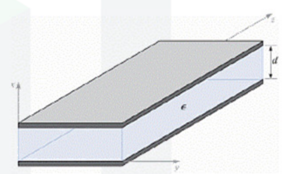
Channel Waveguides for Printed Circuits

Transmission lines are metallic structures that guide electromagnetic waves from DC to very high frequencies.

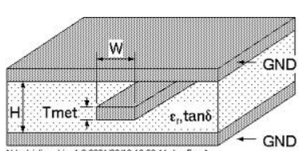
Microstrip



Parallel-Plate Transmission Line



Stripline



Slot Line

Coplanar Line

EMPossible

Slide 12

12

Formulation of Rigorous Full-Vectorial Waveguide Analysis

Starting Point

Start with Maxwell's equations in the following form.

$$\frac{\partial E_z}{\partial y'} - \frac{\partial E_y}{\partial z'} = \mu_{xx}\tilde{H}_x \qquad \frac{\partial \tilde{H}_z}{\partial y'} - \frac{\partial \tilde{H}_y}{\partial z'} = \varepsilon_{xx}E_x$$

$$\frac{\partial E_x}{\partial z'} - \frac{\partial E_z}{\partial x'} = \mu_{yy}\tilde{H}_y \qquad \frac{\partial \tilde{H}_x}{\partial z'} - \frac{\partial \tilde{H}_z}{\partial x'} = \varepsilon_{yy}E_y$$

$$\frac{\partial E_y}{\partial x'} - \frac{\partial E_x}{\partial y'} = \mu_{zz}\tilde{H}_z \qquad \frac{\partial \tilde{H}_y}{\partial x'} - \frac{\partial \tilde{H}_x}{\partial y'} = \varepsilon_{zz}E_z$$

Recall, for the positive sign convention the magnetic field H was normalized according to

$$\vec{\tilde{H}} = j\eta_0\vec{H}$$

and the grid coordinates were normalized according to

$$x' = k_0 x \qquad y' = k_0 y \qquad z' = k_0 z$$

EMPossible

Modal Solution for Waveguides

A mode in a waveguide has the following general mathematical form which is consistent with the Bloch theorem.

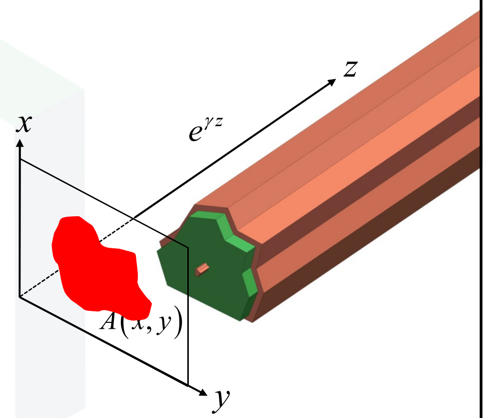
$$\vec{E}(x,y,z) = \vec{A}(x,y)e^{\gamma z}$$

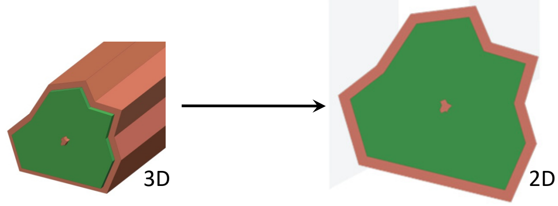
complex amplitude, mode shape

accumulation of phase in z direction

$$\gamma = -\alpha + j\beta \equiv \text{complex progation constant}$$

This means we can solve the problem by just analyzing the cross section in the x-y plane. This reduces to a two-dimensional problem.



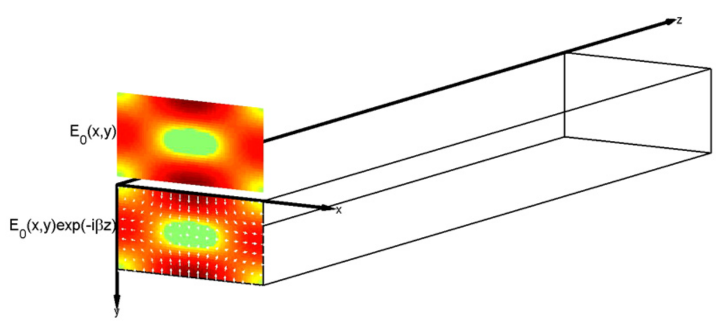


EMPossible

Slide 15

15

Animation of a Waveguide Mode



EMPossible

Slide 16

16

Meaning of Complex Propagation Constant γ

We have written our solution in the following form.

$$\vec{E}(x,y,z)=\vec{A}(x,y)e^{\gamma z}$$

But $\gamma = -\alpha + j\beta$, so this equation can be written as

$$\vec{E}(x,y,z)=\vec{A}(x,y)\underbrace{e^{-\alpha z}}\,\underbrace{e^{j\beta z}}$$

α is responsible for attenuation. β is responsible for wave oscillation.

$$\beta = \frac{2\pi}{\lambda}$$

EMPossible Slide 17

17

The Effective Refractive Index n_{eff}

We can also write our solution in terms of an effective refractive index n_{eff}.

$$\vec{E}(x,y,z)=\vec{A}(x,y)e^{jk_0 n_{\text{eff}} z}$$

The effective refractive index is a complex number to account for loss and/or gain.

$$n_{\text{eff}} = n_{\text{o}} + j\kappa$$

$n_{\text{o}} \equiv$ ordinary refractive index

$\kappa \equiv$ extinction coefficient (loss)

The solution can now be written as

$$\vec{E}(x,y,z)=\vec{A}(x,y)\underbrace{e^{-k_0 \kappa z}}\,\underbrace{e^{jk_0 n_{\text{o}} z}}$$

κ is responsible for attenuation. n_{o} is responsible for wave oscillation.

EMPossible Slide 18

18

Related Between γ and n_{eff}

γ and n_{eff} convey the same information and we can calculate one from the other. Comparing our two forms of the solution, we see that

$$\vec{E}(x,y,z) = \vec{A}(x,y)e^{\gamma z} = \vec{A}(x,y)e^{jk_0 n_{\text{eff}} z}$$

$$\Downarrow$$

$$\boxed{\gamma = jk_0 n_{\text{eff}}}$$

We can further relate α to κ and β to n_0 as follows

$$\vec{E}(x,y,z) = \vec{A}(x,y)e^{-\alpha z}e^{j\beta z} = \vec{A}(x,y)e^{-k_0 \kappa z}e^{jk_0 n_{\text{o}} z}$$

$$\Downarrow$$

$$\boxed{\alpha = k_0 \kappa} \qquad \boxed{\beta = k_0 n_{\text{o}}}$$

EMPossible

Slide 19

19

Substitute Solution into Maxwell's Equations

Given the general form for a mode in a waveguide, the fields have the following form

$$\vec{E}(x',y',z') = \vec{A}(x',y')e^{\gamma z'/k_0} \qquad \vec{\tilde{H}}(x',y',z') = \vec{B}(x',y')e^{\gamma z'/k_0}$$

We substitute our solution form into the first of Maxwell's equations.

$$E_z(x',y',z') = \underbrace{A_z(x',y')e^{\gamma z'/k_0}} \qquad E_y(x',y',z') = \underbrace{A_y(x',y')e^{\gamma z'/k_0}} \qquad \tilde{H}_x(x',y',z') = \underbrace{B_x(x',y')e^{\gamma z'/k_0}}$$

$$\frac{\partial \overbrace{E_z}}{\partial y'} - \frac{\partial \overbrace{E_y}}{\partial z'} = \mu_{xx}\overbrace{\tilde{H}_x}$$

$$\frac{\partial}{\partial y'}\left[A_z(x',y')e^{\gamma z'/k_0}\right] - \frac{\partial}{\partial z'}\left[A_y(x',y')e^{\gamma z'/k_0}\right] = \mu_{xx}\left[B_x(x',y')e^{\gamma z'/k_0}\right]$$

$$\frac{\partial A_z(x',y')}{\partial y'}e^{\gamma z'/k_0} - \frac{\gamma}{k_0}A_y(x',y')e^{\gamma z'/k_0} = \mu_{xx}B_x(x',y')e^{\gamma z'/k_0}$$

$$\frac{\partial A_z(x',y')}{\partial y'} - \frac{\gamma}{k_0}A_y(x',y') = \mu_{xx}B_x(x',y') \longrightarrow \boxed{\frac{\partial A_z}{\partial y'} - \frac{\gamma}{k_0}A_y = \mu_{xx}B_x}$$

EMPossible

Slide 20

20

Maxwell's Equations for Waveguides

We can write the remaining equations by analogy

$E_x \leftrightarrow A_x \quad E_y \leftrightarrow A_y \quad E_z \leftrightarrow A_z \qquad \tilde{H}_x \leftrightarrow B_x \quad \tilde{H}_y \leftrightarrow B_y \quad \tilde{H}_z \leftrightarrow B_z \qquad \frac{\partial}{\partial z'} \leftrightarrow \tilde{\gamma} = \frac{\gamma}{k_0}$

$$\boxed{\begin{aligned} \frac{\partial A_z}{\partial y'} - \tilde{\gamma} A_y &= \mu_{xx} B_x \\ \tilde{\gamma} A_x - \frac{\partial A_z}{\partial x'} &= \mu_{yy} B_y \\ \frac{\partial A_y}{\partial x'} - \frac{\partial A_x}{\partial y'} &= \mu_{zz} B_z \end{aligned}} \qquad \boxed{\begin{aligned} \frac{\partial B_z}{\partial y'} - \tilde{\gamma} B_y &= \varepsilon_{xx} A_x \\ \tilde{\gamma} B_x - \frac{\partial B_z}{\partial x'} &= \varepsilon_{yy} A_y \\ \frac{\partial B_y}{\partial x'} - \frac{\partial B_x}{\partial y'} &= \varepsilon_{zz} A_z \end{aligned}}$$

Note: we have normalized the propagation constant according to

$$\tilde{\gamma} = \frac{\gamma}{k_0} \longrightarrow \tilde{\gamma} = jn_{\text{eff}}$$

EMPossible

Matrix Form

We can now write our six equation in matrix form.

$$\begin{aligned} \frac{\partial A_z}{\partial y'} - \tilde{\gamma} A_y &= \mu_{xx} B_x \\ \tilde{\gamma} A_x - \frac{\partial A_z}{\partial x'} &= \mu_{yy} B_y \\ \frac{\partial A_y}{\partial x'} - \frac{\partial A_x}{\partial y'} &= \mu_{zz} B_z \end{aligned} \qquad \begin{aligned} \mathbf{D}_{y'}^{e}\mathbf{a}_z - \tilde{\gamma}\mathbf{a}_y &= \boldsymbol{\mu}_{xx}\mathbf{b}_x \\ \tilde{\gamma}\mathbf{a}_x - \mathbf{D}_{x'}^{e}\mathbf{a}_z &= \boldsymbol{\mu}_{yy}\mathbf{b}_y \\ \mathbf{D}_{x'}^{e}\mathbf{a}_y - \mathbf{D}_{y'}^{e}\mathbf{a}_x &= \boldsymbol{\mu}_{zz}\mathbf{b}_z \end{aligned}$$

$$\Rightarrow$$

$$\begin{aligned} \frac{\partial B_z}{\partial y'} - \tilde{\gamma} B_y &= \varepsilon_{xx} A_x \\ \tilde{\gamma} B_x - \frac{\partial B_z}{\partial x'} &= \varepsilon_{yy} A_y \\ \frac{\partial B_y}{\partial x'} - \frac{\partial B_x}{\partial y'} &= \varepsilon_{zz} A_z \end{aligned} \qquad \begin{aligned} \mathbf{D}_{y'}^{h}\mathbf{b}_z - \tilde{\gamma}\mathbf{b}_y &= \boldsymbol{\varepsilon}_{xx}\mathbf{a}_x \\ \tilde{\gamma}\mathbf{b}_x - \mathbf{D}_{x'}^{h}\mathbf{b}_z &= \boldsymbol{\varepsilon}_{yy}\mathbf{a}_y \\ \mathbf{D}_{x'}^{h}\mathbf{b}_y - \mathbf{D}_{y'}^{h}\mathbf{b}_x &= \boldsymbol{\varepsilon}_{zz}\mathbf{a}_z \end{aligned}$$

Here we use Dirichlet boundary conditions for these derivative operators. This is valid because the energy in the guided modes will be confined to the center of the grid.

EMPossible

Solve for Longitudinal Field Components

We solve the third and sixth equations for the longitudinal components.

$$\mathbf{D}_{y'}^{e}\mathbf{a}_z - \tilde{\gamma}\mathbf{a}_y = \boldsymbol{\mu}_{xx}\mathbf{b}_x$$
$$\tilde{\gamma}\mathbf{a}_x - \mathbf{D}_{x'}^{e}\mathbf{a}_z = \boldsymbol{\mu}_{yy}\mathbf{b}_y$$
$$\mathbf{D}_{x'}^{e}\mathbf{a}_y - \mathbf{D}_{y'}^{e}\mathbf{a}_x = \boldsymbol{\mu}_{zz}\mathbf{b}_z \quad \rightarrow \quad \mathbf{b}_z = \boldsymbol{\mu}_{zz}^{-1}\left(\mathbf{D}_{x'}^{e}\mathbf{a}_y - \mathbf{D}_{y'}^{e}\mathbf{a}_x\right)$$

$$\mathbf{D}_{y'}^{h}\mathbf{b}_z - \tilde{\gamma}\mathbf{b}_y = \boldsymbol{\varepsilon}_{xx}\mathbf{a}_x$$
$$\tilde{\gamma}\mathbf{b}_x - \mathbf{D}_{x'}^{h}\mathbf{b}_z = \boldsymbol{\varepsilon}_{yy}\mathbf{a}_y$$
$$\mathbf{D}_{x'}^{h}\mathbf{b}_y - \mathbf{D}_{y'}^{h}\mathbf{b}_x = \boldsymbol{\varepsilon}_{zz}\mathbf{a}_z \quad \rightarrow \quad \mathbf{a}_z = \boldsymbol{\varepsilon}_{zz}^{-1}\left(\mathbf{D}_{x'}^{h}\mathbf{b}_y - \mathbf{D}_{y'}^{h}\mathbf{b}_x\right)$$

EMPossible Slide 23

23

Eliminate Longitudinal Field Components

Now we substitute the expressions for $\mathbf{a}_z$ and $\mathbf{b}_z$ into the remaining equations.

$$\left.\begin{aligned}\mathbf{D}_{y'}^{e}\mathbf{a}_z - \tilde{\gamma}\mathbf{a}_y &= \boldsymbol{\mu}_{xx}\mathbf{b}_x \\ \tilde{\gamma}\mathbf{a}_x - \mathbf{D}_{x'}^{e}\mathbf{a}_z &= \boldsymbol{\mu}_{yy}\mathbf{b}_y\end{aligned}\right\}$$
$$\mathbf{b}_z = \boldsymbol{\mu}_{zz}^{-1}\left(\mathbf{D}_{x'}^{e}\mathbf{a}_y - \mathbf{D}_{y'}^{e}\mathbf{a}_x\right)$$

$$\left.\begin{aligned}\mathbf{D}_{y'}^{h}\mathbf{b}_z - \tilde{\gamma}\mathbf{b}_y &= \boldsymbol{\varepsilon}_{xx}\mathbf{a}_x \\ \tilde{\gamma}\mathbf{b}_x - \mathbf{D}_{x'}^{h}\mathbf{b}_z &= \boldsymbol{\varepsilon}_{yy}\mathbf{a}_y\end{aligned}\right\}$$
$$\mathbf{a}_z = \boldsymbol{\varepsilon}_{zz}^{-1}\left(\mathbf{D}_{x'}^{h}\mathbf{b}_y - \mathbf{D}_{y'}^{h}\mathbf{b}_x\right)$$

$$\Rightarrow$$

$$\mathbf{D}_{y'}^{e}\boldsymbol{\varepsilon}_{zz}^{-1}\left(\mathbf{D}_{x'}^{h}\mathbf{b}_y - \mathbf{D}_{y'}^{h}\mathbf{b}_x\right) - \tilde{\gamma}\mathbf{a}_y = \boldsymbol{\mu}_{xx}\mathbf{b}_x$$
$$\tilde{\gamma}\mathbf{a}_x - \mathbf{D}_{x'}^{e}\boldsymbol{\varepsilon}_{zz}^{-1}\left(\mathbf{D}_{x'}^{h}\mathbf{b}_y - \mathbf{D}_{y'}^{h}\mathbf{b}_x\right) = \boldsymbol{\mu}_{yy}\mathbf{b}_y$$

$$\mathbf{D}_{y'}^{h}\boldsymbol{\mu}_{zz}^{-1}\left(\mathbf{D}_{x'}^{e}\mathbf{a}_y - \mathbf{D}_{y'}^{e}\mathbf{a}_x\right) - \tilde{\gamma}\mathbf{b}_y = \boldsymbol{\varepsilon}_{xx}\mathbf{a}_x$$
$$\tilde{\gamma}\mathbf{b}_x - \mathbf{D}_{x'}^{h}\boldsymbol{\mu}_{zz}^{-1}\left(\mathbf{D}_{x'}^{e}\mathbf{a}_y - \mathbf{D}_{y'}^{e}\mathbf{a}_x\right) = \boldsymbol{\varepsilon}_{yy}\mathbf{a}_y$$

We now have four equations that just contain the transverse field components E_x, E_y, H_x, and H_y.

EMPossible Slide 24

24

Rearrange the Terms

We rearrange our four equations to put the $\tilde{\gamma}$ term on the right. We also fully expand the equations and collect the common terms that are multiplying the field components.

$$\mathbf{D}_{y'}^{e}\boldsymbol{\varepsilon}_{zz}^{-1}\left(\mathbf{D}_{x'}^{h}\mathbf{b}_{y}-\mathbf{D}_{y'}^{h}\mathbf{b}_{x}\right)-\tilde{\gamma}\mathbf{a}_{y}=\boldsymbol{\mu}_{xx}\mathbf{b}_{x}$$
$$\tilde{\gamma}\mathbf{a}_{x}-\mathbf{D}_{x'}^{e}\boldsymbol{\varepsilon}_{zz}^{-1}\left(\mathbf{D}_{x'}^{h}\mathbf{b}_{y}-\mathbf{D}_{y'}^{h}\mathbf{b}_{x}\right)=\boldsymbol{\mu}_{yy}\mathbf{b}_{y}$$

$$\Rightarrow$$

$$\mathbf{D}_{x'}^{e}\boldsymbol{\varepsilon}_{zz}^{-1}\mathbf{D}_{y'}^{h}\mathbf{b}_{x}-\left(\mathbf{D}_{x'}^{e}\boldsymbol{\varepsilon}_{zz}^{-1}\mathbf{D}_{x'}^{h}+\boldsymbol{\mu}_{yy}\right)\mathbf{b}_{y}=-\tilde{\gamma}\mathbf{a}_{x}$$
$$\left(\mathbf{D}_{y'}^{e}\boldsymbol{\varepsilon}_{zz}^{-1}\mathbf{D}_{y'}^{h}+\boldsymbol{\mu}_{xx}\right)\mathbf{b}_{x}-\mathbf{D}_{y'}^{e}\boldsymbol{\varepsilon}_{zz}^{-1}\mathbf{D}_{x'}^{h}\mathbf{b}_{y}=-\tilde{\gamma}\mathbf{a}_{y}$$

$$\mathbf{D}_{y'}^{h}\boldsymbol{\mu}_{zz}^{-1}\left(\mathbf{D}_{x'}^{e}\mathbf{a}_{y}-\mathbf{D}_{y'}^{e}\mathbf{a}_{x}\right)-\tilde{\gamma}\mathbf{b}_{y}=\boldsymbol{\varepsilon}_{xx}\mathbf{a}_{x}$$
$$\tilde{\gamma}\mathbf{b}_{x}-\mathbf{D}_{x'}^{h}\boldsymbol{\mu}_{zz}^{-1}\left(\mathbf{D}_{x'}^{e}\mathbf{a}_{y}-\mathbf{D}_{y'}^{e}\mathbf{a}_{x}\right)=\boldsymbol{\varepsilon}_{yy}\mathbf{a}_{y}$$

$$\mathbf{D}_{x'}^{h}\boldsymbol{\mu}_{zz}^{-1}\mathbf{D}_{y'}^{e}\mathbf{a}_{x}-\left(\mathbf{D}_{x'}^{h}\boldsymbol{\mu}_{zz}^{-1}\mathbf{D}_{x'}^{e}+\boldsymbol{\varepsilon}_{yy}\right)\mathbf{a}_{y}=-\tilde{\gamma}\mathbf{b}_{x}$$
$$\left(\mathbf{D}_{y'}^{h}\boldsymbol{\mu}_{zz}^{-1}\mathbf{D}_{y'}^{e}+\boldsymbol{\varepsilon}_{xx}\right)\mathbf{a}_{x}-\mathbf{D}_{y'}^{h}\boldsymbol{\mu}_{zz}^{-1}\mathbf{D}_{x'}^{e}\mathbf{a}_{y}=-\tilde{\gamma}\mathbf{b}_{y}$$

Block Matrix Form

Now we can write our four matrix equations in block matrix form.

$$\mathbf{D}_{x'}^{e}\boldsymbol{\varepsilon}_{zz}^{-1}\mathbf{D}_{y'}^{h}\mathbf{b}_{x}-\left(\mathbf{D}_{x'}^{e}\boldsymbol{\varepsilon}_{zz}^{-1}\mathbf{D}_{x'}^{h}+\boldsymbol{\mu}_{yy}\right)\mathbf{b}_{y}=-\tilde{\gamma}\mathbf{a}_{x}$$
$$\left(\mathbf{D}_{y'}^{e}\boldsymbol{\varepsilon}_{zz}^{-1}\mathbf{D}_{y'}^{h}+\boldsymbol{\mu}_{xx}\right)\mathbf{b}_{x}-\mathbf{D}_{y'}^{e}\boldsymbol{\varepsilon}_{zz}^{-1}\mathbf{D}_{x'}^{h}\mathbf{b}_{y}=-\tilde{\gamma}\mathbf{a}_{y}$$

$$\begin{bmatrix}\mathbf{D}_{x'}^{e}\boldsymbol{\varepsilon}_{zz}^{-1}\mathbf{D}_{y'}^{h} & -\left(\mathbf{D}_{x'}^{e}\boldsymbol{\varepsilon}_{zz}^{-1}\mathbf{D}_{x'}^{h}+\boldsymbol{\mu}_{yy}\right)\\ \left(\mathbf{D}_{y'}^{e}\boldsymbol{\varepsilon}_{zz}^{-1}\mathbf{D}_{y'}^{h}+\boldsymbol{\mu}_{xx}\right) & -\mathbf{D}_{y'}^{e}\boldsymbol{\varepsilon}_{zz}^{-1}\mathbf{D}_{x'}^{h}\end{bmatrix}\begin{bmatrix}\mathbf{b}_{x}\\ \mathbf{b}_{y}\end{bmatrix}=-\tilde{\gamma}\begin{bmatrix}\mathbf{a}_{x}\\ \mathbf{a}_{y}\end{bmatrix}$$

$$\mathbf{D}_{x'}^{h}\boldsymbol{\mu}_{zz}^{-1}\mathbf{D}_{y'}^{e}\mathbf{a}_{x}-\left(\mathbf{D}_{x'}^{h}\boldsymbol{\mu}_{zz}^{-1}\mathbf{D}_{x'}^{e}+\boldsymbol{\varepsilon}_{yy}\right)\mathbf{a}_{y}=-\tilde{\gamma}\mathbf{b}_{x}$$
$$\left(\mathbf{D}_{y'}^{h}\boldsymbol{\mu}_{zz}^{-1}\mathbf{D}_{y'}^{e}+\boldsymbol{\varepsilon}_{xx}\right)\mathbf{a}_{x}-\mathbf{D}_{y'}^{h}\boldsymbol{\mu}_{zz}^{-1}\mathbf{D}_{x'}^{e}\mathbf{a}_{y}=-\tilde{\gamma}\mathbf{b}_{y}$$

$$\begin{bmatrix}\mathbf{D}_{x'}^{h}\boldsymbol{\mu}_{zz}^{-1}\mathbf{D}_{y'}^{e} & -\left(\mathbf{D}_{x'}^{h}\boldsymbol{\mu}_{zz}^{-1}\mathbf{D}_{x'}^{e}+\boldsymbol{\varepsilon}_{yy}\right)\\ \left(\mathbf{D}_{y'}^{h}\boldsymbol{\mu}_{zz}^{-1}\mathbf{D}_{y'}^{e}+\boldsymbol{\varepsilon}_{xx}\right) & -\mathbf{D}_{y'}^{h}\boldsymbol{\mu}_{zz}^{-1}\mathbf{D}_{x'}^{e}\end{bmatrix}\begin{bmatrix}\mathbf{a}_{x}\\ \mathbf{a}_{y}\end{bmatrix}=-\tilde{\gamma}\begin{bmatrix}\mathbf{b}_{x}\\ \mathbf{b}_{y}\end{bmatrix}$$

Standard $\mathbf{PQ}$ Form

We can write our block matrix equations in a more compact form as

$$\begin{bmatrix} \mathbf{D}_{x'}^{e}\boldsymbol{\varepsilon}_{zz}^{-1}\mathbf{D}_{y'}^{h} & -\left(\mathbf{D}_{x'}^{e}\boldsymbol{\varepsilon}_{zz}^{-1}\mathbf{D}_{x'}^{h}+\boldsymbol{\mu}_{yy}\right) \\ \left(\mathbf{D}_{y'}^{e}\boldsymbol{\varepsilon}_{zz}^{-1}\mathbf{D}_{y'}^{h}+\boldsymbol{\mu}_{xx}\right) & -\mathbf{D}_{y'}^{e}\boldsymbol{\varepsilon}_{zz}^{-1}\mathbf{D}_{x'}^{h} \end{bmatrix}\begin{bmatrix}\mathbf{b}_x\\ \mathbf{b}_y\end{bmatrix} = -\tilde{\gamma}\begin{bmatrix}\mathbf{a}_x\\ \mathbf{a}_y\end{bmatrix} \Rightarrow \mathbf{P}\begin{bmatrix}\mathbf{b}_x\\ \mathbf{b}_y\end{bmatrix} = -\tilde{\gamma}\begin{bmatrix}\mathbf{a}_x\\ \mathbf{a}_y\end{bmatrix}$$

$$\begin{bmatrix} \mathbf{D}_{x'}^{h}\boldsymbol{\mu}_{zz}^{-1}\mathbf{D}_{y'}^{e} & -\left(\mathbf{D}_{x'}^{h}\boldsymbol{\mu}_{zz}^{-1}\mathbf{D}_{x'}^{e}+\boldsymbol{\varepsilon}_{yy}\right) \\ \left(\mathbf{D}_{y'}^{h}\boldsymbol{\mu}_{zz}^{-1}\mathbf{D}_{y'}^{e}+\boldsymbol{\varepsilon}_{xx}\right) & -\mathbf{D}_{y'}^{h}\boldsymbol{\mu}_{zz}^{-1}\mathbf{D}_{x'}^{e} \end{bmatrix}\begin{bmatrix}\mathbf{a}_x\\ \mathbf{a}_y\end{bmatrix} = -\tilde{\gamma}\begin{bmatrix}\mathbf{b}_x\\ \mathbf{b}_y\end{bmatrix} \Rightarrow \mathbf{Q}\begin{bmatrix}\mathbf{a}_x\\ \mathbf{a}_y\end{bmatrix} = -\tilde{\gamma}\begin{bmatrix}\mathbf{b}_x\\ \mathbf{b}_y\end{bmatrix}$$

$$\mathbf{P} = \begin{bmatrix} \mathbf{D}_{x'}^{e}\boldsymbol{\varepsilon}_{zz}^{-1}\mathbf{D}_{y'}^{h} & -\left(\mathbf{D}_{x'}^{e}\boldsymbol{\varepsilon}_{zz}^{-1}\mathbf{D}_{x'}^{h}+\boldsymbol{\mu}_{yy}\right) \\ \left(\mathbf{D}_{y'}^{e}\boldsymbol{\varepsilon}_{zz}^{-1}\mathbf{D}_{y'}^{h}+\boldsymbol{\mu}_{xx}\right) & -\mathbf{D}_{y'}^{e}\boldsymbol{\varepsilon}_{zz}^{-1}\mathbf{D}_{x'}^{h} \end{bmatrix} \qquad \mathbf{Q} = \begin{bmatrix} \mathbf{D}_{x'}^{h}\boldsymbol{\mu}_{zz}^{-1}\mathbf{D}_{y'}^{e} & -\left(\mathbf{D}_{x'}^{h}\boldsymbol{\mu}_{zz}^{-1}\mathbf{D}_{x'}^{e}+\boldsymbol{\varepsilon}_{yy}\right) \\ \left(\mathbf{D}_{y'}^{h}\boldsymbol{\mu}_{zz}^{-1}\mathbf{D}_{y'}^{e}+\boldsymbol{\varepsilon}_{xx}\right) & -\mathbf{D}_{y'}^{h}\boldsymbol{\mu}_{zz}^{-1}\mathbf{D}_{x'}^{e} \end{bmatrix}$$

Eigen-Value Problem

We now derive a standard eigen-value problem as follows:

$$\mathbf{Q}\begin{bmatrix}\mathbf{a}_x\\ \mathbf{a}_y\end{bmatrix} = -\tilde{\gamma}\begin{bmatrix}\mathbf{b}_x\\ \mathbf{b}_y\end{bmatrix} \qquad \mathbf{P}\begin{bmatrix}\mathbf{b}_x\\ \mathbf{b}_y\end{bmatrix} = -\tilde{\gamma}\begin{bmatrix}\mathbf{a}_x\\ \mathbf{a}_y\end{bmatrix}$$

$$\begin{bmatrix}\mathbf{b}_x\\ \mathbf{b}_y\end{bmatrix} = -\frac{1}{\tilde{\gamma}}\mathbf{Q}\begin{bmatrix}\mathbf{a}_x\\ \mathbf{a}_y\end{bmatrix}$$

Solve first equation for **b**

Substitute expression for **b** into second equation.

$$\mathbf{P}\left(-\frac{1}{\tilde{\gamma}}\mathbf{Q}\begin{bmatrix}\mathbf{a}_x\\ \mathbf{a}_y\end{bmatrix}\right) = -\tilde{\gamma}\begin{bmatrix}\mathbf{a}_x\\ \mathbf{a}_y\end{bmatrix}$$

$$\mathbf{PQ}\begin{bmatrix}\mathbf{a}_x\\ \mathbf{a}_y\end{bmatrix} = \tilde{\gamma}^2\begin{bmatrix}\mathbf{a}_x\\ \mathbf{a}_y\end{bmatrix}$$

$$\boldsymbol{\Omega}^2\begin{bmatrix}\mathbf{a}_x\\ \mathbf{a}_y\end{bmatrix} = \tilde{\gamma}^2\begin{bmatrix}\mathbf{a}_x\\ \mathbf{a}_y\end{bmatrix}$$

$$\boldsymbol{\Omega}^2 = \mathbf{PQ}$$

This is a standard eigen-value problem.

$$\mathbf{Ax} = \lambda\mathbf{x}$$

$$\mathbf{A} = \boldsymbol{\Omega}^2 \qquad \lambda = \tilde{\gamma}^2$$

Summary of Formulation

Start with normalized Maxwell's equations.

$$\frac{\partial E_z}{\partial y'} - \frac{\partial E_y}{\partial z'} = \mu_{xx}\tilde{H}_x$$
$$\frac{\partial E_x}{\partial z'} - \frac{\partial E_z}{\partial x'} = \mu_{yy}\tilde{H}_y$$
$$\frac{\partial E_y}{\partial x'} - \frac{\partial E_x}{\partial y'} = \mu_{zz}\tilde{H}_z$$

$$\frac{\partial \tilde{H}_z}{\partial y'} - \frac{\partial \tilde{H}_y}{\partial z'} = \varepsilon_{xx}E_x$$
$$\frac{\partial \tilde{H}_x}{\partial z'} - \frac{\partial \tilde{H}_z}{\partial x'} = \varepsilon_{yy}E_y$$
$$\frac{\partial \tilde{H}_y}{\partial x'} - \frac{\partial \tilde{H}_x}{\partial y'} = \varepsilon_{zz}E_z$$

$$\frac{\partial A_z}{\partial y'} - \tilde{\gamma}A_y = \mu_{xx}B_x$$
$$\tilde{\gamma}A_x - \frac{\partial A_z}{\partial x'} = \mu_{yy}B_y$$
$$\frac{\partial A_y}{\partial x'} - \frac{\partial A_x}{\partial y'} = \mu_{zz}B_z$$

$$\frac{\partial B_z}{\partial y'} - \tilde{\gamma}B_y = \varepsilon_{xx}A_x$$
$$\tilde{\gamma}B_x - \frac{\partial B_z}{\partial x'} = \varepsilon_{yy}A_y$$
$$\frac{\partial B_y}{\partial x'} - \frac{\partial B_x}{\partial y'} = \varepsilon_{zz}A_z$$

Maxwell's equations with assumed solution.

Maxwell's equations in matrix form.

$$\mathbf{D}_{y'}^{e}\mathbf{a}_z - \tilde{\gamma}\mathbf{a}_y = \boldsymbol{\mu}_{xx}\mathbf{b}_x$$
$$\tilde{\gamma}\mathbf{a}_x - \mathbf{D}_{x'}^{e}\mathbf{a}_z = \boldsymbol{\mu}_{yy}\mathbf{b}_y$$
$$\mathbf{D}_{x'}^{e}\mathbf{a}_y - \mathbf{D}_{y'}^{e}\mathbf{a}_x = \boldsymbol{\mu}_{zz}\mathbf{b}_z$$

$$\mathbf{D}_{y'}^{h}\mathbf{b}_z - \tilde{\gamma}\mathbf{b}_y = \boldsymbol{\varepsilon}_{xx}\mathbf{a}_x$$
$$\tilde{\gamma}\mathbf{b}_x - \mathbf{D}_{x'}^{h}\mathbf{b}_z = \boldsymbol{\varepsilon}_{yy}\mathbf{a}_y$$
$$\mathbf{D}_{x'}^{h}\mathbf{b}_y - \mathbf{D}_{y'}^{h}\mathbf{b}_x = \boldsymbol{\varepsilon}_{zz}\mathbf{a}_z$$

Eliminate longitudinal field components.

$$\mathbf{D}_{y'}^{e}\boldsymbol{\varepsilon}_{zz}^{-1}\left(\mathbf{D}_{x'}^{h}\mathbf{b}_y - \mathbf{D}_{y'}^{h}\mathbf{b}_x\right) - \tilde{\gamma}\mathbf{a}_y = \boldsymbol{\mu}_{xx}\mathbf{b}_x$$
$$\tilde{\gamma}\mathbf{a}_x - \mathbf{D}_{x'}^{e}\boldsymbol{\varepsilon}_{zz}^{-1}\left(\mathbf{D}_{x'}^{h}\mathbf{b}_y - \mathbf{D}_{y'}^{h}\mathbf{b}_x\right) = \boldsymbol{\mu}_{yy}\mathbf{b}_y$$

$$\mathbf{D}_{y'}^{h}\boldsymbol{\mu}_{zz}^{-1}\left(\mathbf{D}_{x'}^{e}\mathbf{a}_y - \mathbf{D}_{y'}^{e}\mathbf{a}_x\right) - \tilde{\gamma}\mathbf{b}_y = \boldsymbol{\varepsilon}_{xx}\mathbf{a}_x$$
$$\tilde{\gamma}\mathbf{b}_x - \mathbf{D}_{x'}^{h}\boldsymbol{\mu}_{zz}^{-1}\left(\mathbf{D}_{x'}^{e}\mathbf{a}_y - \mathbf{D}_{y'}^{e}\mathbf{a}_x\right) = \boldsymbol{\varepsilon}_{yy}\mathbf{a}_y$$

Final eigen-value problem.

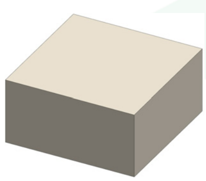
$$\boldsymbol{\Omega}^2\begin{bmatrix}\mathbf{a}_x\\ \mathbf{a}_y\end{bmatrix} = \tilde{\gamma}^2\begin{bmatrix}\mathbf{a}_x\\ \mathbf{a}_y\end{bmatrix}$$
$$\boldsymbol{\Omega}^2 = \mathbf{PQ}$$

$$\mathbf{P} = \begin{bmatrix}\mathbf{D}_{x'}^{e}\boldsymbol{\varepsilon}_{zz}^{-1}\mathbf{D}_{y'}^{h} & -\left(\mathbf{D}_{x'}^{e}\boldsymbol{\varepsilon}_{zz}^{-1}\mathbf{D}_{x'}^{h} + \boldsymbol{\mu}_{yy}\right)\\ \left(\mathbf{D}_{y'}^{e}\boldsymbol{\varepsilon}_{zz}^{-1}\mathbf{D}_{y'}^{h} + \boldsymbol{\mu}_{xx}\right) & -\mathbf{D}_{y'}^{e}\boldsymbol{\varepsilon}_{zz}^{-1}\mathbf{D}_{x'}^{h}\end{bmatrix}$$

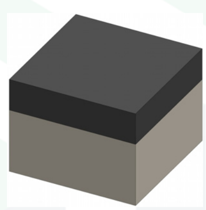
$$\mathbf{Q} = \begin{bmatrix}\mathbf{D}_{x'}^{h}\boldsymbol{\mu}_{zz}^{-1}\mathbf{D}_{y'}^{e} & -\left(\mathbf{D}_{x'}^{h}\boldsymbol{\mu}_{zz}^{-1}\mathbf{D}_{x'}^{e} + \boldsymbol{\varepsilon}_{yy}\right)\\ \left(\mathbf{D}_{y'}^{h}\boldsymbol{\mu}_{zz}^{-1}\mathbf{D}_{y'}^{e} + \boldsymbol{\varepsilon}_{xx}\right) & -\mathbf{D}_{y'}^{h}\boldsymbol{\mu}_{zz}^{-1}\mathbf{D}_{x'}^{e}\end{bmatrix}$$

EMPossible

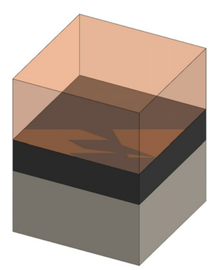
Example – Rib Waveguide (1 of 3)



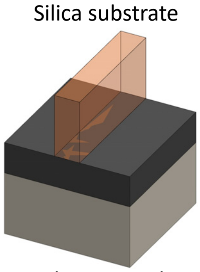
Silica substrate



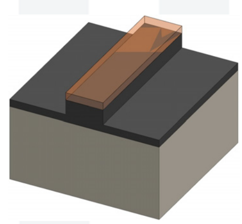
Silica substrate with SiN



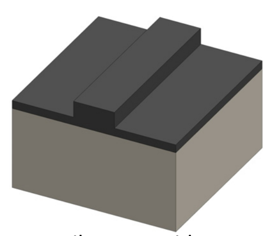
Silica substrate with SiN and photoresist



Silica substrate with SiN and developed photoresist



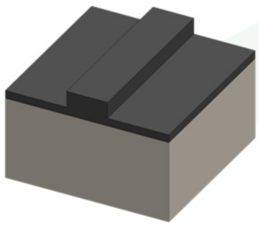
Wafer after etching process



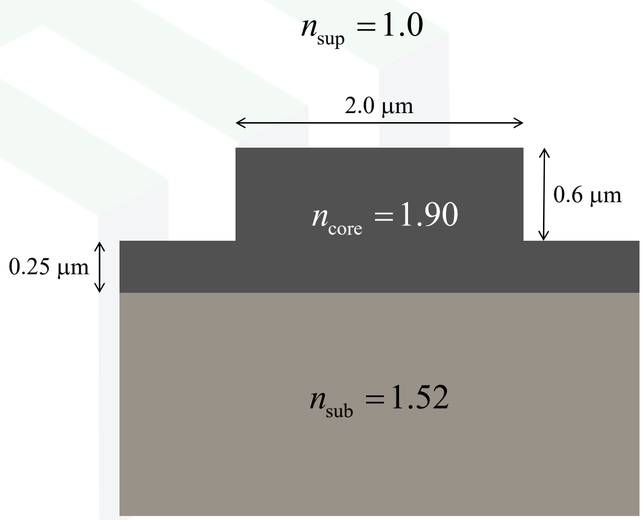
Rib Waveguide

EMPossible

Example – Rib Waveguide (2 of 3)

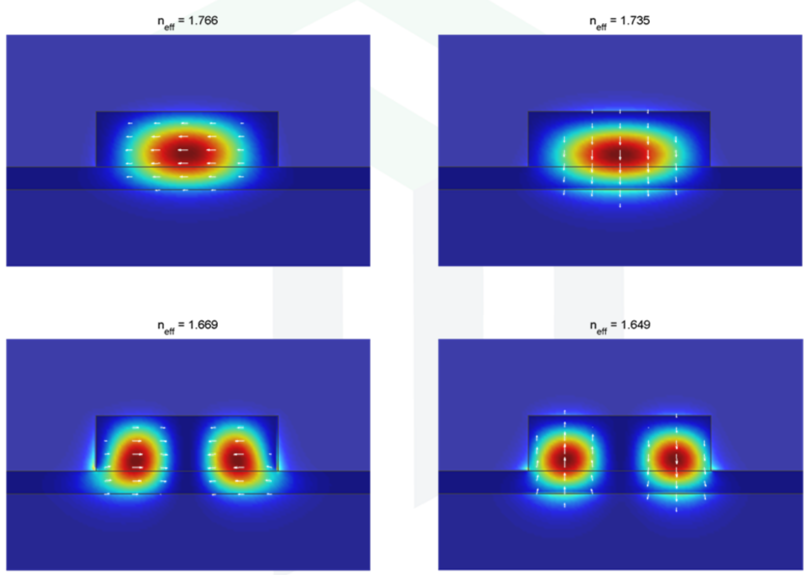


3D View



31

Example – Rib Waveguide (3 of 3)



32

Remarks About Channel Waveguides

- The wave is confined in both transverse directions
- TE and TM modes do not exist in dielectric channel waveguides. Only “hybrid modes” exist.
- Dielectric must be homogeneous, like in metal rectangular waveguide, to support TE and TM modes.
- TEM modes can only exist in transmission lines, which are a special case of multiconductor waveguides.
- Hybrid modes are usually strongly linearly polarized and often components can be ignored to simplify analysis with little loss in accuracy.
  - This leads to quasi-TE and quasi-TM modes

Bonus: Rigorous Finite-Difference Analysis of Anisotropic Waveguides

Tensors

$$\begin{bmatrix} \boldsymbol{\varepsilon}'_{xx} & \boldsymbol{\varepsilon}'_{xy} & \boldsymbol{\varepsilon}'_{xz} \\ \boldsymbol{\varepsilon}'_{yx} & \boldsymbol{\varepsilon}'_{yy} & \boldsymbol{\varepsilon}'_{yz} \\ \boldsymbol{\varepsilon}'_{zx} & \boldsymbol{\varepsilon}'_{zy} & \boldsymbol{\varepsilon}'_{zz} \end{bmatrix} = \begin{bmatrix} \boldsymbol{\varepsilon}_{xx} & \mathbf{R}_x^+\mathbf{R}_y^-\boldsymbol{\varepsilon}_{xy} & \mathbf{R}_x^+\mathbf{R}_z^-\boldsymbol{\varepsilon}_{xz} \\ \mathbf{R}_y^+\mathbf{R}_x^-\boldsymbol{\varepsilon}_{yx} & \boldsymbol{\varepsilon}_{yy} & \mathbf{R}_y^+\mathbf{R}_z^-\boldsymbol{\varepsilon}_{yz} \\ \mathbf{R}_z^+\mathbf{R}_x^-\boldsymbol{\varepsilon}_{zx} & \mathbf{R}_z^+\mathbf{R}_y^-\boldsymbol{\varepsilon}_{zy} & \boldsymbol{\varepsilon}_{zz} \end{bmatrix} \qquad \begin{bmatrix} \boldsymbol{\mu}'_{xx} & \boldsymbol{\mu}'_{xy} & \boldsymbol{\mu}'_{xz} \\ \boldsymbol{\mu}'_{yx} & \boldsymbol{\mu}'_{yy} & \boldsymbol{\mu}'_{yz} \\ \boldsymbol{\mu}'_{zx} & \boldsymbol{\mu}'_{zy} & \boldsymbol{\mu}'_{zz} \end{bmatrix} = \begin{bmatrix} \boldsymbol{\mu}_{xx} & \mathbf{R}_x^-\mathbf{R}_y^+\boldsymbol{\mu}_{xy} & \mathbf{R}_x^-\mathbf{R}_z^+\boldsymbol{\mu}_{xz} \\ \mathbf{R}_y^-\mathbf{R}_x^+\boldsymbol{\mu}_{yx} & \boldsymbol{\mu}_{yy} & \mathbf{R}_y^-\mathbf{R}_z^+\boldsymbol{\mu}_{yz} \\ \mathbf{R}_z^-\mathbf{R}_x^+\boldsymbol{\mu}_{zx} & \mathbf{R}_z^-\mathbf{R}_y^+\boldsymbol{\mu}_{zy} & \boldsymbol{\mu}_{zz} \end{bmatrix}$$

Eigen-Value Problem

$$\mathbf{A}\boldsymbol{\psi} = -\tilde{\gamma}\boldsymbol{\psi} \qquad \boldsymbol{\psi} = \begin{bmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{b}_x & \mathbf{b}_y \end{bmatrix}^T$$

$$\mathbf{A} = \begin{bmatrix} \mathbf{D}_x^e\boldsymbol{\varepsilon}'^{-1}_{zz}\boldsymbol{\varepsilon}'_{zx} + \boldsymbol{\mu}'_{yz}\boldsymbol{\mu}'^{-1}_{zz}\mathbf{D}_y^e & \mathbf{D}_x^e\boldsymbol{\varepsilon}'^{-1}_{zz}\boldsymbol{\varepsilon}'_{zy} - \boldsymbol{\mu}'_{yz}\boldsymbol{\mu}'^{-1}_{zz}\mathbf{D}_x^e & \boldsymbol{\mu}'_{yz}\boldsymbol{\mu}'^{-1}_{zz}\boldsymbol{\mu}'_{zx} - \boldsymbol{\mu}'_{yx} + \mathbf{D}_x^e\boldsymbol{\varepsilon}'^{-1}_{zz}\mathbf{D}_y^h & \boldsymbol{\mu}'_{yz}\boldsymbol{\mu}'^{-1}_{zz}\boldsymbol{\mu}'_{zy} - \boldsymbol{\mu}'_{yy} - \mathbf{D}_x^e\boldsymbol{\varepsilon}'^{-1}_{zz}\mathbf{D}_x^h \\ \mathbf{D}_y^e\boldsymbol{\varepsilon}'^{-1}_{zz}\boldsymbol{\varepsilon}'_{zx} - \boldsymbol{\mu}'_{xz}\boldsymbol{\mu}'^{-1}_{zz}\mathbf{D}_y^e & \boldsymbol{\mu}'_{xz}\boldsymbol{\mu}'^{-1}_{zz}\mathbf{D}_x^e + \mathbf{D}_y^e\boldsymbol{\varepsilon}'^{-1}_{zz}\boldsymbol{\varepsilon}'_{zy} & \boldsymbol{\mu}'_{xx} - \boldsymbol{\mu}'_{xz}\boldsymbol{\mu}'^{-1}_{zz}\boldsymbol{\mu}'_{zx} + \mathbf{D}_y^e\boldsymbol{\varepsilon}'^{-1}_{zz}\mathbf{D}_y^h & \boldsymbol{\mu}'_{xy} - \boldsymbol{\mu}'_{xz}\boldsymbol{\mu}'^{-1}_{zz}\boldsymbol{\mu}'_{zy} - \mathbf{D}_y^e\boldsymbol{\varepsilon}'^{-1}_{zz}\mathbf{D}_x^h \\ \boldsymbol{\varepsilon}'_{yz}\boldsymbol{\varepsilon}'^{-1}_{zz}\boldsymbol{\varepsilon}'_{zx} - \boldsymbol{\varepsilon}'_{yx} + \mathbf{D}_x^h\boldsymbol{\mu}'^{-1}_{zz}\mathbf{D}_y^e & \boldsymbol{\varepsilon}'_{yz}\boldsymbol{\varepsilon}'^{-1}_{zz}\boldsymbol{\varepsilon}'_{zy} - \boldsymbol{\varepsilon}'_{yy} - \mathbf{D}_x^h\boldsymbol{\mu}'^{-1}_{zz}\mathbf{D}_x^e & \mathbf{D}_x^h\boldsymbol{\mu}'^{-1}_{zz}\boldsymbol{\mu}'_{zx} + \boldsymbol{\varepsilon}'_{yz}\boldsymbol{\varepsilon}'^{-1}_{zz}\mathbf{D}_y^h & \mathbf{D}_x^h\boldsymbol{\mu}'^{-1}_{zz}\boldsymbol{\mu}'_{zy} - \boldsymbol{\varepsilon}'_{yz}\boldsymbol{\varepsilon}'^{-1}_{zz}\mathbf{D}_x^h \\ \boldsymbol{\varepsilon}'_{xx} - \boldsymbol{\varepsilon}'_{xz}\boldsymbol{\varepsilon}'^{-1}_{zz}\boldsymbol{\varepsilon}'_{zx} + \mathbf{D}_y^h\boldsymbol{\mu}'^{-1}_{zz}\mathbf{D}_y^e & \boldsymbol{\varepsilon}'_{xy} - \boldsymbol{\varepsilon}'_{xz}\boldsymbol{\varepsilon}'^{-1}_{zz}\boldsymbol{\varepsilon}'_{zy} - \mathbf{D}_y^h\boldsymbol{\mu}'^{-1}_{zz}\mathbf{D}_x^e & \mathbf{D}_y^h\boldsymbol{\mu}'^{-1}_{zz}\boldsymbol{\mu}'_{zx} - \boldsymbol{\varepsilon}'_{xz}\boldsymbol{\varepsilon}'^{-1}_{zz}\mathbf{D}_y^h & \boldsymbol{\varepsilon}'_{xz}\boldsymbol{\varepsilon}'^{-1}_{zz}\mathbf{D}_x^h + \mathbf{D}_y^h\boldsymbol{\mu}'^{-1}_{zz}\boldsymbol{\mu}'_{zy} \end{bmatrix}$$

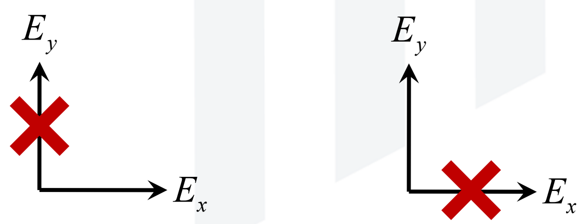
$$\mathbf{A} = \begin{bmatrix} \mathbf{D}_x^e\boldsymbol{\varepsilon}'^{-1}_{zz}\boldsymbol{\varepsilon}'_{zx} & \mathbf{D}_x^e\boldsymbol{\varepsilon}'^{-1}_{zz}\boldsymbol{\varepsilon}'_{zy} & \mathbf{D}_x^e\boldsymbol{\varepsilon}'^{-1}_{zz}\mathbf{D}_y^h & -\mathbf{I} - \mathbf{D}_x^e\boldsymbol{\varepsilon}'^{-1}_{zz}\mathbf{D}_x^h \\ \mathbf{D}_y^e\boldsymbol{\varepsilon}'^{-1}_{zz}\boldsymbol{\varepsilon}'_{zx} & \mathbf{D}_y^e\boldsymbol{\varepsilon}'^{-1}_{zz}\boldsymbol{\varepsilon}'_{zy} & \mathbf{I} + \mathbf{D}_y^e\boldsymbol{\varepsilon}'^{-1}_{zz}\mathbf{D}_y^h & -\mathbf{D}_y^e\boldsymbol{\varepsilon}'^{-1}_{zz}\mathbf{D}_x^h \\ \boldsymbol{\varepsilon}'_{yz}\boldsymbol{\varepsilon}'^{-1}_{zz}\boldsymbol{\varepsilon}'_{zx} - \boldsymbol{\varepsilon}'_{yx} + \mathbf{D}_x^h\mathbf{D}_y^e & \boldsymbol{\varepsilon}'_{yz}\boldsymbol{\varepsilon}'^{-1}_{zz}\boldsymbol{\varepsilon}'_{zy} - \boldsymbol{\varepsilon}'_{yy} - \mathbf{D}_x^h\mathbf{D}_x^e & \boldsymbol{\varepsilon}'_{yz}\boldsymbol{\varepsilon}'^{-1}_{zz}\mathbf{D}_y^h & -\boldsymbol{\varepsilon}'_{yz}\boldsymbol{\varepsilon}'^{-1}_{zz}\mathbf{D}_x^h \\ \boldsymbol{\varepsilon}'_{xx} - \boldsymbol{\varepsilon}'_{xz}\boldsymbol{\varepsilon}'^{-1}_{zz}\boldsymbol{\varepsilon}'_{zx} + \mathbf{D}_y^h\mathbf{D}_y^e & \boldsymbol{\varepsilon}'_{xy} - \boldsymbol{\varepsilon}'_{xz}\boldsymbol{\varepsilon}'^{-1}_{zz}\boldsymbol{\varepsilon}'_{zy} - \mathbf{D}_y^h\mathbf{D}_x^e & -\boldsymbol{\varepsilon}'_{xz}\boldsymbol{\varepsilon}'^{-1}_{zz}\mathbf{D}_y^h & \boldsymbol{\varepsilon}'_{xz}\boldsymbol{\varepsilon}'^{-1}_{zz}\mathbf{D}_x^h \end{bmatrix}$$

No magnetic response

Longitudinal Field Components

$$\mathbf{a}_z = \boldsymbol{\varepsilon}'^{-1}_{zz}\left(\mathbf{D}_x^h\mathbf{b}_y - \mathbf{D}_y^h\mathbf{b}_x - \boldsymbol{\varepsilon}'_{zx}\mathbf{a}_x - \boldsymbol{\varepsilon}'_{zy}\mathbf{a}_y\right) \qquad \mathbf{b}_z = \boldsymbol{\mu}'^{-1}_{zz}\left(\mathbf{D}_x^e\mathbf{a}_y - \mathbf{D}_y^e\mathbf{a}_x - \boldsymbol{\mu}'_{zx}\mathbf{b}_x - \boldsymbol{\mu}'_{zy}\mathbf{b}_y\right)$$

Formulation of Quasi-Vectorial Waveguide Analysis



35

Alternate Form of Full Vector Analysis

Our full vector eigen-value problem can also be written as

$$\begin{bmatrix} \mathbf{\Omega}_{xx}^2 & \mathbf{\Omega}_{xy}^2 \\ \mathbf{\Omega}_{yx}^2 & \mathbf{\Omega}_{yy}^2 \end{bmatrix} \begin{bmatrix} \mathbf{a}_x \\ \mathbf{a}_y \end{bmatrix} = \tilde{\gamma}^2 \begin{bmatrix} \mathbf{a}_x \\ \mathbf{a}_y \end{bmatrix} \qquad \mathbf{\Omega}^2 = \mathbf{PQ} = \begin{bmatrix} \mathbf{\Omega}_{xx}^2 & \mathbf{\Omega}_{xy}^2 \\ \mathbf{\Omega}_{yx}^2 & \mathbf{\Omega}_{yy}^2 \end{bmatrix}$$

$$\mathbf{\Omega}_{xx}^2 = \mathbf{D}_{x'}^e \boldsymbol{\varepsilon}_{zz}^{-1} \mathbf{D}_{y'}^h \mathbf{D}_{x'}^h \boldsymbol{\mu}_{zz}^{-1} \mathbf{D}_{y'}^e - \left(\mathbf{D}_{x'}^e \boldsymbol{\varepsilon}_{zz}^{-1} \mathbf{D}_{x'}^h + \boldsymbol{\mu}_{yy} \right)\left(\mathbf{D}_{y'}^h \boldsymbol{\mu}_{zz}^{-1} \mathbf{D}_{y'}^e + \boldsymbol{\varepsilon}_{xx} \right)$$

$$\mathbf{\Omega}_{xy}^2 = \left(\mathbf{D}_{x'}^e \boldsymbol{\varepsilon}_{zz}^{-1} \mathbf{D}_{x'}^h + \boldsymbol{\mu}_{yy} \right) \mathbf{D}_{y'}^h \boldsymbol{\mu}_{zz}^{-1} \mathbf{D}_{x'}^e - \mathbf{D}_{x'}^e \boldsymbol{\varepsilon}_{zz}^{-1} \mathbf{D}_{y'}^h \left(\mathbf{D}_{x'}^h \boldsymbol{\mu}_{zz}^{-1} \mathbf{D}_{x'}^e + \boldsymbol{\varepsilon}_{yy} \right)$$

$$\mathbf{\Omega}_{yx}^2 = \left(\mathbf{D}_{y'}^e \boldsymbol{\varepsilon}_{zz}^{-1} \mathbf{D}_{y'}^h + \boldsymbol{\mu}_{xx} \right) \mathbf{D}_{x'}^h \boldsymbol{\mu}_{zz}^{-1} \mathbf{D}_{y'}^e - \mathbf{D}_{y'}^e \boldsymbol{\varepsilon}_{zz}^{-1} \mathbf{D}_{x'}^h \left(\mathbf{D}_{y'}^h \boldsymbol{\mu}_{zz}^{-1} \mathbf{D}_{y'}^e + \boldsymbol{\varepsilon}_{xx} \right)$$

$$\mathbf{\Omega}_{yy}^2 = \mathbf{D}_{y'}^e \boldsymbol{\varepsilon}_{zz}^{-1} \mathbf{D}_{x'}^h \mathbf{D}_{y'}^h \boldsymbol{\mu}_{zz}^{-1} \mathbf{D}_{x'}^e - \left(\mathbf{D}_{y'}^e \boldsymbol{\varepsilon}_{zz}^{-1} \mathbf{D}_{y'}^h + \boldsymbol{\mu}_{xx} \right)\left(\mathbf{D}_{x'}^h \boldsymbol{\mu}_{zz}^{-1} \mathbf{D}_{x'}^e + \boldsymbol{\varepsilon}_{yy} \right)$$

36

Two Coupled Matrix Equations

Our alternate full-vector eigen-value problem can be written as two coupled matrix equations.

$$\begin{bmatrix} \mathbf{\Omega}_{xx}^2 & \mathbf{\Omega}_{xy}^2 \\ \mathbf{\Omega}_{yx}^2 & \mathbf{\Omega}_{yy}^2 \end{bmatrix} \begin{bmatrix} \mathbf{a}_x \\ \mathbf{a}_y \end{bmatrix} = \tilde{\gamma}^2 \begin{bmatrix} \mathbf{a}_x \\ \mathbf{a}_y \end{bmatrix}$$

$$\mathbf{\Omega}_{xx}^2 \mathbf{a}_x + \mathbf{\Omega}_{xy}^2 \mathbf{a}_y = \tilde{\gamma}^2 \mathbf{a}_x \qquad \mathbf{\Omega}_{yx}^2 \mathbf{a}_x + \mathbf{\Omega}_{yy}^2 \mathbf{a}_y = \tilde{\gamma}^2 \mathbf{a}_y$$

Cross coupling between $\mathbf{a}_x$ and $\mathbf{a}_y$.

Cross coupling between $\mathbf{a}_y$ and $\mathbf{a}_x$.

Self-coupling term for $\mathbf{a}_x$.

Self-coupling term for $\mathbf{a}_y$.

EMPossible

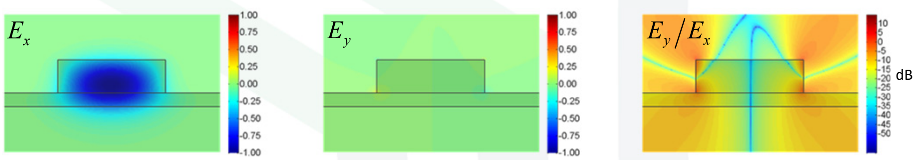
Slide 37

37

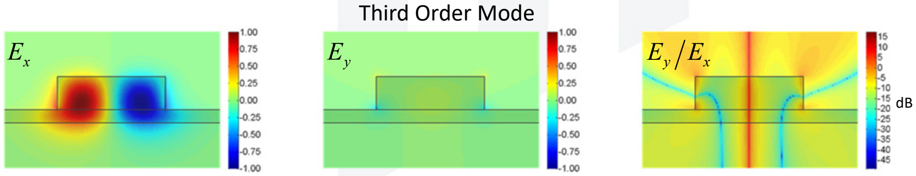
Strong Linear Polarization

Observe how strongly linearly polarized the modes are…

First Order Mode



Third Order Mode



EMPossible

Slide 38

38

Quasi-Vectorial Approximation

When the modes are strongly linearly polarized along x or y, it is a good approximation to neglect the cross-coupling terms.

$$\mathbf{\Omega}_{xx}^{2}\mathbf{a}_{x}+\cancel{\mathbf{\Omega}_{xy}^{2}\mathbf{a}_{y}}=\tilde{\gamma}^{2}\mathbf{a}_{x} \qquad \cancel{\mathbf{\Omega}_{yx}^{2}\mathbf{a}_{x}}+\mathbf{\Omega}_{yy}^{2}\mathbf{a}_{y}=\tilde{\gamma}^{2}\mathbf{a}_{y}$$

We now have two independent eigen-value problems that can be solved independently.

E_x Polarized Mode

$$\mathbf{\Omega}_{xx}^{2}\mathbf{a}_{x}=\tilde{\gamma}^{2}\mathbf{a}_{x}$$

$$\mathbf{\Omega}_{xx}^{2}=\mathbf{D}_{x'}^{e}\boldsymbol{\varepsilon}_{zz}^{-1}\mathbf{D}_{y'}^{h}\mathbf{D}_{x'}^{h}\boldsymbol{\mu}_{zz}^{-1}\mathbf{D}_{y'}^{e}-\left(\mathbf{D}_{x'}^{e}\boldsymbol{\varepsilon}_{zz}^{-1}\mathbf{D}_{x'}^{h}+\boldsymbol{\mu}_{yy}\right)\left(\mathbf{D}_{y'}^{h}\boldsymbol{\mu}_{zz}^{-1}\mathbf{D}_{y'}^{e}+\boldsymbol{\varepsilon}_{xx}\right)$$

E_y Polarized Mode

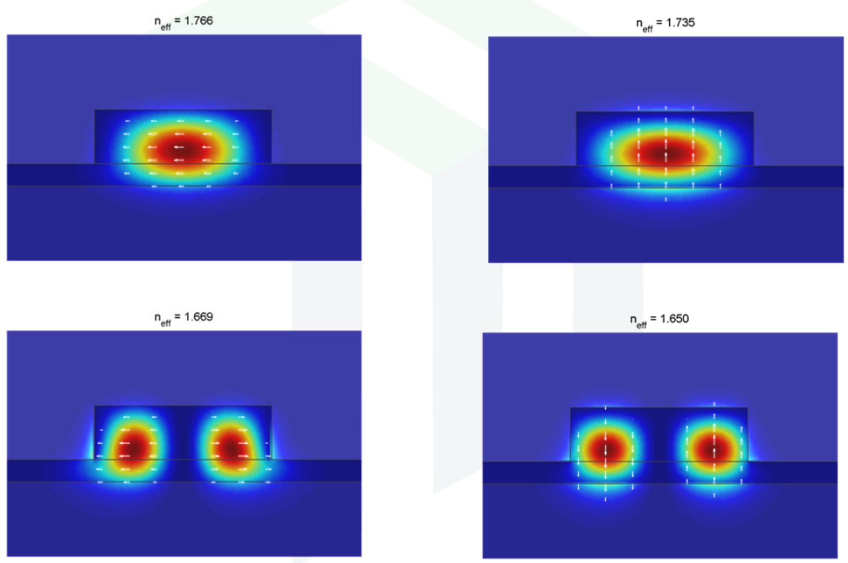
$$\mathbf{\Omega}_{yy}^{2}\mathbf{a}_{y}=\tilde{\gamma}^{2}\mathbf{a}_{y}$$

$$\mathbf{\Omega}_{yy}^{2}=\mathbf{D}_{y'}^{e}\boldsymbol{\varepsilon}_{zz}^{-1}\mathbf{D}_{x'}^{h}\mathbf{D}_{y'}^{h}\boldsymbol{\mu}_{zz}^{-1}\mathbf{D}_{x'}^{e}-\left(\mathbf{D}_{y'}^{e}\boldsymbol{\varepsilon}_{zz}^{-1}\mathbf{D}_{y'}^{h}+\boldsymbol{\mu}_{xx}\right)\left(\mathbf{D}_{x'}^{h}\boldsymbol{\mu}_{zz}^{-1}\mathbf{D}_{x'}^{e}+\boldsymbol{\varepsilon}_{yy}\right)$$

EMPossible Slide 39

39

Example – Same Rib Waveguide

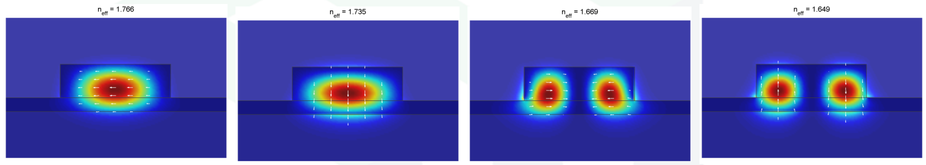


EMPossible Slide 40

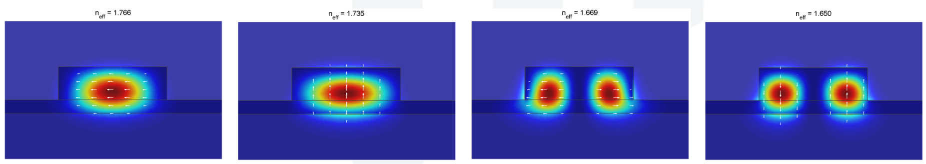
40

Full-Vector Vs. Quasi-Vectorial

Full-Vector Analysis (12 second run time @ $\lambda/30$ resolution)



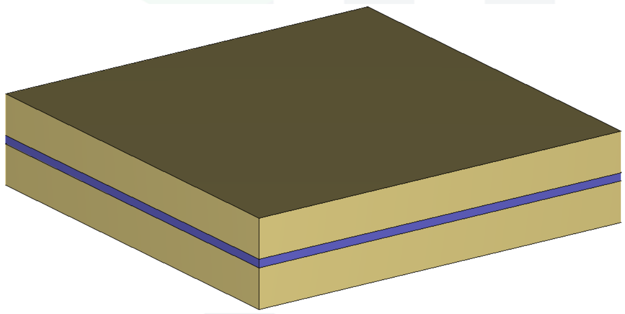
Quasi-Vectorial Analysis (7 second run time @ $\lambda/30$ resolution)



Remarks About Quasi-Vectorial Analysis

- Quasi-vectorial analysis is an approximation.
- Quasi-TE and quasi-TM modes do not exist.
- For many waveguides, this is an extremely good approximation.

Formulation of Slab Waveguide Analysis

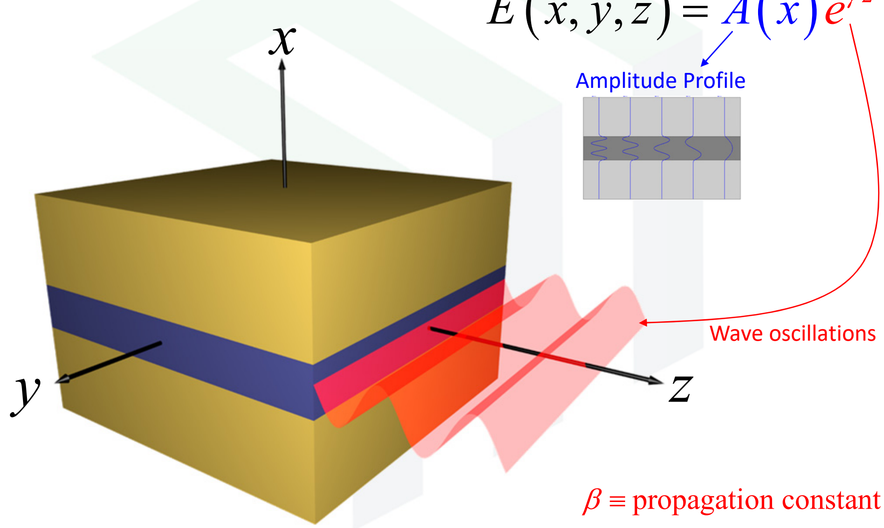


Slide 43

43

Mathematical Form of Solution

$$\vec{E}(x,y,z)=\vec{A}(x)e^{\gamma z}$$



$\beta \equiv$ propagation constant

EMPossible

Slide 44

44

Maxwell's Equations for Slab Waveguides

For slab waveguides, the device is uniform along the y direction. Therefore, the field is uniform as well and

$$\frac{\partial}{\partial y'} = 0$$

Our six waveguide equations reduce to

$$\cancel{\frac{\partial A_z}{\partial y'}} - \tilde{\gamma} A_y = \mu_{xx} B_x \qquad \cancel{\frac{\partial B_z}{\partial y'}} - \tilde{\gamma} B_y = \varepsilon_{xx} A_x$$

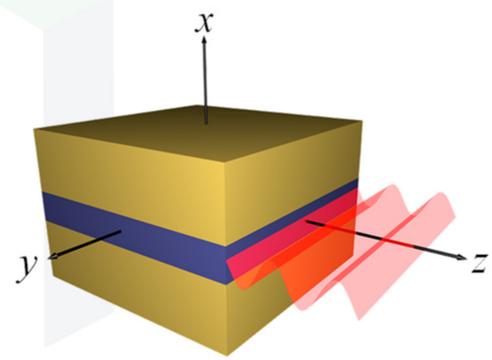
$$\tilde{\gamma} A_x - \frac{\partial A_z}{\partial x'} = \mu_{yy} B_y \qquad \tilde{\gamma} B_x - \frac{\partial B_z}{\partial x'} = \varepsilon_{yy} A_y$$

$$\frac{\partial A_y}{\partial x'} - \cancel{\frac{\partial A_x}{\partial y'}} = \mu_{zz} B_z \qquad \frac{\partial B_y}{\partial x'} - \cancel{\frac{\partial B_x}{\partial y'}} = \varepsilon_{zz} A_z$$

$$-\tilde{\gamma} A_y = \mu_{xx} B_x \qquad -\tilde{\gamma} B_y = \varepsilon_{xx} A_x$$

$$\tilde{\gamma} A_x - \frac{\partial A_z}{\partial x'} = \mu_{yy} B_y \qquad \tilde{\gamma} B_x - \frac{\partial B_z}{\partial x'} = \varepsilon_{yy} A_y$$

$$\frac{\partial A_y}{\partial x'} = \mu_{zz} B_z \qquad \frac{\partial B_y}{\partial x'} = \varepsilon_{zz} A_z$$



Two Independent Modes

Our six equations have decoupled into two distinct modes.

E Mode

$$\tilde{\gamma} B_x - \frac{\partial B_z}{\partial x'} = \varepsilon_{yy} A_y$$

$$-\tilde{\gamma} A_y = \mu_{xx} B_x$$

$$\frac{\partial A_y}{\partial x'} = \mu_{zz} B_z$$

H Mode

$$\tilde{\gamma} A_x - \frac{\partial A_z}{\partial x'} = \mu_{yy} B_y$$

$$-\tilde{\gamma} B_y = \varepsilon_{xx} A_x$$

$$\frac{\partial B_y}{\partial x'} = \varepsilon_{zz} A_z$$

Note: In contrast to the quasi-vectorial analysis which used an approximation to split Maxwell's equations into two modes, Maxwell's equations rigorously split into two modes for slab waveguides.

Matrix Form

We can write our six equations in matrix form as

E Mode

$$\tilde{\gamma} B_x - \frac{\partial B_z}{\partial x'} = \varepsilon_{yy} A_y$$
$$-\tilde{\gamma} A_y = \mu_{xx} B_x$$
$$\frac{\partial A_y}{\partial x'} = \mu_{zz} B_z$$

$$\tilde{\gamma}\mathbf{b}_x - \mathbf{D}_{x'}^{h}\mathbf{b}_z = \boldsymbol{\varepsilon}_{yy}\mathbf{a}_y$$
$$-\tilde{\gamma}\mathbf{a}_y = \boldsymbol{\mu}_{xx}\mathbf{b}_x$$
$$\mathbf{D}_{x'}^{e}\mathbf{a}_y = \boldsymbol{\mu}_{zz}\mathbf{b}_z$$

H Mode

$$\tilde{\gamma} A_x - \frac{\partial A_z}{\partial x'} = \mu_{yy} B_y$$
$$-\tilde{\gamma} B_y = \varepsilon_{xx} A_x$$
$$\frac{\partial B_y}{\partial x'} = \varepsilon_{zz} A_z$$

$$\tilde{\gamma}\mathbf{a}_x - \mathbf{D}_{x'}^{e}\mathbf{a}_z = \boldsymbol{\mu}_{yy}\mathbf{b}_y$$
$$-\tilde{\gamma}\mathbf{b}_y = \boldsymbol{\varepsilon}_{xx}\mathbf{a}_x$$
$$\mathbf{D}_{x'}^{h}\mathbf{b}_y = \boldsymbol{\varepsilon}_{zz}\mathbf{a}_z$$

EMPossible Slide 47

47

Two Eigen-Value Problems

We can formulate two matrix wave equations by solving the last two equations for the x and z components and substituting those expressions into the first equations.

E Mode

$$\tilde{\gamma}\mathbf{b}_x - \mathbf{D}_{x'}^{h}\mathbf{b}_z = \boldsymbol{\varepsilon}_{yy}\mathbf{a}_y$$
$$-\tilde{\gamma}\mathbf{a}_y = \boldsymbol{\mu}_{xx}\mathbf{b}_x \rightarrow \mathbf{b}_x = -\tilde{\gamma}\boldsymbol{\mu}_{xx}^{-1}\mathbf{a}_y$$
$$\mathbf{D}_{x'}^{e}\mathbf{a}_y = \boldsymbol{\mu}_{zz}\mathbf{b}_z \rightarrow \mathbf{b}_z = \boldsymbol{\mu}_{zz}^{-1}\mathbf{D}_{x'}^{e}\mathbf{a}_y$$

$$-\left(\mathbf{D}_{x'}^{h}\boldsymbol{\mu}_{zz}^{-1}\mathbf{D}_{x'}^{e} + \boldsymbol{\varepsilon}_{yy}\right)\mathbf{a}_y = \tilde{\gamma}^2\boldsymbol{\mu}_{xx}^{-1}\mathbf{a}_y$$

H Mode

$$\tilde{\gamma}\mathbf{a}_x - \mathbf{D}_{x'}^{e}\mathbf{a}_z = \boldsymbol{\mu}_{yy}\mathbf{b}_y$$
$$-\tilde{\gamma}\mathbf{b}_y = \boldsymbol{\varepsilon}_{xx}\mathbf{a}_x \rightarrow \mathbf{a}_x = -\tilde{\gamma}\boldsymbol{\varepsilon}_{xx}^{-1}\mathbf{b}_y$$
$$\mathbf{D}_{x'}^{h}\mathbf{b}_y = \boldsymbol{\varepsilon}_{zz}\mathbf{a}_z \rightarrow \mathbf{a}_z = \boldsymbol{\varepsilon}_{zz}^{-1}\mathbf{D}_{x'}^{h}\mathbf{b}_y$$

$$-\left(\mathbf{D}_{x'}^{e}\boldsymbol{\varepsilon}_{zz}^{-1}\mathbf{D}_{x'}^{h} + \boldsymbol{\mu}_{yy}\right)\mathbf{b}_y = \tilde{\gamma}^2\boldsymbol{\varepsilon}_{xx}^{-1}\mathbf{b}_y$$

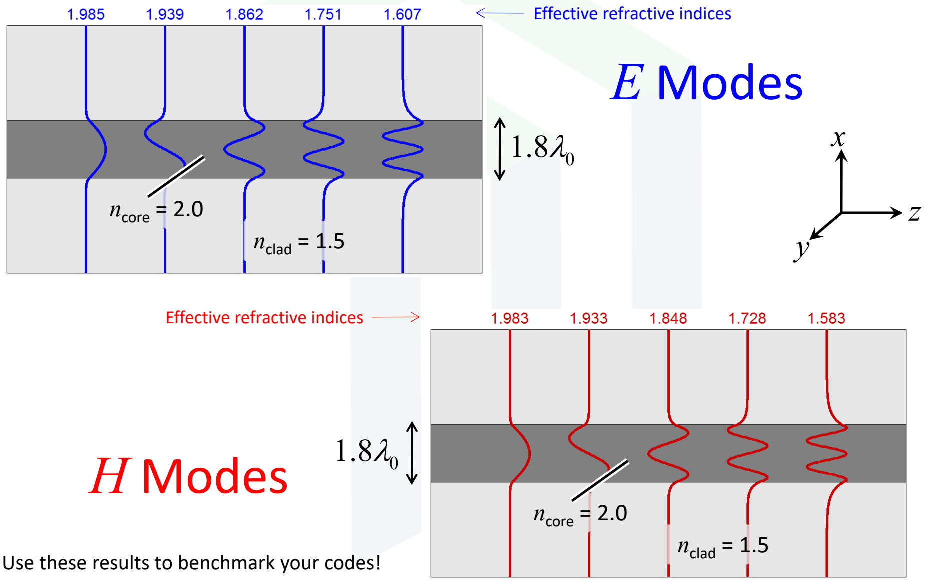
These equations are generalized eigen-value problems.

$$\mathbf{Ax} = \lambda\mathbf{Bx}$$

EMPossible Slide 48

48

Typical Modes in a Slab Waveguide



Use these results to benchmark your codes!

EMPossible

Slide 49

49

Remarks About Slab Waveguide Analysis

- Waves are confined in only one transverse direction.
- Waves are free to spread out in the uniform transverse direction
- Propagation within the slab can be restricted to a single direction without loss of generality.
- Maxwell's equations rigorously decouple into two distinct modes.
- No approximations are necessary

EMPossible

Slide 50

50

Implementation

Slide 51

51

Summary of Formulations

Full Vector Analysis

$$\boldsymbol{\Omega}^2 \begin{bmatrix} \mathbf{a}_x \\ \mathbf{a}_y \end{bmatrix} = \tilde{\gamma}^2 \begin{bmatrix} \mathbf{a}_x \\ \mathbf{a}_y \end{bmatrix}$$

$$\boldsymbol{\Omega}^2 = \mathbf{PQ}$$

$$\mathbf{P} = \begin{bmatrix} \mathbf{D}_{x'}^{e}\boldsymbol{\varepsilon}_{zz}^{-1}\mathbf{D}_{y'}^{h} & -\left(\mathbf{D}_{x'}^{e}\boldsymbol{\varepsilon}_{zz}^{-1}\mathbf{D}_{x'}^{h} + \boldsymbol{\mu}_{yy}\right) \\ \left(\mathbf{D}_{y'}^{e}\boldsymbol{\varepsilon}_{zz}^{-1}\mathbf{D}_{y'}^{h} + \boldsymbol{\mu}_{xx}\right) & -\mathbf{D}_{y'}^{e}\boldsymbol{\varepsilon}_{zz}^{-1}\mathbf{D}_{x'}^{h} \end{bmatrix}$$

$$\mathbf{Q} = \begin{bmatrix} \mathbf{D}_{x'}^{h}\boldsymbol{\mu}_{zz}^{-1}\mathbf{D}_{y'}^{e} & -\left(\mathbf{D}_{x'}^{h}\boldsymbol{\mu}_{zz}^{-1}\mathbf{D}_{x'}^{e} + \boldsymbol{\varepsilon}_{yy}\right) \\ \left(\mathbf{D}_{y'}^{h}\boldsymbol{\mu}_{zz}^{-1}\mathbf{D}_{y'}^{e} + \boldsymbol{\varepsilon}_{xx}\right) & -\mathbf{D}_{y'}^{h}\boldsymbol{\mu}_{zz}^{-1}\mathbf{D}_{x'}^{e} \end{bmatrix}$$

Quasi-Vectorial Analysis

E_x Mode: $\boldsymbol{\Omega}_{xx}^2\mathbf{a}_x = \tilde{\gamma}^2\mathbf{a}_x$ $\quad \boldsymbol{\Omega}_{xx}^2 = \mathbf{D}_{x'}^{e}\boldsymbol{\varepsilon}_{zz}^{-1}\mathbf{D}_{y'}^{h}\mathbf{D}_{x'}^{h}\boldsymbol{\mu}_{zz}^{-1}\mathbf{D}_{y'}^{e} - \left(\mathbf{D}_{x'}^{e}\boldsymbol{\varepsilon}_{zz}^{-1}\mathbf{D}_{x'}^{h} + \boldsymbol{\mu}_{yy}\right)\left(\mathbf{D}_{y'}^{h}\boldsymbol{\mu}_{zz}^{-1}\mathbf{D}_{y'}^{e} + \boldsymbol{\varepsilon}_{xx}\right)$

E_y Mode: $\boldsymbol{\Omega}_{yy}^2\mathbf{a}_y = \tilde{\gamma}^2\mathbf{a}_y$ $\quad \boldsymbol{\Omega}_{yy}^2 = \mathbf{D}_{y'}^{e}\boldsymbol{\varepsilon}_{zz}^{-1}\mathbf{D}_{x'}^{h}\mathbf{D}_{y'}^{h}\boldsymbol{\mu}_{zz}^{-1}\mathbf{D}_{x'}^{e} - \left(\mathbf{D}_{y'}^{e}\boldsymbol{\varepsilon}_{zz}^{-1}\mathbf{D}_{y'}^{h} + \boldsymbol{\mu}_{xx}\right)\left(\mathbf{D}_{x'}^{h}\boldsymbol{\mu}_{zz}^{-1}\mathbf{D}_{x'}^{e} + \boldsymbol{\varepsilon}_{yy}\right)$

Slab Waveguide Analysis

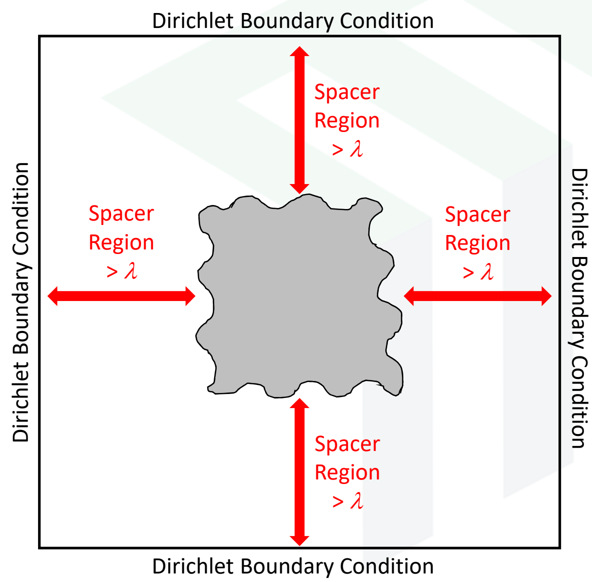
E Mode: $-\left(\mathbf{D}_{x'}^{h}\boldsymbol{\mu}_{zz}^{-1}\mathbf{D}_{x'}^{e} + \boldsymbol{\varepsilon}_{yy}\right)\mathbf{a}_y = \tilde{\gamma}^2\boldsymbol{\mu}_{xx}^{-1}\mathbf{a}_y$ $\quad$ H Mode: $-\left(\mathbf{D}_{x'}^{e}\boldsymbol{\varepsilon}_{zz}^{-1}\mathbf{D}_{x'}^{h} + \boldsymbol{\mu}_{yy}\right)\mathbf{b}_y = \tilde{\gamma}^2\boldsymbol{\varepsilon}_{xx}^{-1}\mathbf{b}_y$

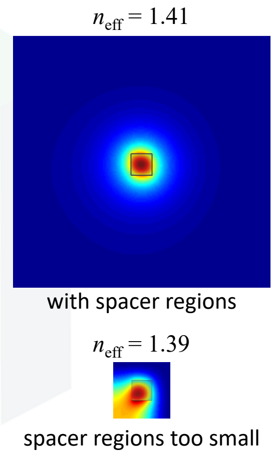
EMPossible

Slide 52

52

Grid Scheme





The spacer region provides enough room that the fields decay to almost zero before reaching the boundary where we have implemented Dirichlet boundary conditions.

EMPossible

Slide 53

53

Solution in MATLAB Using `eig()`

We can use MATLAB's built-in `eig()` function to solve this eigen-value problem for <u>all</u> possible modes.

```
[V,D] = eig(A,B);
```

The solution can be interpreted as

$$\mathbf{V} = \begin{bmatrix} \begin{bmatrix} \tilde{E}_y^{(1)}(1) \\ \tilde{E}_y^{(1)}(2) \\ \tilde{E}_y^{(1)}(3) \\ \vdots \\ \tilde{E}_y^{(1)}(N_x-1) \\ \tilde{E}_y^{(1)}(N_x) \end{bmatrix} & \begin{bmatrix} \tilde{E}_y^{(2)}(1) \\ \tilde{E}_y^{(2)}(2) \\ \tilde{E}_y^{(2)}(3) \\ \vdots \\ \tilde{E}_y^{(2)}(N_x-1) \\ \tilde{E}_y^{(2)}(N_x) \end{bmatrix} & \cdots & \begin{bmatrix} \tilde{E}_y^{(M)}(1) \\ \tilde{E}_y^{(M)}(2) \\ \tilde{E}_y^{(M)}(3) \\ \vdots \\ \tilde{E}_y^{(M)}(N_x-1) \\ \tilde{E}_y^{(M)}(N_x) \end{bmatrix} \end{bmatrix}$$

$$\mathbf{D} = \begin{bmatrix} \tilde{\gamma}_1^2 & & & \\ & \tilde{\gamma}_2^2 & & \\ & & \ddots & \\ & & & \tilde{\gamma}_M^2 \end{bmatrix}$$

The eigen-vectors describe the amplitude profile of the modes.

$$\tilde{E}_y(x) \cdot e^{\tilde{\gamma} z'}$$

The eigen-values describe attenuation and the accumulation of phase.

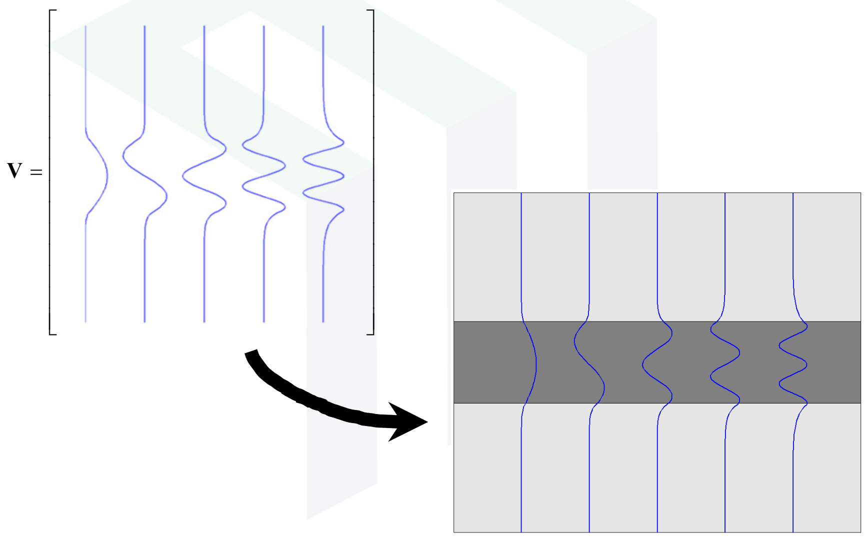
EMPossible

Slide 54

54

Concept of the Eigen-Vector Matrix

The columns of the eigen-vector matrix are the "modes" of the waveguide.



Solution in MATLAB Using `eigs()`

Typically we do NOT want to calculate all of the eigen-modes. This would take a prohibitively long time and most of the solutions will have no meaning to a waveguide problem.

We need to control MATLAB so as to calculate only the guided modes. We do this by telling MATLAB to calculate all the modes with eigen-values close to some estimated effective refractive index. A good estimate is something slightly less than the refractive index of the core.

$$n_{\text{eff}}\Big|_{\text{guess}} \approx n_{\text{core}}$$

This implies our guess at the complex propagation constant is

$$\tilde{\gamma}\Big|_{\text{guess}} \approx j\, n_{\text{eff}}\Big|_{\text{guess}} = jn_{\text{core}}$$

$$\tilde{\gamma}^2\Big|_{\text{guess}} \approx -n_{\text{core}}^2$$

```
% SOLVE EIGEN-VALUE PROBLEM
% NSOL is the number of solutions
[V,D] = eigs(OMEGA_SQ,NSOL,-ncore^2);
```

Calculating the Meaningful Parameters

This step can be tricky due to maintain proper signs with the various complex numbers. The eigen-value problem returns $\tilde{\gamma}_i^2$.

The effective refractive index is

$$\tilde{\gamma}_i^2 = -n_{\text{eff},i}^2 \quad \rightarrow \quad \boxed{n_{\text{eff},i} = \sqrt{-\tilde{\gamma}_i^2}}$$

The complex propagation constant is

$$\gamma_i = k_0 \tilde{\gamma}_i = j k_0 n_{\text{eff},i} = j k_0 \sqrt{-\tilde{\gamma}_i^2} \quad \rightarrow \quad \boxed{\gamma_i = -k_0 \sqrt{\tilde{\gamma}_i^2}}$$

The attenuation coefficient α and phase constant β are

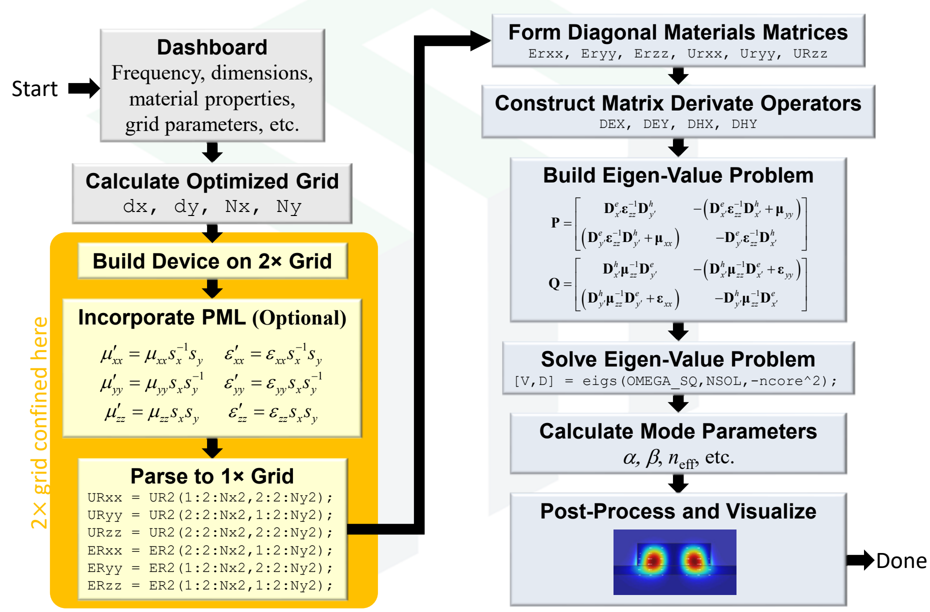
$$\begin{aligned} \gamma_i &= j k_0 n_{\text{eff},i} = -\alpha_i + j\beta_i \\ n_{\text{eff},i} &= n_{\text{o},i} + j\kappa_i \end{aligned} \quad \rightarrow \quad \begin{aligned} &\boxed{\alpha_i = -\operatorname{Re}\left[\gamma_i\right] = k_0 \operatorname{Im}\left[n_{\text{eff},i}\right] = k_0 \kappa_i} \\ &\boxed{\beta_i = \operatorname{Im}\left[\gamma_i\right] = k_0 \operatorname{Re}\left[n_{\text{eff},i}\right] = k_0 n_{\text{o},i}} \end{aligned}$$

```
% CALCULATE MEANINGFUL
% PARAMETERS
neff  = sqrt(-D);
gamma = -k0*sqrt(D);
no    = real(neff);
kappa = imag(neff);
alpha = -real(gamma);
beta  = imag(gamma);
```

EMPossible Slide 57

57

Block Diagram of Waveguide Analysis

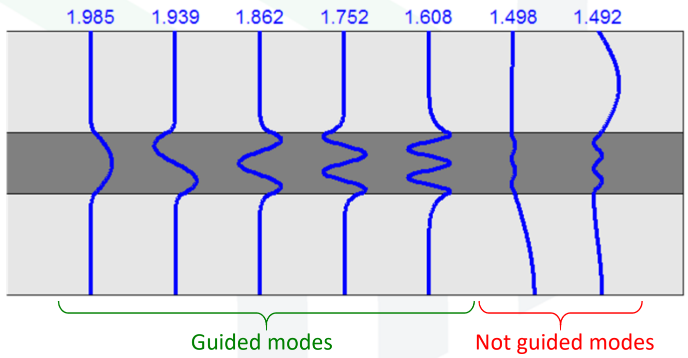


EMPossible Slide 58

58

Identifying Guided Modes (1 of 2)

Slab Waveguides



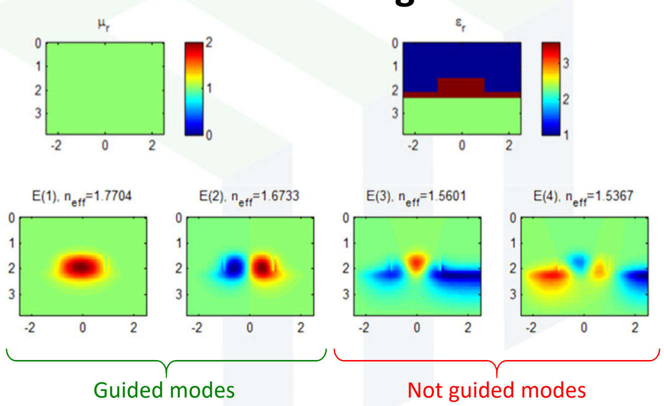
The guided modes are confined to the waveguide and approach zero well before the boundaries.

EMPossible Slide 59

59

Identifying Guided Modes (2 of 2)

Channel Waveguides



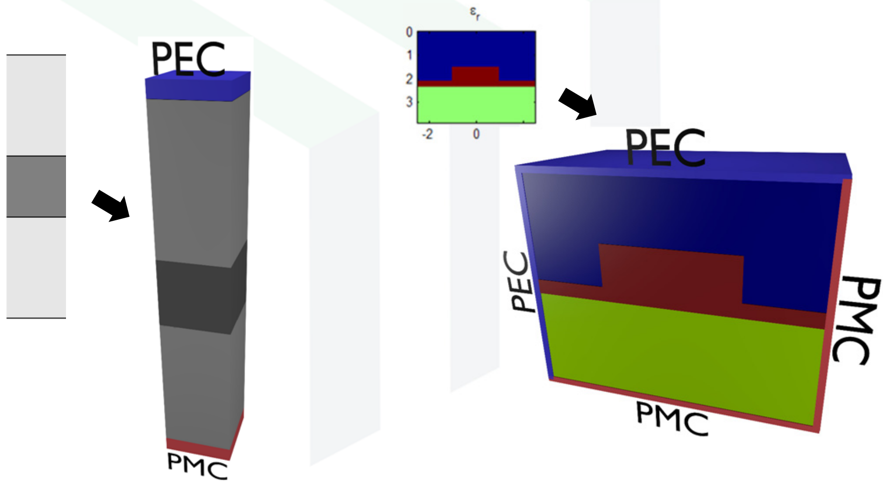
The guided modes are confined to the waveguide and approach zero well before the boundaries.

EMPossible Slide 60

60

Origin of the "Not Guided Modes"

Remember, we used Dirichlet boundary conditions for this analysis. This forces the electric field to zero (PEC) at the x-lo and y-lo boundaries and forces the magnetic field to zero (PMC) at the x-hi and y-hi boundaries. We are actually modeling huge metallic waveguides stuffed with dielectric structures. The "not guided modes" are higher-order modes of the huge metal waveguide.



EMPossible

Slide 61

61

Transmission Line Analysis

Slide 62

62

Calculating Voltage on Line

To calculate the voltage across the line, perform a line integration from conductor to conductor.

$$V_0 = \int_a^b \vec{E} \bullet d\vec{\ell}$$

Calculating Current on Line

To calculate the current in the line, perform a close-contour line integration around one of the conductors.

$$I_0 = \oint_L \vec{H} \bullet d\vec{\ell}$$

Characteristic Impedance, Z_0

The characteristic impedance Z_0 is simply

$$Z_0 = \frac{V_0}{I_0}$$

Distributed Parameters R, L, G, and C

In the positive sign convention, we have

$$Z_0 = \sqrt{\frac{X}{A}} \qquad \gamma = \sqrt{XA} \qquad \begin{aligned} X &= -R + j\omega L \\ A &= -G + j\omega C \end{aligned}$$

Solving for X and A gives

$$X = \gamma Z_0 \qquad A = \frac{\gamma}{Z_0}$$

The R, L, G, and C parameters are then

$$L = \frac{\mathrm{Im}[X]}{\omega} = \frac{\mathrm{Im}[\gamma Z_0]}{\omega} \qquad R = -\mathrm{Re}[X] = -\mathrm{Re}[\gamma Z_0]$$

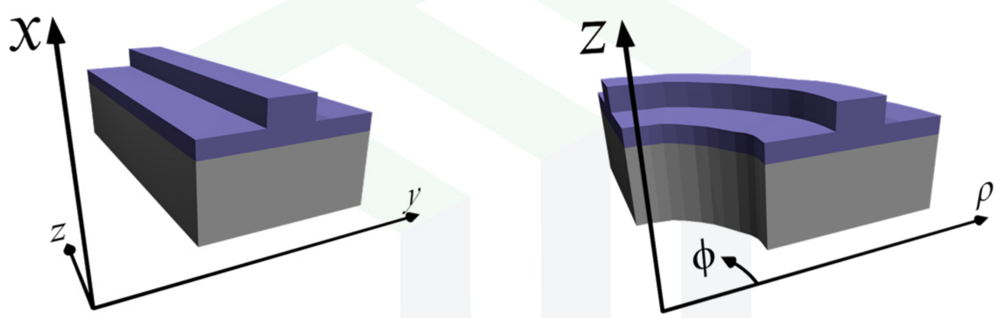
$$C = \frac{\mathrm{Im}[A]}{\omega} = \frac{\mathrm{Im}[\gamma / Z_0]}{\omega} \qquad G = -\mathrm{Re}[A] = -\mathrm{Re}\left[\frac{\gamma}{Z_0}\right]$$

Bent Waveguides

Slide 67

67

Geometry of a Bent Waveguide



Straight waveguides are best analyzed using standard Cartesian coordinates.

Propagation is in $+z$ direction.

Bent waveguides are best analyzed using cylindrical coordinates.

Propagation is in $+\phi$ direction.

EMPossible

Slide 68

68

Maxwell's Equations in Cylindrical Coordinates

$$\nabla \times \vec{E} = k_0 \mu_r \vec{\tilde{H}} \quad \Rightarrow \quad \begin{aligned} \frac{1}{\rho}\frac{\partial E_z}{\partial \phi} - \frac{\partial E_\phi}{\partial z} &= k_0 \mu_{\rho\rho} \tilde{H}_\rho \\ \frac{\partial E_\rho}{\partial z} - \frac{\partial E_z}{\partial \rho} &= k_0 \mu_{\phi\phi} \tilde{H}_\phi \\ \frac{1}{\rho}\left[\frac{\partial\left(\rho E_\phi\right)}{\partial \rho} - \frac{\partial E_\rho}{\partial \phi}\right] &= k_0 \mu_{zz} \tilde{H}_z \end{aligned}$$

$$\nabla \times \vec{\tilde{H}} = k_0 \varepsilon_r \vec{E} \quad \Rightarrow \quad \begin{aligned} \frac{1}{\rho}\frac{\partial \tilde{H}_z}{\partial \phi} - \frac{\partial \tilde{H}_\phi}{\partial z} &= k_0 \varepsilon_{\rho\rho} E_\rho \\ \frac{\partial \tilde{H}_\rho}{\partial z} - \frac{\partial \tilde{H}_z}{\partial \rho} &= k_0 \varepsilon_{\phi\phi} E_\phi \\ \frac{1}{\rho}\left[\frac{\partial\left(\rho \tilde{H}_\phi\right)}{\partial \rho} - \frac{\partial \tilde{H}_\rho}{\partial \phi}\right] &= k_0 \varepsilon_{zz} E_z \end{aligned}$$

EMPossible Slide 69

69

Maxwell's Equations in Cylindrical Coordinates with PML

$$\nabla \times \vec{E} = k_0 \mu_r [s] \vec{\tilde{H}} \quad \Rightarrow \quad \begin{aligned} \frac{1}{\rho}\frac{\partial E_z}{\partial \phi} - \frac{\partial E_\phi}{\partial z} &= k_0 \mu_{\rho\rho} \frac{s_\phi s_z}{s_\rho} \tilde{H}_\rho \\ \frac{\partial E_\rho}{\partial z} - \frac{\partial E_z}{\partial \rho} &= k_0 \mu_{\phi\phi} \frac{s_\rho s_z}{s_\phi} \tilde{H}_\phi \\ \frac{1}{\rho}\left[\frac{\partial\left(\rho E_\phi\right)}{\partial \rho} - \frac{\partial E_\rho}{\partial \phi}\right] &= k_0 \mu_{zz} \frac{s_\phi s_\rho}{s_z} \tilde{H}_z \end{aligned}$$

$$\nabla \times \vec{\tilde{H}} = k_0 \varepsilon_r [s] \vec{E} \quad \Rightarrow \quad \begin{aligned} \frac{1}{\rho}\frac{\partial \tilde{H}_z}{\partial \phi} - \frac{\partial \tilde{H}_\phi}{\partial z} &= k_0 \varepsilon_{\rho\rho} \frac{s_\phi s_z}{s_\rho} E_\rho \\ \frac{\partial \tilde{H}_\rho}{\partial z} - \frac{\partial \tilde{H}_z}{\partial \rho} &= k_0 \varepsilon_{\phi\phi} \frac{s_\rho s_z}{s_\phi} E_\phi \\ \frac{1}{\rho}\left[\frac{\partial\left(\rho \tilde{H}_\phi\right)}{\partial \rho} - \frac{\partial \tilde{H}_\rho}{\partial \phi}\right] &= k_0 \varepsilon_{zz} \frac{s_\phi s_\rho}{s_z} E_z \end{aligned}$$

EMPossible Slide 70

70

Assumed Form of Solution

$$E_\rho(\rho,\phi,z) = A_\rho(\rho,z)e^{j\beta\rho\phi}$$
$$E_\phi(\rho,\phi,z) = A_\phi(\rho,z)e^{j\beta\rho\phi}$$
$$E_z(\rho,\phi,z) = A_z(\rho,z)e^{j\beta\rho\phi}$$

$$\tilde{H}_\rho(\rho,\phi,z) = B_\rho(\rho,z)e^{j\beta\rho\phi}$$
$$\tilde{H}_\phi(\rho,\phi,z) = B_\phi(\rho,z)e^{j\beta\rho\phi}$$
$$\tilde{H}_z(\rho,\phi,z) = B_z(\rho,z)e^{j\beta\rho\phi}$$

EMPossible

Slide 71

71

Substitute Solution Into Maxwell's Equations

$$\frac{1}{\rho}\frac{\partial}{\partial\phi}A_z e^{j\beta\rho\phi} - \frac{\partial}{\partial z}A_\phi e^{j\beta\rho\phi} = k_0\mu'_{\rho\rho}B_\rho e^{j\beta\rho\phi}$$
$$\frac{\partial}{\partial z}A_\rho e^{j\beta\rho\phi} - \frac{\partial}{\partial\rho}A_z e^{j\beta\rho\phi} = k_0\mu'_{\phi\phi}B_\phi e^{j\beta\rho\phi}$$
$$\frac{1}{\rho}\left[\frac{\partial}{\partial\rho}\rho A_\phi e^{j\beta\rho\phi} - \frac{\partial}{\partial\phi}A_\rho e^{j\beta\rho\phi}\right] = k_0\mu'_{zz}B_z e^{j\beta\rho\phi}$$

$$\frac{1}{\rho}\frac{\partial}{\partial\phi}B_z e^{j\beta\rho\phi} - \frac{\partial}{\partial z}B_\phi e^{j\beta\rho\phi} = k_0\varepsilon'_{\rho\rho}A_\rho e^{j\beta\rho\phi}$$
$$\frac{\partial}{\partial z}B_\rho e^{j\beta\rho\phi} - \frac{\partial}{\partial\rho}B_z e^{j\beta\rho\phi} = k_0\varepsilon'_{\phi\phi}A_\phi e^{j\beta\rho\phi}$$
$$\frac{1}{\rho}\left[\frac{\partial}{\partial\rho}\rho B_\phi e^{j\beta\rho\phi} - \frac{\partial}{\partial\phi}B_\rho e^{j\beta\rho\phi}\right] = k_0\varepsilon'_{zz}A_z e^{j\beta\rho\phi}$$

EMPossible

Slide 72

72

Simplify Equations

$$j\beta A_z - \frac{\partial A_\phi}{\partial z} = k_0 \mu'_{\rho\rho} B_\rho$$
$$\frac{\partial A_\rho}{\partial z} - j\beta\phi A_z - \frac{\partial A_z}{\partial \rho} = k_0 \mu'_{\phi\phi} B_\phi$$
$$j\beta\phi A_\phi + \frac{\partial A_\phi}{\partial \rho} + \frac{1}{\rho} A_\phi - j\beta A_\rho = k_0 \mu'_{zz} B_z$$

$$j\beta B_z - \frac{\partial B_\phi}{\partial z} = k_0 \varepsilon'_{\rho\rho} A_\rho$$
$$\frac{\partial B_\rho}{\partial z} - j\beta\phi B_z - \frac{\partial B_z}{\partial \rho} = k_0 \varepsilon'_{\phi\phi} A_\phi$$
$$j\beta\phi B_\phi + \frac{\partial B_\phi}{\partial \rho} + \frac{1}{\rho} B_\phi - j\beta B_\rho = k_0 \varepsilon'_{zz} A_z$$

EMPossible

Slide 73

73

Normalize Variables

The following parameters are normalized

$$\rho' = k_0 \rho \qquad z' = k_0 z \qquad \beta = k_0 n_{\text{eff}}$$

Our six equations become

$$j n_{\text{eff}} A_z - \frac{\partial A_\phi}{\partial z'} = \mu'_{\rho\rho} B_\rho$$
$$\frac{\partial A_\rho}{\partial z'} - j n_{\text{eff}} \phi A_z - \frac{\partial A_z}{\partial \rho'} = \mu'_{\phi\phi} B_\phi$$
$$j n_{\text{eff}} \phi A_\phi + \frac{\partial A_\phi}{\partial \rho'} + \frac{1}{\rho'} A_\phi - j n_{\text{eff}} A_\rho = \mu'_{zz} B_z$$

$$j n_{\text{eff}} B_z - \frac{\partial B_\phi}{\partial z'} = \varepsilon'_{\rho\rho} A_\rho$$
$$\frac{\partial B_\rho}{\partial z'} - j n_{\text{eff}} \phi B_z - \frac{\partial B_z}{\partial \rho'} = \varepsilon'_{\phi\phi} A_\phi$$
$$j n_{\text{eff}} \phi B_\phi + \frac{\partial B_\phi}{\partial \rho'} + \frac{1}{\rho'} B_\phi - j n_{\text{eff}} B_\rho = \varepsilon'_{zz} A_z$$

EMPossible

Slide 74

74

Analyze Cross Section

We are free to choose any cross section. For convenience, we choose $\phi = 0$.

$$jn_{\text{eff}} A_z - \frac{\partial A_\phi}{\partial z'} = \mu'_{\rho\rho} B_\rho$$
$$\frac{\partial A_\rho}{\partial z'} - \frac{\partial A_z}{\partial \rho'} = \mu'_{\phi\phi} B_\phi$$
$$\frac{\partial A_\phi}{\partial \rho'} + \frac{1}{\rho'} A_\phi - jn_{\text{eff}} A_\rho = \mu'_{zz} B_z$$

$$jn_{\text{eff}} B_z - \frac{\partial B_\phi}{\partial z'} = \varepsilon'_{\rho\rho} A_\rho$$
$$\frac{\partial B_\rho}{\partial z'} - \frac{\partial B_z}{\partial \rho'} = \varepsilon'_{\phi\phi} A_\phi$$
$$\frac{\partial B_\phi}{\partial \rho'} + \frac{1}{\rho'} B_\phi - jn_{\text{eff}} B_\rho = \varepsilon'_{zz} A_z$$

EMPossible

Slide 75

75

Finite-Difference Form

$$jn_{\text{eff}} A_z - \frac{\partial A_\phi}{\partial z'} = \mu'_{\rho\rho} B_\rho$$
$$\frac{\partial A_\rho}{\partial z'} - \frac{\partial A_z}{\partial \rho'} = \mu'_{\phi\phi} B_\phi$$
$$\frac{\partial A_\phi}{\partial \rho'} + \frac{1}{\rho'} A_\phi - jn_{\text{eff}} A_\rho = \mu'_{zz} B_z$$

$\Rightarrow$

$$jn_{\text{eff}} \mathbf{a}_z - \mathbf{D}^e_{z'} \mathbf{a}_\phi = \boldsymbol{\mu}'_{\rho\rho} \mathbf{b}_\rho$$
$$\mathbf{D}^e_{z'} \mathbf{a}_\rho - \mathbf{D}^e_{\rho'} \mathbf{a}_z = \boldsymbol{\mu}'_{\phi\phi} \mathbf{b}_\phi$$
$$\mathbf{D}^e_{\rho'} \mathbf{a}_\phi + \boldsymbol{\rho}'^{-1} \mathbf{a}_\phi - jn_{\text{eff}} \mathbf{a}_\rho = \boldsymbol{\mu}'_{zz} \mathbf{b}_z$$

$$jn_{\text{eff}} B_z - \frac{\partial B_\phi}{\partial z'} = \varepsilon'_{\rho\rho} A_\rho$$
$$\frac{\partial B_\rho}{\partial z'} - \frac{\partial B_z}{\partial \rho'} = \varepsilon'_{\phi\phi} A_\phi$$
$$\frac{\partial B_\phi}{\partial \rho'} + \frac{1}{\rho'} B_\phi - jn_{\text{eff}} B_\rho = \varepsilon'_{zz} A_z$$

$\Rightarrow$

$$jn_{\text{eff}} \mathbf{b}_z - \mathbf{D}^h_{z'} \mathbf{b}_\phi = \boldsymbol{\varepsilon}'_{\rho\rho} \mathbf{a}_\rho$$
$$\mathbf{D}^h_{z'} \mathbf{b}_{\rho'} - \mathbf{D}^h_{\rho'} \mathbf{b}_{z'} = \boldsymbol{\varepsilon}'_{\phi\phi} \mathbf{a}_\phi$$
$$\mathbf{D}^h_{\rho'} \mathbf{b}_\phi + \boldsymbol{\rho}'^{-1} \mathbf{b}_\phi - jn_{\text{eff}} \mathbf{b}_\rho = \boldsymbol{\varepsilon}'_{zz} \mathbf{a}_z$$

EMPossible

Slide 76

76

Solve for Longitudinal Component

$$
\begin{aligned}
jn_{\text{eff}}\mathbf{a}_z - \mathbf{D}^e_{z'}\mathbf{a}_\phi &= \boldsymbol{\mu}'_{\rho\rho}\mathbf{b}_\rho \\
\mathbf{D}^e_{z'}\mathbf{a}_\rho - \mathbf{D}^e_{\rho'}\mathbf{a}_z &= \boldsymbol{\mu}'_{\phi\phi}\mathbf{b}_\phi \quad \longrightarrow \quad \mathbf{b}_\phi = \boldsymbol{\mu}'^{-1}_{\phi\phi}\left(\mathbf{D}^e_{z'}\mathbf{a}_\rho - \mathbf{D}^e_{\rho'}\mathbf{a}_z\right) \\
\mathbf{D}^e_{\rho'}\mathbf{a}_\phi + \boldsymbol{\rho}'^{-1}\mathbf{a}_\phi - jn_{\text{eff}}\mathbf{a}_\rho &= \boldsymbol{\mu}'_{zz}\mathbf{b}_z
\end{aligned}
$$

$$
\begin{aligned}
jn_{\text{eff}}\mathbf{b}_z - \mathbf{D}^h_{z'}\mathbf{b}_\phi &= \boldsymbol{\varepsilon}'_{\rho\rho}\mathbf{a}_\rho \\
\mathbf{D}^h_{z'}\mathbf{b}_{\rho'} - \mathbf{D}^h_{\rho'}\mathbf{b}_{z'} &= \boldsymbol{\varepsilon}'_{\phi\phi}\mathbf{a}_\phi \quad \longrightarrow \quad \mathbf{a}_\phi = \boldsymbol{\varepsilon}'^{-1}_{\phi\phi}\left(\mathbf{D}^h_{z'}\mathbf{b}_{\rho'} - \mathbf{D}^h_{\rho'}\mathbf{b}_{z'}\right) \\
\mathbf{D}^h_{\rho'}\mathbf{b}_\phi + \boldsymbol{\rho}'^{-1}\mathbf{b}_\phi - jn_{\text{eff}}\mathbf{b}_\rho &= \boldsymbol{\varepsilon}'_{zz}\mathbf{a}_z
\end{aligned}
$$

EMPossible

Slide 77

77

Eliminate ϕ Components

$$
jn_{\text{eff}}\mathbf{a}_z - \mathbf{D}^e_{z'}\boldsymbol{\varepsilon}'^{-1}_{\phi\phi}\left(\mathbf{D}^h_{z'}\mathbf{b}_{\rho'} - \mathbf{D}^h_{\rho'}\mathbf{b}_{z'}\right) = \boldsymbol{\mu}'_{\rho\rho}\mathbf{b}_\rho
$$

$$
\mathbf{D}^e_{\rho'}\boldsymbol{\varepsilon}'^{-1}_{\phi\phi}\left(\mathbf{D}^h_{z'}\mathbf{b}_{\rho'} - \mathbf{D}^h_{\rho'}\mathbf{b}_{z'}\right) + \boldsymbol{\rho}'^{-1}\boldsymbol{\varepsilon}'^{-1}_{\phi\phi}\left(\mathbf{D}^h_{z'}\mathbf{b}_{\rho'} - \mathbf{D}^h_{\rho'}\mathbf{b}_{z'}\right) - jn_{\text{eff}}\mathbf{a}_\rho = \boldsymbol{\mu}'_{zz}\mathbf{b}_z
$$

$$
jn_{\text{eff}}\mathbf{b}_z - \mathbf{D}^h_{z'}\boldsymbol{\mu}'^{-1}_{\phi\phi}\left(\mathbf{D}^e_{z'}\mathbf{a}_\rho - \mathbf{D}^e_{\rho'}\mathbf{a}_z\right) = \boldsymbol{\varepsilon}'_{\rho\rho}\mathbf{a}_\rho
$$

$$
\mathbf{D}^h_{\rho'}\boldsymbol{\mu}'^{-1}_{\phi\phi}\left(\mathbf{D}^e_{z'}\mathbf{a}_\rho - \mathbf{D}^e_{\rho'}\mathbf{a}_z\right) + \boldsymbol{\rho}'^{-1}\boldsymbol{\mu}'^{-1}_{\phi\phi}\left(\mathbf{D}^e_{z'}\mathbf{a}_\rho - \mathbf{D}^e_{\rho'}\mathbf{a}_z\right) - jn_{\text{eff}}\mathbf{b}_\rho = \boldsymbol{\varepsilon}'_{zz}\mathbf{a}_z
$$

EMPossible

Slide 78

78

Rearrange Equations

$$\left(\boldsymbol{\mu}'_{\rho\rho}+\mathbf{D}^{e}_{z'}\boldsymbol{\varepsilon}'^{-1}_{\phi\phi}\mathbf{D}^{h}_{z'}\right)\mathbf{b}_{\rho'}-\left(\mathbf{D}^{e}_{z'}\boldsymbol{\varepsilon}'^{-1}_{\phi\phi}\mathbf{D}^{h}_{\rho'}\right)\mathbf{b}_{z'}=jn_{\text{eff}}\mathbf{a}_{z}$$

$$\left(\mathbf{D}^{e}_{\rho'}\boldsymbol{\varepsilon}'^{-1}_{\phi\phi}\mathbf{D}^{h}_{z'}+\boldsymbol{\rho}'^{-1}\boldsymbol{\varepsilon}'^{-1}_{\phi\phi}\mathbf{D}^{h}_{z'}\right)\mathbf{b}_{\rho'}-\left(\boldsymbol{\mu}'_{zz}+\mathbf{D}^{e}_{\rho'}\boldsymbol{\varepsilon}'^{-1}_{\phi\phi}\mathbf{D}^{h}_{\rho'}+\boldsymbol{\rho}'^{-1}\boldsymbol{\varepsilon}'^{-1}_{\phi\phi}\mathbf{D}^{h}_{\rho'}\right)\mathbf{b}_{z'}=jn_{\text{eff}}\mathbf{a}_{\rho}$$

$$\left(\boldsymbol{\varepsilon}'_{\rho\rho}+\mathbf{D}^{h}_{z'}\boldsymbol{\mu}'^{-1}_{\phi\phi}\mathbf{D}^{e}_{z'}\right)\mathbf{a}_{\rho}-\left(\mathbf{D}^{h}_{z'}\boldsymbol{\mu}'^{-1}_{\phi\phi}\mathbf{D}^{e}_{\rho'}\right)\mathbf{a}_{z}=jn_{\text{eff}}\mathbf{b}_{z}$$

$$\left(\mathbf{D}^{h}_{\rho'}\boldsymbol{\mu}'^{-1}_{\phi\phi}\mathbf{D}^{e}_{z'}+\boldsymbol{\rho}'^{-1}\boldsymbol{\mu}'^{-1}_{\phi\phi}\mathbf{D}^{e}_{z'}\right)\mathbf{a}_{\rho}-\left(\boldsymbol{\varepsilon}'_{zz}+\mathbf{D}^{h}_{\rho'}\boldsymbol{\mu}'^{-1}_{\phi\phi}\mathbf{D}^{e}_{\rho'}+\boldsymbol{\rho}'^{-1}\boldsymbol{\mu}'^{-1}_{\phi\phi}\mathbf{D}^{e}_{\rho'}\right)\mathbf{a}_{z}=jn_{\text{eff}}\mathbf{b}_{\rho}$$

EMPossible

Slide 79

79

Block Matrix Form

$$\begin{bmatrix}\left(\mathbf{D}^{e}_{\rho'}\boldsymbol{\varepsilon}'^{-1}_{\phi\phi}\mathbf{D}^{h}_{z'}+\boldsymbol{\rho}'^{-1}\boldsymbol{\varepsilon}'^{-1}_{\phi\phi}\mathbf{D}^{h}_{z'}\right) & -\left(\boldsymbol{\mu}'_{zz}+\mathbf{D}^{e}_{\rho'}\boldsymbol{\varepsilon}'^{-1}_{\phi\phi}\mathbf{D}^{h}_{\rho'}+\boldsymbol{\rho}'^{-1}\boldsymbol{\varepsilon}'^{-1}_{\phi\phi}\mathbf{D}^{h}_{\rho'}\right)\\ \left(\boldsymbol{\mu}'_{\rho\rho}+\mathbf{D}^{e}_{z'}\boldsymbol{\varepsilon}'^{-1}_{\phi\phi}\mathbf{D}^{h}_{z'}\right) & -\left(\mathbf{D}^{e}_{z'}\boldsymbol{\varepsilon}'^{-1}_{\phi\phi}\mathbf{D}^{h}_{\rho'}\right)\end{bmatrix}\begin{bmatrix}\mathbf{b}_{\rho'}\\ \mathbf{b}_{z'}\end{bmatrix}=jn_{\text{eff}}\begin{bmatrix}\mathbf{a}_{\rho}\\ \mathbf{a}_{z}\end{bmatrix}$$

$$\begin{bmatrix}\left(\mathbf{D}^{h}_{\rho'}\boldsymbol{\mu}'^{-1}_{\phi\phi}\mathbf{D}^{e}_{z'}+\boldsymbol{\rho}'^{-1}\boldsymbol{\mu}'^{-1}_{\phi\phi}\mathbf{D}^{e}_{z'}\right) & -\left(\boldsymbol{\varepsilon}'_{zz}+\mathbf{D}^{h}_{\rho'}\boldsymbol{\mu}'^{-1}_{\phi\phi}\mathbf{D}^{e}_{\rho'}+\boldsymbol{\rho}'^{-1}\boldsymbol{\mu}'^{-1}_{\phi\phi}\mathbf{D}^{e}_{\rho'}\right)\\ \left(\boldsymbol{\varepsilon}'_{\rho\rho}+\mathbf{D}^{h}_{z'}\boldsymbol{\mu}'^{-1}_{\phi\phi}\mathbf{D}^{e}_{z'}\right) & -\left(\mathbf{D}^{h}_{z'}\boldsymbol{\mu}'^{-1}_{\phi\phi}\mathbf{D}^{e}_{\rho'}\right)\end{bmatrix}\begin{bmatrix}\mathbf{a}_{\rho}\\ \mathbf{a}_{z}\end{bmatrix}=jn_{\text{eff}}\begin{bmatrix}\mathbf{b}_{\rho'}\\ \mathbf{b}_{z'}\end{bmatrix}$$

EMPossible

Slide 80

80

Standard PQ Form

$$\mathbf{P}\begin{bmatrix}\mathbf{b}_{\rho'}\\ \mathbf{b}_{z'}\end{bmatrix} = jn_{\text{eff}}\begin{bmatrix}\mathbf{a}_{\rho}\\ \mathbf{a}_{z}\end{bmatrix}$$

$$\mathbf{P} = \begin{bmatrix} \left(\mathbf{D}_{\rho'}^{e}\boldsymbol{\varepsilon}_{\phi\phi}'^{-1}\mathbf{D}_{z'}^{h} + \boldsymbol{\rho}'^{-1}\boldsymbol{\varepsilon}_{\phi\phi}'^{-1}\mathbf{D}_{z'}^{h}\right) & -\left(\boldsymbol{\mu}_{zz}' + \mathbf{D}_{\rho'}^{e}\boldsymbol{\varepsilon}_{\phi\phi}'^{-1}\mathbf{D}_{\rho'}^{h} + \boldsymbol{\rho}'^{-1}\boldsymbol{\varepsilon}_{\phi\phi}'^{-1}\mathbf{D}_{\rho'}^{h}\right) \\ \left(\boldsymbol{\mu}_{\rho\rho}' + \mathbf{D}_{z'}^{e}\boldsymbol{\varepsilon}_{\phi\phi}'^{-1}\mathbf{D}_{z'}^{h}\right) & -\left(\mathbf{D}_{z'}^{e}\boldsymbol{\varepsilon}_{\phi\phi}'^{-1}\mathbf{D}_{\rho'}^{h}\right) \end{bmatrix}$$

$$\mathbf{Q}\begin{bmatrix}\mathbf{a}_{\rho}\\ \mathbf{a}_{z}\end{bmatrix} = jn_{\text{eff}}\begin{bmatrix}\mathbf{b}_{\rho'}\\ \mathbf{b}_{z'}\end{bmatrix}$$

$$\mathbf{Q} = \begin{bmatrix} \left(\mathbf{D}_{\rho'}^{h}\boldsymbol{\mu}_{\phi\phi}'^{-1}\mathbf{D}_{z'}^{e} + \boldsymbol{\rho}'^{-1}\boldsymbol{\mu}_{\phi\phi}'^{-1}\mathbf{D}_{z'}^{e}\right) & -\left(\boldsymbol{\varepsilon}_{zz}' + \mathbf{D}_{\rho'}^{h}\boldsymbol{\mu}_{\phi\phi}'^{-1}\mathbf{D}_{\rho'}^{e} + \boldsymbol{\rho}'^{-1}\boldsymbol{\mu}_{\phi\phi}'^{-1}\mathbf{D}_{\rho'}^{e}\right) \\ \left(\boldsymbol{\varepsilon}_{\rho\rho}' + \mathbf{D}_{z'}^{h}\boldsymbol{\mu}_{\phi\phi}'^{-1}\mathbf{D}_{z'}^{e}\right) & -\left(\mathbf{D}_{z'}^{h}\boldsymbol{\mu}_{\phi\phi}'^{-1}\mathbf{D}_{\rho'}^{e}\right) \end{bmatrix}$$

Eigen-Value Problem

$$\mathbf{P}\begin{bmatrix}\mathbf{b}_{\rho'}\\ \mathbf{b}_{z'}\end{bmatrix} = jn_{\text{eff}}\begin{bmatrix}\mathbf{a}_{\rho}\\ \mathbf{a}_{z}\end{bmatrix} \qquad \mathbf{Q}\begin{bmatrix}\mathbf{a}_{\rho}\\ \mathbf{a}_{z}\end{bmatrix} = jn_{\text{eff}}\begin{bmatrix}\mathbf{b}_{\rho'}\\ \mathbf{b}_{z'}\end{bmatrix}$$

Solve for **b** terms

$$\begin{bmatrix}\mathbf{b}_{\rho'}\\ \mathbf{b}_{z'}\end{bmatrix} = \frac{1}{jn_{\text{eff}}}\mathbf{Q}\begin{bmatrix}\mathbf{a}_{\rho}\\ \mathbf{a}_{z}\end{bmatrix}$$

Replace **b** terms with new expression

$$\mathbf{PQ}\begin{bmatrix}\mathbf{a}_{\rho}\\ \mathbf{a}_{z}\end{bmatrix} = -n_{\text{eff}}^{2}\begin{bmatrix}\mathbf{a}_{\rho}\\ \mathbf{a}_{z}\end{bmatrix}$$

Final eigen-value problem

$$\boldsymbol{\Omega}^{2}\begin{bmatrix}\mathbf{a}_{\rho}\\ \mathbf{a}_{z}\end{bmatrix} = -n_{\text{eff}}^{2}\begin{bmatrix}\mathbf{a}_{\rho}\\ \mathbf{a}_{z}\end{bmatrix} \qquad \boldsymbol{\Omega}^{2} = \mathbf{PQ}$$

Compare to Ordinary Waveguide Problem

$$\boldsymbol{\Omega}^2 \begin{bmatrix} \mathbf{a}_\rho \\ \mathbf{a}_z \end{bmatrix} = -n_{\text{eff}}^2 \begin{bmatrix} \mathbf{a}_\rho \\ \mathbf{a}_z \end{bmatrix} \qquad \boldsymbol{\Omega}^2 = \mathbf{PQ}$$

$$\mathbf{P}_{\text{straight}} = \begin{bmatrix} \mathbf{D}_{x'}^e \boldsymbol{\varepsilon}_{zz}^{-1} \mathbf{D}_{y'}^h & -\left(\mathbf{D}_{x'}^e \boldsymbol{\varepsilon}_{zz}^{-1} \mathbf{D}_{x'}^h + \boldsymbol{\mu}_{yy}\right) \\ \left(\mathbf{D}_{y'}^e \boldsymbol{\varepsilon}_{zz}^{-1} \mathbf{D}_{y'}^h + \boldsymbol{\mu}_{xx}\right) & -\mathbf{D}_{y'}^e \boldsymbol{\varepsilon}_{zz}^{-1} \mathbf{D}_{x'}^h \end{bmatrix}$$

$$\mathbf{Q}_{\text{straight}} = \begin{bmatrix} \mathbf{D}_{x'}^h \boldsymbol{\mu}_{zz}^{-1} \mathbf{D}_{y'}^e & -\left(\mathbf{D}_{x'}^h \boldsymbol{\mu}_{zz}^{-1} \mathbf{D}_{x'}^e + \boldsymbol{\varepsilon}_{yy}\right) \\ \left(\mathbf{D}_{y'}^h \boldsymbol{\mu}_{zz}^{-1} \mathbf{D}_{y'}^e + \boldsymbol{\varepsilon}_{xx}\right) & -\mathbf{D}_{y'}^h \boldsymbol{\mu}_{zz}^{-1} \mathbf{D}_{x'}^e \end{bmatrix}$$

$$\mathbf{P}_{\text{bent}} = \begin{bmatrix} \left(\mathbf{D}_{\rho'}^e \boldsymbol{\varepsilon}_{\phi\phi}'^{-1} \mathbf{D}_{z'}^h + \boldsymbol{\rho}'^{-1} \boldsymbol{\varepsilon}_{\phi\phi}'^{-1} \mathbf{D}_{z'}^h\right) & -\left(\boldsymbol{\mu}_{zz}' + \mathbf{D}_{\rho'}^e \boldsymbol{\varepsilon}_{\phi\phi}'^{-1} \mathbf{D}_{\rho'}^h + \boldsymbol{\rho}'^{-1} \boldsymbol{\varepsilon}_{\phi\phi}'^{-1} \mathbf{D}_{\rho'}^h\right) \\ \left(\boldsymbol{\mu}_{\rho\rho}' + \mathbf{D}_{z'}^e \boldsymbol{\varepsilon}_{\phi\phi}'^{-1} \mathbf{D}_{z'}^h\right) & -\left(\mathbf{D}_{z'}^e \boldsymbol{\varepsilon}_{\phi\phi}'^{-1} \mathbf{D}_{\rho'}^h\right) \end{bmatrix}$$

$$\mathbf{Q}_{\text{bent}} = \begin{bmatrix} \left(\mathbf{D}_{\rho'}^h \boldsymbol{\mu}_{\phi\phi}'^{-1} \mathbf{D}_{z'}^e + \boldsymbol{\rho}'^{-1} \boldsymbol{\mu}_{\phi\phi}'^{-1} \mathbf{D}_{z'}^e\right) & -\left(\boldsymbol{\varepsilon}_{zz}' + \mathbf{D}_{\rho'}^h \boldsymbol{\mu}_{\phi\phi}'^{-1} \mathbf{D}_{\rho'}^e + \boldsymbol{\rho}'^{-1} \boldsymbol{\mu}_{\phi\phi}'^{-1} \mathbf{D}_{\rho'}^e\right) \\ \left(\boldsymbol{\varepsilon}_{\rho\rho}' + \mathbf{D}_{z'}^h \boldsymbol{\mu}_{\phi\phi}'^{-1} \mathbf{D}_{z'}^e\right) & -\left(\mathbf{D}_{z'}^h \boldsymbol{\mu}_{\phi\phi}'^{-1} \mathbf{D}_{\rho'}^e\right) \end{bmatrix}$$

Just Modify Your Straight Code

$$\mathbf{P}_{\text{bent}} = \mathbf{P}_{\text{straight}} + \begin{bmatrix} \mathbf{X}'^{-1} \boldsymbol{\varepsilon}_{zz}'^{-1} \mathbf{D}_{y'}^h & -\mathbf{X}'^{-1} \boldsymbol{\varepsilon}_{zz}'^{-1} \mathbf{D}_{x'}^h \\ \mathbf{0} & \mathbf{0} \end{bmatrix}$$

$$\mathbf{Q}_{\text{bent}} = \mathbf{Q}_{\text{straight}} + \begin{bmatrix} \mathbf{X}'^{-1} \boldsymbol{\mu}_{zz}'^{-1} \mathbf{D}_{y'}^e & -\mathbf{X}'^{-1} \boldsymbol{\mu}_{zz}'^{-1} \mathbf{D}_{x'}^e \\ \mathbf{0} & \mathbf{0} \end{bmatrix}$$

$\mathbf{X}' \equiv$ diagonal matrix of normalized x-coordinates throughout grid.

Now you are simulating bent waveguides!