Electrostatics:
Energy in Electrostatic Fields

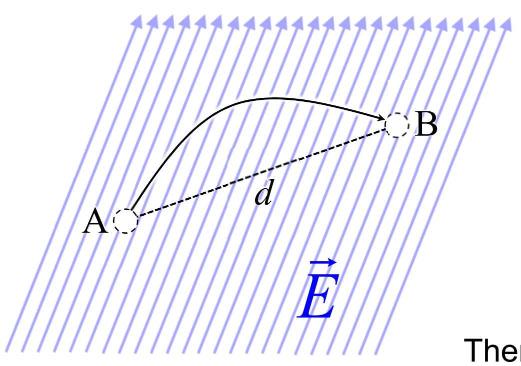
EE3321
Electromagnetic Field Theory

Outline

- Energy in terms of potential
- Energy in terms of the field
- Power and energy in conductors

Energy in Terms of Potential

Recall Potential Difference



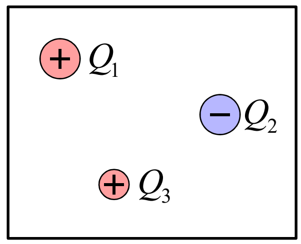
Recall the relation between potential difference, work, and charge.

$$V_{AB} = V_B - V_A = \frac{W}{Q}$$

Therefore, the work it takes to move charge Q from A to B is

$$W = QV_{AB}$$

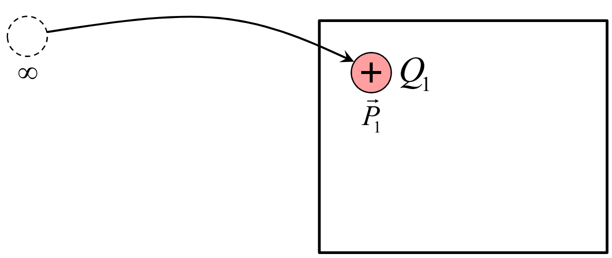
Energy in an Ensemble of Charges



An ensemble of charges contains energy because the charges are putting a force on each other and so they have the potential to do work.

We will calculate how much energy the ensemble contains by calculating how much energy it took to assemble it.

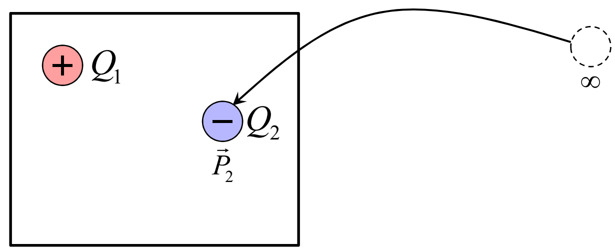
Point Charge #1



No other charges are present, so placing Q_1 at P_1 takes no work.

$$W_1 = 0$$

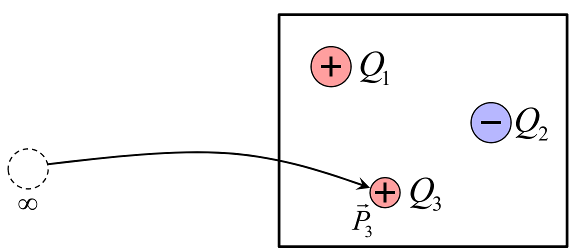
Point Charge #2



Placing Q_2 at P_2 takes work because charge Q_1 is present.

$$W_2 = Q_2 V_{21}$$

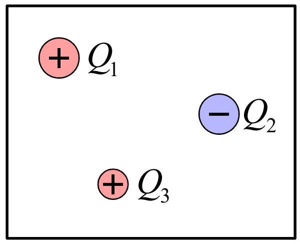
Point Charge #3



Placing Q_3 at P_3 takes work because charges Q_1 and Q_2 are present.

$$W_3 = Q_3 V_{31} + Q_3 V_{32}$$

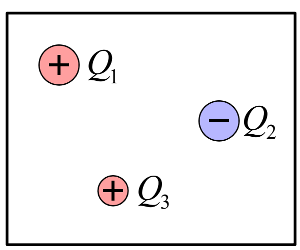
Total Work So Far



The total work placing all three charges is

$$W = W_1 + W_2 + W_3$$
$$= 0 + Q_2 V_{21} + Q_3 \left(V_{31} + V_{32} \right)$$

Assembly in Reverse Order



If we had placed the charges in the reverse order,

$$W = W_3 + W_2 + W_1$$
$$= 0 + Q_2 V_{23} + Q_1 \left(V_{12} + V_{13} \right)$$

Add Both Approaches

$$W = 0 + Q_2 V_{21} + Q_3\left(V_{31} + V_{32}\right)$$

Equation obtained by placing Q_1, then Q_2, and then Q_3.

$$W = 0 + Q_2 V_{23} + Q_1\left(V_{12} + V_{13}\right)$$

Equation obtained by placing Q_3, then Q_2, and then Q_1.

$$2W = 0 + Q_2 V_{21} + Q_3\left(V_{31} + V_{32}\right) + 0 + Q_2 V_{23} + Q_1\left(V_{12} + V_{13}\right)$$

Add the two equations above.

$$2W = Q_1\underbrace{\left(V_{12} + V_{13}\right)}_{V_1} + Q_2\underbrace{\left(V_{21} + V_{23}\right)}_{V_2} + Q_3\underbrace{\left(V_{31} + V_{32}\right)}_{V_3}$$

Total potentials → V_1, V_2, V_3

$$2W = Q_1 V_1 + Q_2 V_2 + Q_3 V_3$$

Final Expression

$$2W = Q_1 V_1 + Q_2 V_2 + Q_3 V_3$$

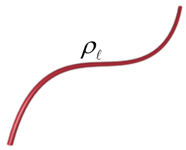
$$W = \frac{1}{2} Q_1 V_1 + \frac{1}{2} Q_2 V_2 + \frac{1}{2} Q_3 V_3$$

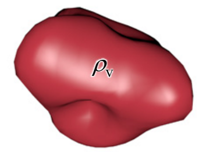
Solve for W.

It is straightforward to generalize this for any number of charges.

$$W = \frac{1}{2}\sum_{i=1}^{N} Q_i V_i \qquad (\text{joules})$$

Energy in Charge Distributions





Point Charge	Line Charge	Sheet Charge	Volume Charge
Q	ρ_ℓ	ρ_s	ρ_v
Charge $Q\ (\text{C})$	Line Charge Density $\rho_\ell\ (\text{C/m})$	Surface Charge Density $\rho_s\ (\text{C/m}^2)$	Volume Charge Density $\rho_v\ (\text{C/m}^3)$
Total Charge $Q_{\text{Total}} = \sum_{i=1}^{N} Q_i$	Total Charge $Q_{\text{Total}} = \int_\ell \rho_\ell d\ell \cong \rho_\ell L$	Total Charge $Q_{\text{Total}} = \iint_S \rho_s ds \cong \rho_s S$	Total Charge $Q_{\text{Total}} = \iiint_v \rho_v dv = \rho_v V$
Total Energy $W = \frac{1}{2}\sum_{i=1}^{N} Q_i V_i$	Total Energy $W = \frac{1}{2}\int_L \rho_\ell V d\ell$	Total Energy $W = \frac{1}{2}\iint_s \rho_s V ds$	Total Energy $W = \frac{1}{2}\iiint_v \rho_v V dv$

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Energy in Terms of the Field

Derivation (1 of 5)

The energy in a volume charge is

$$W = \frac{1}{2}\iiint_{\mathrm{v}} \rho_{\mathrm{v}} V dv$$

Recall from Maxwell's equations that $\rho_{\mathrm{v}} = \nabla \bullet \vec{D}$.

$$W = \frac{1}{2}\iiint_{\mathrm{v}} \left(\nabla \bullet \vec{D}\right) V dv$$

Recall the product rule for divergence $\nabla \bullet \left(f\vec{A}\right) = f\left(\nabla \bullet \vec{A}\right) + \vec{A} \bullet \nabla f$

$$\nabla \bullet \left(V\vec{D}\right) = V\left(\nabla \bullet \vec{D}\right) + \vec{D} \bullet \nabla V$$

$$\left(\nabla \bullet \vec{D}\right)V = \nabla \bullet \left(V\vec{D}\right) - \vec{D} \bullet \nabla V$$

Derivation (2 of 5)

Apply the product rule for our equation for work.

$$W = \frac{1}{2}\iiint_{\mathrm{v}} \left(\nabla \bullet \vec{D}\right) V dv$$

$$= \frac{1}{2}\iiint_{\mathrm{v}} \left[\nabla \bullet \left(V\vec{D}\right) - \vec{D} \bullet \nabla V\right] dv$$

$$= \frac{1}{2}\iiint_{\mathrm{v}} \left[\nabla \bullet \left(V\vec{D}\right)\right] dv - \frac{1}{2}\iiint_{\mathrm{v}} \left[\vec{D} \bullet \nabla V\right] dv$$

Derivation (3 of 5)

Recall the divergence theorem

$$\oiint_S \vec{F} \bullet d\vec{s} = \iiint_V \left(\nabla \bullet \vec{F} \right) dv$$

Apply this to our equation for work.

$$W = \underbrace{\frac{1}{2} \iiint_{\mathrm{v}} \left[\nabla \bullet \left(V\vec{D} \right) \right] dv}_{\frac{1}{2} \oiint_S \left(V\vec{D} \right) \bullet d\vec{s}} - \frac{1}{2} \iiint_{\mathrm{v}} \left[\vec{D} \bullet \nabla V \right] dv$$

$$W = \frac{1}{2} \oiint_S \left(V\vec{D} \right) \bullet d\vec{s} - \frac{1}{2} \iiint_{\mathrm{v}} \left[\vec{D} \bullet \nabla V \right] dv$$

Derivation (4 of 5)

Look more closely at the surface integral.

$$W = \frac{1}{2} \oiint_S \left(V\vec{D} \right) \bullet d\vec{s} - \frac{1}{2} \iiint_{\mathrm{v}} \left[\vec{D} \bullet \nabla V \right] dv$$

$V \propto \frac{1}{r}$

$\left| \vec{D} \right| \propto \frac{1}{r^2}$

$\left| d\vec{s} \right| \propto r^2$

$$\text{Overall} \propto \frac{1}{r} \frac{1}{r^2} r^2 = \frac{1}{r}$$

We are free to choose whatever surface S we wish.

As we enlarge the surface out to infinity, the surface integral becomes negligible relative to the volume integral.

$$W = \cancel{\frac{1}{2} \oiint_S \left(V\vec{D} \right) \bullet d\vec{s}} - \frac{1}{2} \iiint_{\mathrm{v}} \left[\vec{D} \bullet \nabla V \right] dv$$

Derivation (5 of 5)

Our equation for work is now

$$W = -\frac{1}{2}\iiint_V \left[\vec{D} \bullet \nabla V\right] dv$$

Associate the negative sign with ∇V.

$$W = \frac{1}{2}\iiint_V \left[\vec{D} \bullet \left(-\nabla V\right)\right] dv$$

This is the electric field intensity $\vec{E}$.

$$W = \frac{1}{2}\iiint_V \left(\vec{D} \bullet \vec{E}\right) dv$$

This is the general equation for energy stored in the electrostatic field.

It is valid for anisotropic and inhomogeneous media.

Electrostatic Energy in LHI Media

The more common expression for energy in the electrostatic field is for the special case of linear, homogeneous, and isotropic (LHI) media.

In isotropic media we have $\vec{D} = \varepsilon \vec{E}$.

$$W = \frac{1}{2}\iiint_V \left(\vec{D} \bullet \vec{E}\right) dv$$

$$= \frac{1}{2}\iiint_V \left(\varepsilon \vec{E} \bullet \vec{E}\right) dv$$

$$W = \frac{1}{2}\iiint_V \varepsilon \left|\vec{E}\right|^2 dv$$

Simpler equation that is only valid in LHI media.

Electrostatic Energy Density

Observe what we have been integrating to get total energy.

$$W = \iiint_{v} \left(\frac{1}{2} \vec{D} \bullet \vec{E} \right) dv \qquad W = \iiint_{v} \left(\frac{1}{2} \varepsilon \left| \vec{E} \right|^2 \right) dv$$

These expressions must be energy density w.

We can now think of calculating total energy by integrating the energy density w.

$$W = \iiint_{v} w dv \qquad w = \begin{cases} \frac{1}{2} \vec{D} \bullet \vec{E} & \text{General case} \\ \frac{1}{2} \varepsilon \left| \vec{E} \right|^2 & \text{LHI media} \end{cases}$$

Power & Energy in Conductors

Joule's Law

Joule's law states that

$$P = \iiint_V \left(\vec{E} \bullet \vec{J} \right) dv$$

This is equivalent to $P = VI$ in circuit theory.

From this, we can extract the energy density in a conductor.

$$w = \vec{E} \bullet \vec{J}$$

Applying Ohm's law for electromagnetics $\vec{J} = \sigma \vec{E}$ gives

$$w = \vec{E} \bullet \vec{J} = \vec{E} \bullet \sigma \vec{E} = \sigma \left| \vec{E} \right|^2$$

$$\boxed{P = \iiint_V \sigma \left| \vec{E} \right|^2 dv}$$

Most common form.