Advanced Computation:
Computational Electromagnetics

Finite-Difference Time-Domain (FDTD)

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Outline

- Introduction to FDTD
- Concept of the "update equation"
- Time-domain UPML
- Derivation of the update equations
- Total-field/scattered-field source
- Calculating transmission and reflection
- Block diagram of FDTD
- Sequence of code development

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Introduction to Finite-Difference Time-Domain

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Flow of Maxwell's Equations

$$\nabla \times \vec{E}(t) = -\frac{\partial \vec{B}(t)}{\partial t}$$

A circulating ***E*** field induces a change in the ***B*** field at the center of circulation.

$$\vec{B}(t) = [\mu(t)] * \vec{H}(t)$$

A ***B*** field induces an ***H*** field in proportion to the permeability.

$$\nabla \times \vec{H}(t) = \frac{\partial \vec{D}(t)}{\partial t}$$

A circulating ***H*** field induces a change in the ***D*** field at the center of circulation.

$$\vec{D}(t) = [\varepsilon(t)] * \vec{E}(t)$$

A ***D*** field induces an ***E*** field in proportion to the permittivity.

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Flow of Maxwell's Equations Inside Linear, Isotropic and Non-Dispersive Materials

In materials that are linear, isotropic and non-dispersive we have

$$\left[\mu(t)\right]* = \mu \cdot \qquad \left[\varepsilon(t)\right]* = \varepsilon \cdot$$

In this case, the flow of Maxwell's equations reduces to

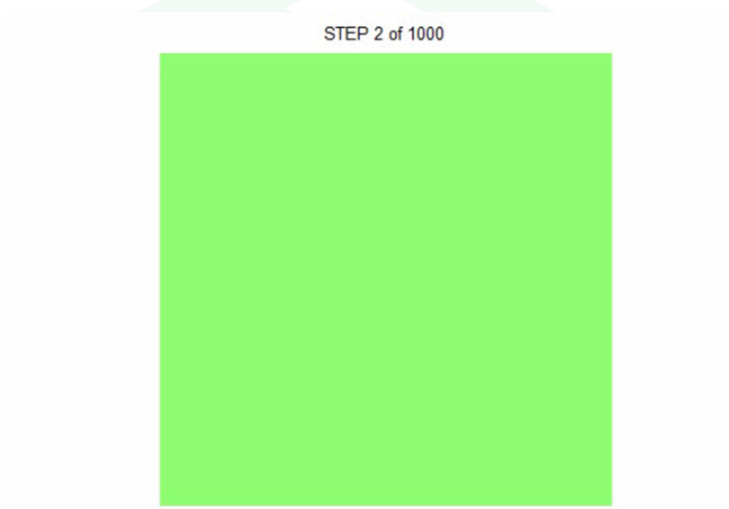
$$\nabla \times \vec{E}(t) = -\mu \frac{\partial \vec{H}(t)}{\partial t}$$

A circulating ***E*** field induces a change in the ***H*** field at the center of circulation in proportion to the permeability.

$$\nabla \times \vec{H}(t) = \varepsilon \frac{\partial \vec{E}(t)}{\partial t}$$

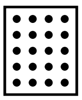
A circulating ***H*** field induces a change in the ***E*** field at the center of circulation in proportion to the permittivity.

Simple FDTD Simulation



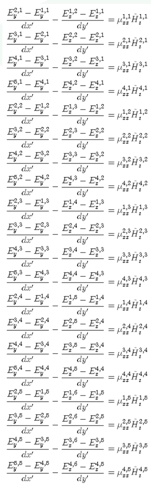
FDFD Vs. FDTD

Yee Grid



The Yee grid, finite-differences, and numerical behavior are almost identical for FDFD and FDTD.

FDFD



FDFD assembles the large set of finite-difference equations into a single matrix equation and solves them simultaneously.

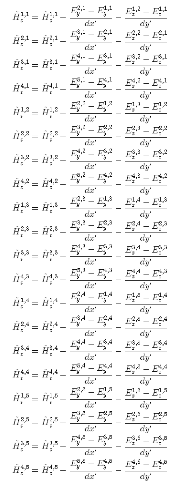
$$\mathbf{Ax} = \mathbf{b}$$

$$\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$$

```
x = A\b;
```

FDTD

FDTD loops through the large set of finite-difference equations and updates the fields in small time steps.



```
for T = 1 : TIME
    for ny = 1 : Ny
    for nx = 1 : Nx
        Hz(nx,ny) = Hx(nx,ny) ...
                  + (Ey(nx+1,ny) - Ey(nx,ny))/dx ...
                  - (Ex(nx,ny+1) - Ex(nx,ny))/dy;
    end
    end
end
```

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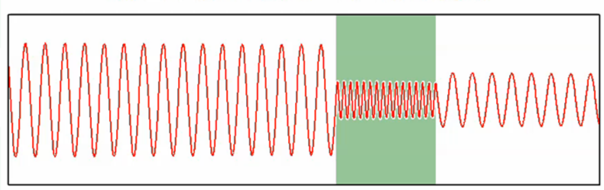
FDFD Vs. FDTD

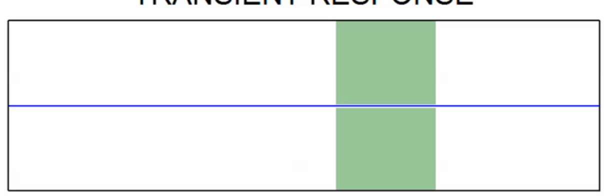
STEADY-STATE RESPONSE

This is what FDFD calculates.



TRANSIENT RESPONSE

This is what FDTD calculates.



FDFD obtains a solution at a single frequency. FDTD inherently simulates a broad range of frequencies so a transient response is always observed.

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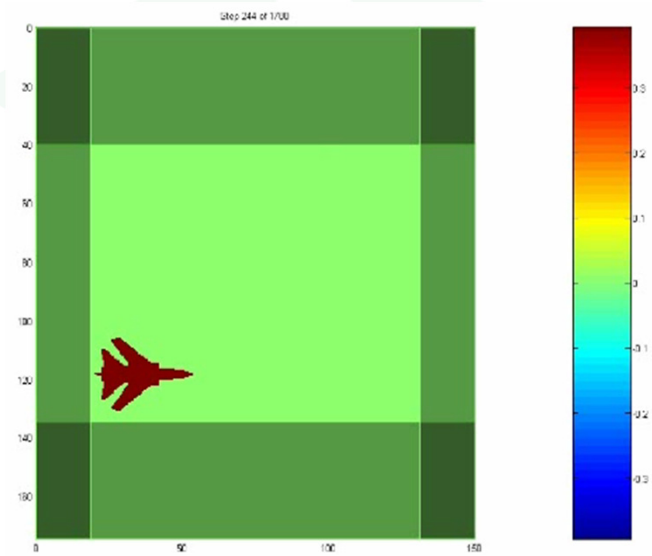
Example Simulation: Pulsed Radar



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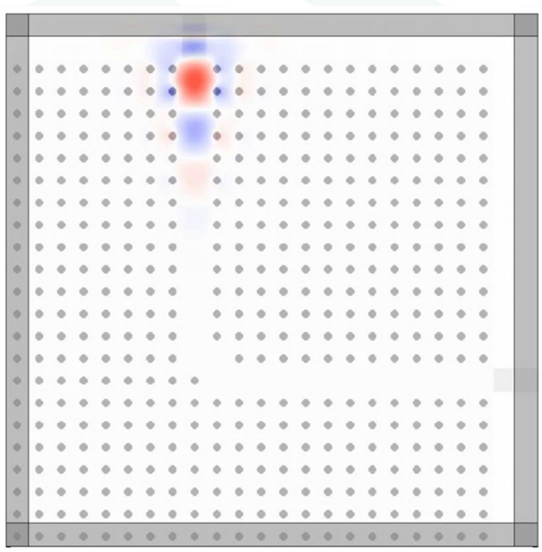
Example Simulation:

Photonic Crystal Waveguide Bend



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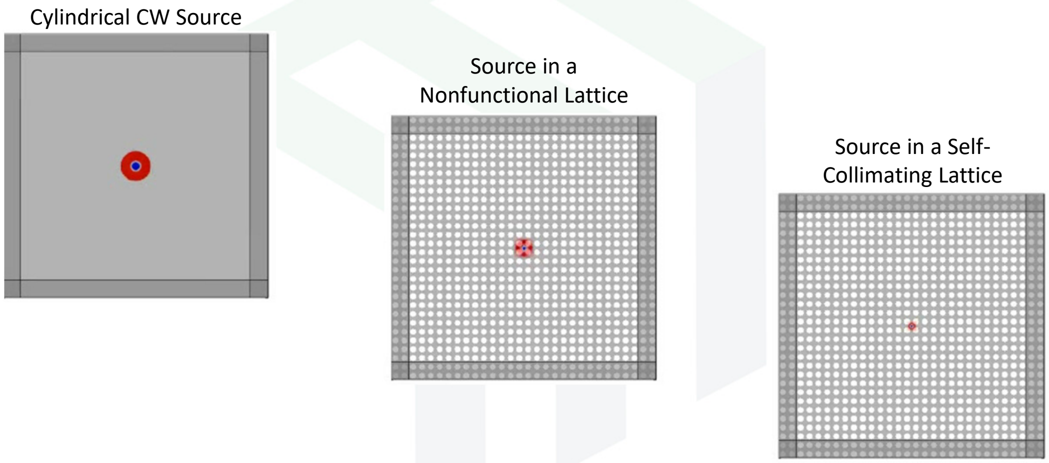
Example Simulation: Self-Collimation



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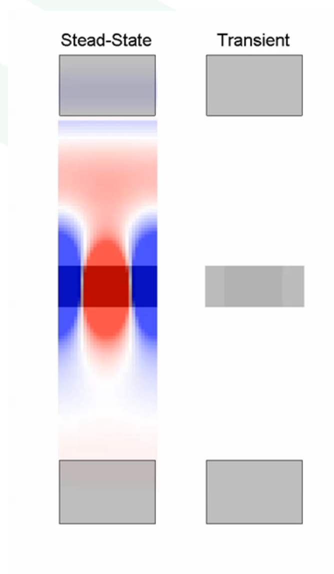
Highly Resonant Devices

FDTD is very slow for highly resonant devices.

Energy gets "stuck" in the device.

FDTD has to keep iterating until that energy escapes.



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Concept of the "Update Equation"

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Approximating the Time Derivative (1 of 3)

An intuitive first guess at approximating the time derivatives in Maxwell's equations is:

$$\nabla \times \vec{E}(t) = -\mu \frac{\partial \vec{H}(t)}{\partial t} \quad \Rightarrow \quad \underbrace{\nabla \times \vec{E}(t)}_{\text{Exists at } t} \cong -\mu \underbrace{\frac{\vec{H}(t+\Delta t) - \vec{H}(t)}{\Delta t}}_{\text{Exists at } t+\Delta t/2}$$

$$\nabla \times \vec{H}(t) = \varepsilon \frac{\partial \vec{E}(t)}{\partial t} \quad \Rightarrow \quad \underbrace{\nabla \times \vec{H}(t)}_{\text{Exists at } t} \cong \varepsilon \underbrace{\frac{\vec{E}(t+\Delta t) - \vec{E}(t)}{\Delta t}}_{\text{Exists at } t+\Delta t/2}$$

This is an unstable formulation.

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Approximating the Time Derivative (2 of 3)

We adjust the finite-difference equations so that each term exists at the same point in time.

$$\nabla\times\vec{E}(t)=-\mu\frac{\partial\vec{H}(t)}{\partial t}\quad\Rightarrow\quad\nabla\times\vec{E}(t)=-\mu\frac{\vec{H}(t+\Delta t/2)-\vec{H}(t-\Delta t/2)}{\Delta t}$$

$$\nabla\times\vec{H}(t)=\varepsilon\frac{\partial\vec{E}(t)}{\partial t}\quad\Rightarrow\quad\nabla\times\vec{H}(t+\Delta t/2)=\varepsilon\frac{\vec{E}(t+\Delta t)-\vec{E}(t)}{\Delta t}$$

These equations will get messy if we include interpolations.

Is there a simpler approach?

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Approximating the Time Derivative (3 of 3)

We stagger **E** and **H** in time so that **E** exists at integer time steps $(0, \Delta t, 2\Delta t, \ldots)$ and **H** exists at half time steps $(\Delta t/2, t+\Delta t/2, 2t+\Delta t/2,\ldots)$.

$$\nabla\times\vec{E}(t)=-\mu\frac{\partial\vec{H}(t)}{\partial t}\quad\Rightarrow\quad\nabla\times\vec{E}\Big|_{t}=-\mu\frac{\vec{H}\Big|_{t+\Delta t/2}-\vec{H}\Big|_{t-\Delta t/2}}{\Delta t}$$

$$\nabla\times\vec{H}(t)=\varepsilon\frac{\partial\vec{E}(t)}{\partial t}\quad\Rightarrow\quad\nabla\times\vec{H}\Big|_{t+\Delta t/2}=\varepsilon\frac{\vec{E}\Big|_{t+\Delta t}-\vec{E}\Big|_{t}}{\Delta t}$$

The spatial derivatives are handled exactly like they are handled in FDFD.

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Derivation of the Update Equations

The “update equations” are the equations used inside the main FDTD loop to calculate the field values at the next time step.

They are derived by solving Maxwell’s equations for the field at the future time value.

$$\nabla \times \vec{E}\Big|_t = -\mu \frac{\vec{H}\big|_{t+\Delta t/2} - \vec{H}\big|_{t-\Delta t/2}}{\Delta t} \quad \Rightarrow \quad \vec{H}\Big|_{t+\Delta t/2} = \vec{H}\Big|_{t-\Delta t/2} - \frac{\Delta t}{\mu}\left(\nabla \times \vec{E}\Big|_t\right)$$

$$\nabla \times \vec{H}\Big|_{t+\Delta t/2} = \varepsilon \frac{\vec{E}\big|_{t+\Delta t} - \vec{E}\big|_{t}}{\Delta t} \quad \Rightarrow \quad \vec{E}\Big|_{t+\Delta t} = \vec{E}\Big|_{t} + \frac{\Delta t}{\varepsilon}\left(\nabla \times \vec{H}\Big|_{t+\Delta t/2}\right)$$

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Anatomy of the FDTD Update Equation

Update coefficient

To speed simulation, calculate these before the main loop.

$$\vec{E}\Big|_{t+\Delta t} = \vec{E}\Big|_{t} + \frac{\Delta t}{\varepsilon}\left(\nabla \times \vec{H}\Big|_{t+\Delta t/2}\right)$$

Field at the next time step.

Field at the previous time step.

Curl of the “other” field at an intermediate time step

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Time-Domain UPML

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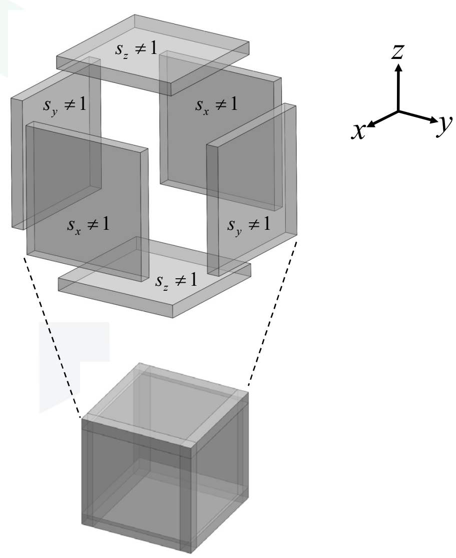
Recall the Uniaxial PML

The 3D PML can be visualized this way…

$$[s]=\begin{bmatrix} \frac{s_y s_z}{s_x} & 0 & 0 \\ 0 & \frac{s_x s_z}{s_y} & 0 \\ 0 & 0 & \frac{s_x s_y}{s_z} \end{bmatrix}$$



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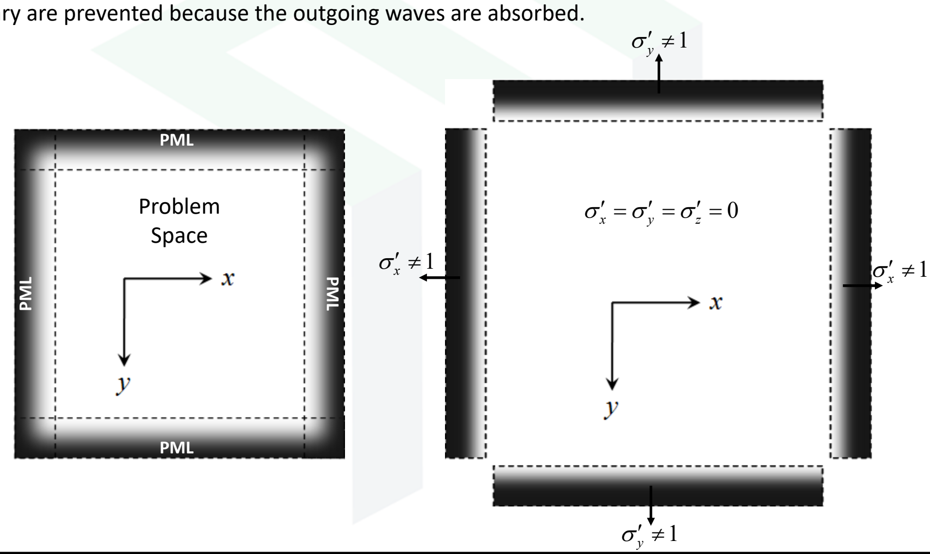
Fictitious Conductivities

The perfectly matched layer (PML) is an absorbing boundary condition (ABC) where the impedance is perfectly matched to the problem space. Reflections entering the lossy regions are prevented because impedance is matched. Reflections from the grid boundary are prevented because the outgoing waves are absorbed.



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Calculating the PML Loss Terms

For best performance, the loss terms should increase gradually into the PMLs.

$$s_x(x) = 1 + \frac{\sigma'_x(x)}{j\omega\varepsilon_0} \qquad \sigma'_x(x) = \frac{\varepsilon_0}{2\Delta t}\left(\frac{x}{L_x}\right)^3$$

$$s_y(y) = 1 + \frac{\sigma'_y(y)}{j\omega\varepsilon_0} \qquad \sigma'_y(y) = \frac{\varepsilon_0}{2\Delta t}\left(\frac{y}{L_y}\right)^3$$

$$s_z(z) = 1 + \frac{\sigma'_z(z)}{j\omega\varepsilon_0} \qquad \sigma'_z(z) = \frac{\varepsilon_0}{2\Delta t}\left(\frac{z}{L_z}\right)^3$$

$L_? \equiv$ length of the PML in the ? direction

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Incorporating a UPML into Maxwell's Equations

Before incorporating a UPML, Maxwell's equations in the frequency-domain are

$$\nabla \times \vec{E}(\omega) = -j\omega\mu_0[\mu_r]\vec{H}(\omega)$$
$$\nabla \times \vec{H}(\omega) = \sigma\vec{E}(\omega) + j\omega\vec{D}(\omega)$$

$$\vec{D}(\omega) = \varepsilon_0[\varepsilon_r]\vec{E}(\omega)$$

We can incorporate a UPML independent of the actual materials on the grid as follows:

$$\nabla \times \vec{E}(\omega) = -j\omega\mu_0[\mu_r][s]\vec{H}(\omega)$$
$$\nabla \times \vec{H}(\omega) = \sigma\vec{E}(\omega) + j\omega[s]\vec{D}(\omega)$$

$$\vec{D}(\omega) = \varepsilon_0[\varepsilon_r]\vec{E}(\omega)$$

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Normalize Maxwell's Equations

We normalize the electric field quantities according to

$$\vec{\tilde{E}} = \sqrt{\frac{\varepsilon_0}{\mu_0}}\vec{E} = \frac{1}{\eta_0}\vec{E}$$

$$\vec{\tilde{D}} = \frac{1}{\sqrt{\mu_0\varepsilon_0}}\vec{D} = c_0\vec{D}$$

Maxwell's equations with the UPML and normalized fields are

$$\nabla \times \vec{\tilde{E}}(\omega) = -j\omega\frac{[\mu_r]}{c_0}[s]\vec{H}(\omega)$$
$$\nabla \times \vec{H}(\omega) = \eta_0\sigma\vec{\tilde{E}}(\omega) + \frac{j\omega}{c_0}[s]\vec{\tilde{D}}(\omega)$$

$$\vec{\tilde{D}}(\omega) = [\varepsilon_r]\vec{\tilde{E}}(\omega)$$

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Vector Expansion of Maxwell's Equations

$$\nabla \times \vec{\tilde{E}}(\omega) = -j\omega \frac{[\mu_r]}{c_0}[s]\vec{H}(\omega) \quad \Rightarrow \quad
\begin{aligned}
\frac{\partial \tilde{E}_z(\omega)}{\partial y} - \frac{\partial \tilde{E}_y(\omega)}{\partial z} &= -j\omega \frac{\mu_{xx}}{c_0}\frac{s_y s_z}{s_x} H_x(\omega) \\
\frac{\partial \tilde{E}_x(\omega)}{\partial z} - \frac{\partial \tilde{E}_z(\omega)}{\partial x} &= -j\omega \frac{\mu_{yy}}{c_0}\frac{s_x s_z}{s_y} H_y(\omega) \\
\frac{\partial \tilde{E}_y(\omega)}{\partial x} - \frac{\partial \tilde{E}_x(\omega)}{\partial y} &= -j\omega \frac{\mu_{zz}}{c_0}\frac{s_x s_y}{s_z} H_z(\omega)
\end{aligned}$$

$$\nabla \times \vec{H}(\omega) = \eta_0 \sigma \vec{\tilde{E}}(\omega) + \frac{j\omega}{c_0}[s]\vec{\tilde{D}}(\omega) \quad \Rightarrow \quad
\begin{aligned}
\frac{\partial H_z(\omega)}{\partial y} - \frac{\partial H_y(\omega)}{\partial z} &= \eta_0 \sigma_{xx}\tilde{E}_x(\omega) + \frac{j\omega}{c_0}\frac{s_y s_z}{s_x}\tilde{D}_x(\omega) \\
\frac{\partial H_x(\omega)}{\partial z} - \frac{\partial H_z(\omega)}{\partial x} &= \eta_0 \sigma_{yy}\tilde{E}_y(\omega) + \frac{j\omega}{c_0}\frac{s_x s_z}{s_y}\tilde{D}_y(\omega) \\
\frac{\partial H_y(\omega)}{\partial x} - \frac{\partial H_x(\omega)}{\partial y} &= \eta_0 \sigma_{zz}\tilde{E}_z(\omega) + \frac{j\omega}{c_0}\frac{s_x s_y}{s_z}\tilde{D}_z(\omega)
\end{aligned}$$

$$\vec{\tilde{D}}(\omega) = [\varepsilon_r]\vec{\tilde{E}}(\omega) \quad \Rightarrow \quad
\begin{aligned}
\tilde{D}_x(\omega) &= \varepsilon_{xx}\tilde{E}_x(\omega) \\
\tilde{D}_y(\omega) &= \varepsilon_{yy}\tilde{E}_y(\omega) \\
\tilde{D}_z(\omega) &= \varepsilon_{zz}\tilde{E}_z(\omega)
\end{aligned}$$

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Maxwell's Equations with a UPML

Neglecting the loss term, we have

$$\nabla \times \vec{\tilde{E}}(\omega) = -j\omega \frac{[\mu_r]}{c_0}[s]\vec{H}(\omega) \quad \Rightarrow \quad
\begin{aligned}
j\omega\left(1+\frac{\sigma'_x}{j\omega\varepsilon_0}\right)^{-1}\left(1+\frac{\sigma'_y}{j\omega\varepsilon_0}\right)\left(1+\frac{\sigma'_z}{j\omega\varepsilon_0}\right)H_x(\omega) &= -\frac{c_0}{\mu_{xx}}\left[\frac{\partial \tilde{E}_z(\omega)}{\partial y} - \frac{\partial \tilde{E}_y(\omega)}{\partial z}\right] \\
j\omega\left(1+\frac{\sigma'_x}{j\omega\varepsilon_0}\right)\left(1+\frac{\sigma'_y}{j\omega\varepsilon_0}\right)^{-1}\left(1+\frac{\sigma'_z}{j\omega\varepsilon_0}\right)H_y(\omega) &= -\frac{c_0}{\mu_{yy}}\left[\frac{\partial \tilde{E}_x(\omega)}{\partial z} - \frac{\partial \tilde{E}_z(\omega)}{\partial x}\right] \\
j\omega\left(1+\frac{\sigma'_x}{j\omega\varepsilon_0}\right)\left(1+\frac{\sigma'_y}{j\omega\varepsilon_0}\right)\left(1+\frac{\sigma'_z}{j\omega\varepsilon_0}\right)^{-1}H_z(\omega) &= -\frac{c_0}{\mu_{zz}}\left[\frac{\partial \tilde{E}_y(\omega)}{\partial x} - \frac{\partial \tilde{E}_x(\omega)}{\partial y}\right]
\end{aligned}$$

$$\nabla \times \vec{H}(\omega) = \eta_0 \sigma \vec{\tilde{E}}(\omega) + \frac{j\omega}{c_0}[s]\vec{\tilde{D}}(\omega) \quad \Rightarrow \quad
\begin{aligned}
j\omega\left(1+\frac{\sigma'_x}{j\omega\varepsilon_0}\right)^{-1}\left(1+\frac{\sigma'_y}{j\omega\varepsilon_0}\right)\left(1+\frac{\sigma'_z}{j\omega\varepsilon_0}\right)\tilde{D}_x(\omega) &= c_0\left[\frac{\partial H_z(\omega)}{\partial y} - \frac{\partial H_y(\omega)}{\partial z}\right] \\
j\omega\left(1+\frac{\sigma'_x}{j\omega\varepsilon_0}\right)\left(1+\frac{\sigma'_y}{j\omega\varepsilon_0}\right)^{-1}\left(1+\frac{\sigma'_z}{j\omega\varepsilon_0}\right)\tilde{D}_y(\omega) &= c_0\left[\frac{\partial H_x(\omega)}{\partial z} - \frac{\partial H_z(\omega)}{\partial x}\right] \\
j\omega\left(1+\frac{\sigma'_x}{j\omega\varepsilon_0}\right)\left(1+\frac{\sigma'_y}{j\omega\varepsilon_0}\right)\left(1+\frac{\sigma'_z}{j\omega\varepsilon_0}\right)^{-1}\tilde{D}_z(\omega) &= c_0\left[\frac{\partial H_y(\omega)}{\partial x} - \frac{\partial H_x(\omega)}{\partial y}\right]
\end{aligned}$$

$$\vec{\tilde{D}}(\omega) = [\varepsilon_r]\vec{\tilde{E}}(\omega) \quad \Rightarrow \quad
\begin{aligned}
\tilde{D}_x(\omega) &= \varepsilon_{xx}\tilde{E}_x(\omega) \\
\tilde{D}_y(\omega) &= \varepsilon_{yy}\tilde{E}_y(\omega) \\
\tilde{D}_z(\omega) &= \varepsilon_{zz}\tilde{E}_z(\omega)
\end{aligned}$$

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Same Equations in the Time-Domain

$$\frac{\partial}{\partial t}H_x(t)+\frac{\sigma'_y+\sigma'_z}{\varepsilon_0}H_x(t)+\frac{\sigma'_y\sigma'_z}{\varepsilon_0^2}\int_{-\infty}^{t}H_x(\tau)d\tau=-\frac{c_0}{\mu_{xx}}\left[\frac{\partial\tilde{E}_z(t)}{\partial y}-\frac{\partial\tilde{E}_y(t)}{\partial z}\right]-\frac{c_0\sigma'_x}{\varepsilon_0\mu_{xx}}\int_{-\infty}^{t}\left[\frac{\partial\tilde{E}_z(\tau)}{\partial y}-\frac{\partial\tilde{E}_y(\tau)}{\partial z}\right]d\tau$$

$$\frac{\partial}{\partial t}H_y(t)+\frac{\sigma'_x+\sigma'_z}{\varepsilon_0}H_y(t)+\frac{\sigma'_x\sigma'_z}{\varepsilon_0^2}\int_{-\infty}^{t}H_y(\tau)d\tau=-\frac{c_0}{\mu_{yy}}\left[\frac{\partial\tilde{E}_x(t)}{\partial z}-\frac{\partial\tilde{E}_z(t)}{\partial x}\right]-\frac{c_0\sigma'_y}{\varepsilon_0\mu_{yy}}\int_{-\infty}^{t}\left[\frac{\partial\tilde{E}_x(\tau)}{\partial z}-\frac{\partial\tilde{E}_z(\tau)}{\partial x}\right]d\tau$$

$$\frac{\partial}{\partial t}H_z(t)+\frac{\sigma'_x+\sigma'_y}{\varepsilon_0}H_z(t)+\frac{\sigma'_x\sigma'_y}{\varepsilon_0^2}\int_{-\infty}^{t}H_z(\tau)d\tau=-\frac{c_0}{\mu_{zz}}\left[\frac{\partial\tilde{E}_y(t)}{\partial x}-\frac{\partial\tilde{E}_x(t)}{\partial y}\right]-\frac{c_0\sigma'_z}{\varepsilon_0\mu_{zz}}\int_{-\infty}^{t}\left[\frac{\partial\tilde{E}_y(\tau)}{\partial x}-\frac{\partial\tilde{E}_x(\tau)}{\partial y}\right]d\tau$$

$$\frac{\partial}{\partial t}\tilde{D}_x(t)+\frac{\sigma'_y+\sigma'_z}{\varepsilon_0}\tilde{D}_x(t)+\frac{\sigma'_y\sigma'_z}{\varepsilon_0^2}\int_{-\infty}^{t}\tilde{D}_x(\tau)d\tau=c_0\left[\frac{\partial H_z(t)}{\partial y}-\frac{\partial H_y(t)}{\partial z}\right]+\frac{c_0\sigma'_x}{\varepsilon_0}\int_{-\infty}^{t}\left[\frac{\partial H_z(\tau)}{\partial y}-\frac{\partial H_y(\tau)}{\partial z}\right]d\tau$$

$$\frac{\partial}{\partial t}\tilde{D}_y(t)+\frac{\sigma'_x+\sigma'_z}{\varepsilon_0}\tilde{D}_y(t)+\frac{\sigma'_x\sigma'_z}{\varepsilon_0^2}\int_{-\infty}^{t}\tilde{D}_y(\tau)d\tau=c_0\left[\frac{\partial H_x(t)}{\partial z}-\frac{\partial H_z(t)}{\partial x}\right]+\frac{c_0\sigma'_y}{\varepsilon_0}\int_{-\infty}^{t}\left[\frac{\partial H_x(\tau)}{\partial z}-\frac{\partial H_z(\tau)}{\partial x}\right]d\tau$$

$$\frac{\partial}{\partial t}\tilde{D}_z(t)+\frac{\sigma'_x+\sigma'_y}{\varepsilon_0}\tilde{D}_z(t)+\frac{\sigma'_x\sigma'_y}{\varepsilon_0^2}\int_{-\infty}^{t}\tilde{D}_z(\tau)d\tau=c_0\left[\frac{\partial H_y(t)}{\partial x}-\frac{\partial H_x(t)}{\partial y}\right]+\frac{c_0\sigma'_z}{\varepsilon_0}\int_{-\infty}^{t}\left[\frac{\partial H_y(\tau)}{\partial x}-\frac{\partial H_x(\tau)}{\partial y}\right]d\tau$$

$$\tilde{D}_x(t)=\varepsilon_{xx}\tilde{E}_x(t)$$
$$\tilde{D}_y(t)=\varepsilon_{yy}\tilde{E}_y(t)$$
$$\tilde{D}_z(t)=\varepsilon_{zz}\tilde{E}_z(t)$$

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Update Equations

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Finite-Difference Approximations

Putting all the terms together, the equation is

$$\frac{\partial H_x(t)}{\partial t}+\frac{\sigma'_y+\sigma'_z}{\varepsilon_0}H_x(t)+\int_{-\infty}^{t}\frac{\sigma'_y\sigma'_z}{\varepsilon_0^2}H_x(\tau)\,d\tau=-\frac{c_0}{\mu_{xx}}C_x^E(t)-\int_{-\infty}^{t}\frac{c_0\sigma'_x}{\varepsilon_0\mu_{xx}}C_x^E(\tau)\,d\tau$$

$$\frac{H_x\big|_{t+\Delta t/2}^{i,j,k}-H_x\big|_{t-\Delta t/2}^{i,j,k}}{\Delta t}+\frac{\sigma'_y\big|^{i,j,k}+\sigma'_z\big|^{i,j,k}}{\varepsilon_0}\frac{H_x\big|_{t+\Delta t/2}^{i,j,k}+H_x\big|_{t-\Delta t/2}^{i,j,k}}{2}+\frac{\sigma'_y\sigma'_z\Delta t}{\varepsilon_0^2}\left(\frac{H_x\big|_{t-\Delta t/2}^{i,j,k}+H_x\big|_{t+\Delta t/2}^{i,j,k}}{4}+\sum_{T=\Delta t/2}^{t-\Delta t/2}H_x\big|_T^{i,j,k}\right)$$

$$=-\frac{c_0}{\mu_{xx}\big|^{i,j,k}}C_x^E\big|_t^{i,j,k}-\frac{c_0\Delta t\,\sigma'_x\big|^{i,j,k}}{\varepsilon_0\,\mu_{xx}\big|^{i,j,k}}\sum_{T=0}^{t}C_x^E\big|_T^{i,j,k}$$

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Summary of All Numerical Equations

$$\frac{H_x\big|_{t+\Delta t/2}^{i,j,k}-H_x\big|_{t-\Delta t/2}^{i,j,k}}{\Delta t}+\frac{\sigma_y^H\big|^{i,j,k}+\sigma_z^H\big|^{i,j,k}}{\varepsilon_0}\left(\frac{H_x\big|_{t+\frac{\Delta t}{2}}^{i,j,k}+H_x\big|_{t-\frac{\Delta t}{2}}^{i,j,k}}{2}\right)+\frac{\left(\sigma_y^H\big|^{i,j,k}\right)\left(\sigma_z^H\big|^{i,j,k}\right)\Delta t}{\varepsilon_0^2}\left[\frac{1}{4}\left(H_x\big|_{t+\frac{\Delta t}{2}}^{i,j,k}+H_x\big|_{t-\frac{\Delta t}{2}}^{i,j,k}\right)+\sum_{T=0}^{t-\frac{\Delta t}{2}}H_x\big|_T^{i,j,k}\right]=-\frac{c_0}{\mu_{xx}\big|^{i,j,k}}C_x^E\big|_t^{i,j,k}-\frac{c_0\Delta t\,\sigma'_x\big|^{i,j,k}}{\varepsilon_0\,\mu_{xx}\big|^{i,j,k}}\sum_{T=0}^{t}C_x^E\big|_T^{i,j,k}$$

$$\frac{H_y\big|_{t+\Delta t/2}^{i,j,k}-H_y\big|_{t-\Delta t/2}^{i,j,k}}{\Delta t}+\frac{\sigma_x^H\big|^{i,j,k}+\sigma_z^H\big|^{i,j,k}}{\varepsilon_0}\left(\frac{H_y\big|_{t+\frac{\Delta t}{2}}^{i,j,k}+H_y\big|_{t-\frac{\Delta t}{2}}^{i,j,k}}{2}\right)+\frac{\left(\sigma_x^H\big|^{i,j,k}\right)\left(\sigma_z^H\big|^{i,j,k}\right)\Delta t}{\varepsilon_0^2}\left[\frac{1}{4}\left(H_y\big|_{t+\frac{\Delta t}{2}}^{i,j,k}+H_y\big|_{t-\frac{\Delta t}{2}}^{i,j,k}\right)+\sum_{T=0}^{t-\frac{\Delta t}{2}}H_y\big|_T^{i,j,k}\right]=-\frac{c_0}{\mu_{yy}\big|^{i,j,k}}C_y^E\big|_t^{i,j,k}-\frac{c_0\Delta t\,\sigma'_y\big|^{i,j,k}}{\varepsilon_0\,\mu_{yy}\big|^{i,j,k}}\sum_{T=0}^{t}C_y^E\big|_T^{i,j,k}$$

$$\frac{H_z\big|_{t+\Delta t/2}^{i,j,k}-H_z\big|_{t-\Delta t/2}^{i,j,k}}{\Delta t}+\frac{\sigma_x^H\big|^{i,j,k}+\sigma_y^H\big|^{i,j,k}}{\varepsilon_0}\left(\frac{H_z\big|_{t+\frac{\Delta t}{2}}^{i,j,k}+H_z\big|_{t-\frac{\Delta t}{2}}^{i,j,k}}{2}\right)+\frac{\left(\sigma_x^H\big|^{i,j,k}\right)\left(\sigma_y^H\big|^{i,j,k}\right)\Delta t}{\varepsilon_0^2}\left[\frac{1}{4}\left(H_z\big|_{t+\frac{\Delta t}{2}}^{i,j,k}+H_z\big|_{t-\frac{\Delta t}{2}}^{i,j,k}\right)+\sum_{T=0}^{t-\frac{\Delta t}{2}}H_z\big|_T^{i,j,k}\right]=-\frac{c_0}{\mu_{zz}\big|^{i,j,k}}C_z^E\big|_t^{i,j,k}-\frac{c_0\Delta t\,\sigma'_z\big|^{i,j,k}}{\varepsilon_0\,\mu_{zz}\big|^{i,j,k}}\sum_{T=0}^{t}C_z^E\big|_T^{i,j,k}$$

$$\frac{\tilde{D}_x\big|_{t+\Delta t}^{i,j,k}-\tilde{D}_x\big|_t^{i,j,k}}{\Delta t}+\frac{\sigma_y^D\big|^{i,j,k}+\sigma_z^D\big|^{i,j,k}}{\varepsilon_0}\left(\frac{\tilde{D}_x\big|_{t+\Delta t}^{i,j,k}+\tilde{D}_x\big|_t^{i,j,k}}{2}\right)+\frac{\left(\sigma_y^D\big|^{i,j,k}\right)\left(\sigma_z^D\big|^{i,j,k}\right)\Delta t}{\varepsilon_0^2}\left[\frac{1}{4}\left(\tilde{D}_x\big|_{t+\Delta t}^{i,j,k}+\tilde{D}_x\big|_t^{i,j,k}\right)+\sum_{T=0}^{t-\Delta t}\tilde{D}_x\big|_T^{i,j,k}\right]=-c_0\,C_x^H\big|_{t+\Delta t/2}^{i,j,k}-\frac{c_0\Delta t\,\sigma_x^D\big|^{i,j,k}}{\varepsilon_0}\sum_{T=0}^{t-\frac{\Delta t}{2}}C_x^H\big|_T^{i,j,k}$$

$$\frac{\tilde{D}_y\big|_{t+\Delta t}^{i,j,k}-\tilde{D}_y\big|_t^{i,j,k}}{\Delta t}+\frac{\sigma_x^D\big|^{i,j,k}+\sigma_z^D\big|^{i,j,k}}{\varepsilon_0}\left(\frac{\tilde{D}_y\big|_{t+\Delta t}^{i,j,k}+\tilde{D}_y\big|_t^{i,j,k}}{2}\right)+\frac{\left(\sigma_x^D\big|^{i,j,k}\right)\left(\sigma_z^D\big|^{i,j,k}\right)\Delta t}{\varepsilon_0^2}\left[\frac{1}{4}\left(\tilde{D}_y\big|_{t+\Delta t}^{i,j,k}+\tilde{D}_y\big|_t^{i,j,k}\right)+\sum_{T=0}^{t-\Delta t}\tilde{D}_y\big|_T^{i,j,k}\right]=-c_0\,C_y^H\big|_{t+\Delta t/2}^{i,j,k}-\frac{c_0\Delta t\,\sigma_y^D\big|^{i,j,k}}{\varepsilon_0}\sum_{T=0}^{t-\frac{\Delta t}{2}}C_y^H\big|_T^{i,j,k}$$

$$\frac{\tilde{D}_z\big|_{t+\Delta t}^{i,j,k}-\tilde{D}_z\big|_t^{i,j,k}}{\Delta t}+\frac{\sigma_x^D\big|^{i,j,k}+\sigma_y^D\big|^{i,j,k}}{\varepsilon_0}\left(\frac{\tilde{D}_z\big|_{t+\Delta t}^{i,j,k}+\tilde{D}_z\big|_t^{i,j,k}}{2}\right)+\frac{\left(\sigma_x^D\big|^{i,j,k}\right)\left(\sigma_y^D\big|^{i,j,k}\right)\Delta t}{\varepsilon_0^2}\left[\frac{1}{4}\left(\tilde{D}_z\big|_{t+\Delta t}^{i,j,k}+\tilde{D}_z\big|_t^{i,j,k}\right)+\sum_{T=0}^{t-\Delta t}\tilde{D}_z\big|_T^{i,j,k}\right]=-c_0\,C_z^H\big|_{t+\Delta t/2}^{i,j,k}-\frac{c_0\Delta t\,\sigma_z^D\big|^{i,j,k}}{\varepsilon_0}\sum_{T=0}^{t-\frac{\Delta t}{2}}C_z^H\big|_T^{i,j,k}$$

$$\tilde{D}_x\big|_t^{i,j,k}=\left(\varepsilon_{xx}\big|^{i,j,k}\right)\tilde{E}_x\big|_t^{i,j,k}$$

$$\tilde{D}_y\big|_t^{i,j,k}=\left(\varepsilon_{yy}\big|^{i,j,k}\right)\tilde{E}_y\big|_t^{i,j,k}$$

$$\tilde{D}_z\big|_t^{i,j,k}=\left(\varepsilon_{zz}\big|^{i,j,k}\right)\tilde{E}_z\big|_t^{i,j,k}$$

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Solve for H_x at the Future Time Step

Solving our numerical equation for the H field at $t+\Delta t/2$ yields

$$\frac{\left.H_x\right|_{t+\Delta t/2}^{i,j,k}-\left.H_x\right|_{t-\Delta t/2}^{i,j,k}}{\Delta t}+\frac{\left.\sigma_y^H\right|^{i,j,k}+\left.\sigma_z^H\right|^{i,j,k}}{\varepsilon_0}\left(\frac{\left.H_x\right|_{t+\frac{\Delta t}{2}}^{i,j,k}+\left.H_x\right|_{t-\frac{\Delta t}{2}}^{i,j,k}}{2}\right)+\frac{\left(\left.\sigma_y^H\right|^{i,j,k}\right)\left(\left.\sigma_z^H\right|^{i,j,k}\right)\Delta t}{\varepsilon_0^2}\left[\frac{1}{4}\left(\left.H_x\right|_{t+\frac{\Delta t}{2}}^{i,j,k}+\left.H_x\right|_{t-\frac{\Delta t}{2}}^{i,j,k}\right)+\sum_{T=0}^{t-\frac{\Delta t}{2}}\left.H_x\right|_T^{i,j,k}\right]=-\frac{c_0}{\left.\mu_{xx}\right|^{i,j,k}}\left.C_x^E\right|_t^{i,j,k}-\frac{c_0\Delta t\,\left.\sigma_x'\right|^{i,j,k}}{\varepsilon_0\,\left.\mu_{xx}\right|^{i,j,k}}\sum_{T=0}^{t}\left.C_x^E\right|_T^{i,j,k}$$

⬇

$$\left.H_x\right|_{t+\frac{\Delta t}{2}}^{i,j,k}=\left[\frac{\frac{1}{\Delta t}-\left(\frac{\left.\sigma_y^H\right|^{i,j,k}+\left.\sigma_z^H\right|^{i,j,k}}{2\varepsilon_0}\right)-\frac{\left(\left.\sigma_y^H\right|^{i,j,k}\right)\left(\left.\sigma_z^H\right|^{i,j,k}\right)\Delta t}{4\varepsilon_0^2}}{\frac{1}{\Delta t}+\left(\frac{\left.\sigma_y^H\right|^{i,j,k}+\left.\sigma_z^H\right|^{i,j,k}}{2\varepsilon_0}\right)+\frac{\left(\left.\sigma_y^H\right|^{i,j,k}\right)\left(\left.\sigma_z^H\right|^{i,j,k}\right)\Delta t}{4\varepsilon_0^2}}\right]\left.H_x\right|_{t-\frac{\Delta t}{2}}^{i,j,k}+\left[-\frac{c_0/\left.\mu_{xx}\right|^{i,j,k}}{\frac{1}{\Delta t}+\left(\frac{\left.\sigma_y^H\right|^{i,j,k}+\left.\sigma_z^H\right|^{i,j,k}}{2\varepsilon_0}\right)+\frac{\left(\left.\sigma_y^H\right|^{i,j,k}\right)\left(\left.\sigma_z^H\right|^{i,j,k}\right)\Delta t}{4\varepsilon_0^2}}\right]\left.C_x^E\right|_t^{i,j,k}$$

$$+\left[-\frac{\frac{c_0\Delta t}{\varepsilon_0}\frac{\left.\sigma_x^H\right|^{i,j,k}}{\left.\mu_{xx}\right|^{i,j,k}}}{\frac{1}{\Delta t}+\left(\frac{\left.\sigma_y^H\right|^{i,j,k}+\left.\sigma_z^H\right|^{i,j,k}}{2\varepsilon_0}\right)+\frac{\left(\left.\sigma_y^H\right|^{i,j,k}\right)\left(\left.\sigma_z^H\right|^{i,j,k}\right)\Delta t}{4\varepsilon_0^2}}\right]\sum_{T=0}^{t}\left.C_x^E\right|_t^{i,j,k}+\left[-\frac{\frac{\Delta t}{\varepsilon_0^2}\left(\left.\sigma_y^H\right|^{i,j,k}\right)\left(\left.\sigma_z^H\right|^{i,j,k}\right)}{\frac{1}{\Delta t}+\left(\frac{\left.\sigma_y^H\right|^{i,j,k}+\left.\sigma_z^H\right|^{i,j,k}}{2\varepsilon_0}\right)+\frac{\left(\left.\sigma_y^H\right|^{i,j,k}\right)\left(\left.\sigma_z^H\right|^{i,j,k}\right)\Delta t}{4\varepsilon_0^2}}\right]\sum_{T=0}^{t-\frac{\Delta t}{2}}\left.H_x\right|_T^{i,j,k}$$

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Final Form of the Update Equation for H_x

The update coefficients are computed before the main FDTD loop.

$$\left.m_{Hx0}\right|^{i,j,k}=\frac{1}{\Delta t}+\left(\frac{\left.\sigma_y^H\right|^{i,j,k}+\left.\sigma_z^H\right|^{i,j,k}}{2\varepsilon_0}\right)+\frac{\left(\left.\sigma_y^H\right|^{i,j,k}\right)\left(\left.\sigma_z^H\right|^{i,j,k}\right)\Delta t}{4\varepsilon_0^2}\qquad \left.m_{Hx1}\right|^{i,j,k}=\frac{1}{\left.m_{Hx0}\right|^{i,j,k}}\left[\frac{1}{\Delta t}-\left(\frac{\left.\sigma_y^H\right|^{i,j,k}+\left.\sigma_z^H\right|^{i,j,k}}{2\varepsilon_0}\right)-\frac{\left(\left.\sigma_y^H\right|^{i,j,k}\right)\left(\left.\sigma_z^H\right|^{i,j,k}\right)\Delta t}{4\varepsilon_0^2}\right]$$

$$\left.m_{Hx2}\right|^{i,j,k}=-\frac{1}{\left.m_{Hx0}\right|^{i,j,k}}\frac{c_0}{\left.\mu_{xx}\right|^{i,j,k}}\qquad \left.m_{Hx3}\right|^{i,j,k}=-\frac{1}{\left.m_{Hx0}\right|^{i,j,k}}\frac{c_0\Delta t}{\varepsilon_0}\frac{\left.\sigma_x^H\right|^{i,j,k}}{\left.\mu_{xx}\right|^{i,j,k}}\qquad \left.m_{Hx4}\right|^{i,j,k}=-\frac{1}{\left.m_{Hx0}\right|^{i,j,k}}\frac{\Delta t}{\varepsilon_0^2}\left(\left.\sigma_y^H\right|^{i,j,k}\right)\left(\left.\sigma_z^H\right|^{i,j,k}\right)$$

The integration terms are computed inside the main FDTD loop, but before the update equation.

$$\left.I_{CEx}\right|_t^{i,j,k}=\sum_{T=0}^{t}\left.C_x^E\right|_t^{i,j,k}\qquad \left.I_{Hx}\right|_{t-\frac{\Delta t}{2}}^{i,j,k}=\sum_{T=0}^{t-\frac{\Delta t}{2}}\left.H_x\right|_T^{i,j,k}\qquad \left.C_x^E\right|_t^{i,j,k}=\frac{\left.\tilde{E}_z\right|_t^{i,j+1,k}-\left.\tilde{E}_z\right|_t^{i,j,k}}{\Delta y}-\frac{\left.\tilde{E}_y\right|_t^{i,j,k+1}-\left.\tilde{E}_y\right|_t^{i,j,k}}{\Delta z}$$

The update equation is computed inside the main FDTD loop immediately after the integration terms are updated.

$$\left.H_x\right|_{t+\Delta t/2}^{i,j,k}=\left(\left.m_{Hx1}\right|^{i,j,k}\right)\left.H_x\right|_{t-\Delta t/2}^{i,j,k}+\left(\left.m_{Hx2}\right|^{i,j,k}\right)\left.C_x^E\right|_t^{i,j,k}+\left(\left.m_{Hx3}\right|^{i,j,k}\right)\left.I_{CEx}\right|_t^{i,j,k}+\left(\left.m_{Hx4}\right|^{i,j,k}\right)\left.I_{Hx}\right|_t^{i,j,k}$$

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Final Form of the Update Equation for H_y

The update coefficients are computed before the main FDTD loop.

$$m_{Hy0}\big|^{i,j,k} = \frac{1}{\Delta t} + \left(\frac{\sigma_x^H\big|^{i,j,k} + \sigma_z^H\big|^{i,j,k}}{2\varepsilon_0}\right) + \frac{\left(\sigma_x^H\big|^{i,j,k}\right)\left(\sigma_z^H\big|^{i,j,k}\right)\Delta t}{4\varepsilon_0^2} \qquad m_{Hy1}\big|^{i,j,k} = \frac{1}{m_{Hy0}\big|^{i,j,k}}\left[\frac{1}{\Delta t} - \left(\frac{\sigma_x^H\big|^{i,j,k} + \sigma_z^H\big|^{i,j,k}}{2\varepsilon_0}\right) - \frac{\left(\sigma_x^H\big|^{i,j,k}\right)\left(\sigma_z^H\big|^{i,j,k}\right)\Delta t}{4\varepsilon_0^2}\right]$$

$$m_{Hy2}\big|^{i,j,k} = -\frac{1}{m_{Hy0}\big|^{i,j,k}}\frac{c_0}{\mu_{yy}\big|^{i,j,k}} \qquad m_{Hy3}\big|^{i,j,k} = -\frac{1}{m_{Hy0}\big|^{i,j,k}}\frac{c_0\Delta t}{\varepsilon_0}\frac{\sigma_y^H\big|^{i,j,k}}{\mu_{yy}\big|^{i,j,k}} \qquad m_{Hy4}\big|^{i,j,k} = -\frac{1}{m_{Hy0}\big|^{i,j,k}}\frac{\Delta t}{\varepsilon_0^2}\left(\sigma_x^H\big|^{i,j,k}\right)\left(\sigma_z^H\big|^{i,j,k}\right)$$

The integration terms are computed inside the main FDTD loop, but before the update equation.

$$I_{CEy}\big|_t^{i,j,k} = \sum_{T=0}^{t} C_y^E\big|_T^{i,j,k} \qquad I_{Hy}\big|_{t-\frac{\Delta t}{2}}^{i,j,k} = \sum_{T=0}^{t-\frac{\Delta t}{2}} H_y\big|_T^{i,j,k} \qquad C_y^E\big|_t^{i,j,k} = \frac{\tilde{E}_x\big|_t^{i,j,k+1} - \tilde{E}_x\big|_t^{i,j,k}}{\Delta z} - \frac{\tilde{E}_z\big|_t^{i+1,j,k} - \tilde{E}_z\big|_t^{i,j,k}}{\Delta x}$$

The update equation is computed inside the main FDTD loop immediately after the integration terms are updated.

$$H_y\big|_{t+\frac{\Delta t}{2}}^{i,j,k} = \left(m_{Hy1}\big|^{i,j,k}\right)H_y\big|_{t-\frac{\Delta t}{2}}^{i,j,k} + \left(m_{Hy2}\big|^{i,j,k}\right)C_y^E\big|_t^{i,j,k} + \left(m_{Hy3}\big|^{i,j,k}\right)I_{CEy}\big|_t^{i,j,k} + \left(m_{Hy4}\big|^{i,j,k}\right)I_{Hy}\big|_{t-\frac{\Delta t}{2}}^{i,j,k}$$

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Final Form of the Update Equation for H_z

The update coefficients are computed before the main FDTD loop.

$$m_{Hz0}\big|^{i,j,k} = \frac{1}{\Delta t} + \left(\frac{\sigma_x^H\big|^{i,j,k} + \sigma_y^H\big|^{i,j,k}}{2\varepsilon_0}\right) + \frac{\left(\sigma_x^H\big|^{i,j,k}\right)\left(\sigma_y^H\big|^{i,j,k}\right)\Delta t}{4\varepsilon_0^2} \qquad m_{Hz1}\big|^{i,j,k} = \frac{1}{m_{Hz0}\big|^{i,j,k}}\left[\frac{1}{\Delta t} - \left(\frac{\sigma_x^H\big|^{i,j,k} + \sigma_y^H\big|^{i,j,k}}{2\varepsilon_0}\right) - \frac{\left(\sigma_x^H\big|^{i,j,k}\right)\left(\sigma_y^H\big|^{i,j,k}\right)\Delta t}{4\varepsilon_0^2}\right]$$

$$m_{Hz2}\big|^{i,j,k} = -\frac{1}{m_{Hz0}\big|^{i,j,k}}\frac{c_0}{\mu_{zz}\big|^{i,j,k}} \qquad m_{Hz3}\big|^{i,j,k} = -\frac{1}{m_{Hz0}\big|^{i,j,k}}\frac{c_0\Delta t}{\varepsilon_0}\frac{\sigma_z^H\big|^{i,j,k}}{\mu_{zz}\big|^{i,j,k}} \qquad m_{Hz4}\big|^{i,j,k} = -\frac{1}{m_{Hz0}\big|^{i,j,k}}\frac{\Delta t}{\varepsilon_0^2}\left(\sigma_x^H\big|^{i,j,k}\right)\left(\sigma_y^H\big|^{i,j,k}\right)$$

The integration terms are computed inside the main FDTD loop, but before the update equation.

$$I_{CEz}\big|_t^{i,j,k} = \sum_{T=0}^{t} C_z^E\big|_t^{i,j,k} \qquad I_{Hz}\big|_{t-\frac{\Delta t}{2}}^{i,j,k} = \sum_{T=0}^{t-\frac{\Delta t}{2}} H_z\big|_T^{i,j,k} \qquad C_z^E\big|_t^{i,j,k} = \frac{\tilde{E}_y\big|_t^{i+1,j,k} - \tilde{E}_y\big|_t^{i,j,k}}{\Delta x} - \frac{\tilde{E}_x\big|_t^{i,j+1,k} - \tilde{E}_x\big|_t^{i,j,k}}{\Delta y}$$

The update equation is computed inside the main FDTD loop immediately after the integration terms are updated.

$$H_z\big|_{t+\frac{\Delta t}{2}}^{i,j,k} = \left(m_{Hz1}\big|^{i,j,k}\right)H_z\big|_{t-\frac{\Delta t}{2}}^{i,j,k} + \left(m_{Hz2}\big|^{i,j,k}\right)C_z^E\big|_t^{i,j,k} + \left(m_{Hz3}\big|^{i,j,k}\right)I_{CEz}\big|_t^{i,j,k} + \left(m_{Hz4}\big|^{i,j,k}\right)I_{Hz}\big|_{t-\frac{\Delta t}{2}}^{i,j,k}$$

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Final Form of the Update Equation for D_x

The update coefficients are computed before the main FDTD loop.

$$m_{Dx0}\Big|^{i,j,k} = \frac{1}{\Delta t} + \frac{\sigma_y^D\big|^{i,j,k} + \sigma_z^D\big|^{i,j,k}}{2\varepsilon_0} + \frac{\left(\sigma_y^D\big|^{i,j,k}\right)\left(\sigma_z^D\big|^{i,j,k}\right)\Delta t}{4\varepsilon_0^2} \qquad m_{Dx1}\Big|^{i,j,k} = \frac{1}{m_{Dx0}\big|^{i,j,k}}\left[\frac{1}{\Delta t} - \frac{\sigma_y^D\big|^{i,j,k} + \sigma_z^D\big|^{i,j,k}}{2\varepsilon_0} - \frac{\left(\sigma_y^D\big|^{i,j,k}\right)\left(\sigma_z^D\big|^{i,j,k}\right)\Delta t}{4\varepsilon_0^2}\right]$$

$$m_{Dx2}\Big|^{i,j,k} = \frac{c_0}{m_{Dx0}\big|^{i,j,k}} \qquad m_{Dx3}\Big|^{i,j,k} = \frac{1}{m_{Dx0}\big|^{i,j,k}}\frac{c_0\Delta t\,\sigma_x^D\big|^{i,j,k}}{\varepsilon_0} \qquad m_{Dx4}\Big|^{i,j,k} = -\frac{1}{m_{Dx0}\big|^{i,j,k}}\frac{\Delta t}{\varepsilon_0^2}\left(\sigma_y^D\big|^{i,j,k}\right)\left(\sigma_z^D\big|^{i,j,k}\right)$$

The integration terms are computed inside the main FDTD loop, but before the update equation.

$$I_{CHx}\Big|_{t-\frac{\Delta t}{2}}^{i,j,k} = \sum_{T=0}^{t-\frac{\Delta t}{2}} C_x^H\Big|_T^{i,j,k} \qquad I_{Dx}\Big|_{t-\Delta t}^{i,j,k} = \sum_{T=0}^{t-\Delta t} \tilde{D}_x\Big|_T^{i,j,k} \qquad C_x^H\Big|_{t+\frac{\Delta t}{2}}^{i,j,k} = \frac{H_z\big|_{t+\frac{\Delta t}{2}}^{i,j,k} - H_z\big|_{t+\frac{\Delta t}{2}}^{i,j-1,k}}{\Delta y} - \frac{H_y\big|_{t+\frac{\Delta t}{2}}^{i,j,k} - H_y\big|_{t+\frac{\Delta t}{2}}^{i,j,k-1}}{\Delta z}$$

The update equation is computed inside the main FDTD loop immediately after the integration terms are updated.

$$\tilde{D}_x\Big|_{t+\Delta t}^{i,j,k} = \left(m_{Dx1}\big|^{i,j,k}\right)\tilde{D}_x\Big|_t^{i,j,k} + \left(m_{Dx2}\big|^{i,j,k}\right)C_x^H\Big|_{t+\frac{\Delta t}{2}}^{i,j,k} + \left(m_{Dx3}\big|^{i,j,k}\right)I_{CHx}\Big|_{t-\frac{\Delta t}{2}}^{i,j,k} + \left(m_{Dx4}\big|^{i,j,k}\right)I_{Dx}\Big|_{t-\Delta t}^{i,j,k}$$

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Final Form of the Update Equation for D_y

The update coefficients are computed before the main FDTD loop.

$$m_{Dy0}\Big|^{i,j,k} = \frac{1}{\Delta t} + \frac{\sigma_x^D\big|^{i,j,k} + \sigma_z^D\big|^{i,j,k}}{2\varepsilon_0} + \frac{\left(\sigma_x^D\big|^{i,j,k}\right)\left(\sigma_z^D\big|^{i,j,k}\right)\Delta t}{4\varepsilon_0^2} \qquad m_{Dy1}\Big|^{i,j,k} = \frac{1}{m_{Dy0}\big|^{i,j,k}}\left[\frac{1}{\Delta t} - \frac{\sigma_x^D\big|^{i,j,k} + \sigma_z^D\big|^{i,j,k}}{2\varepsilon_0} - \frac{\left(\sigma_x^D\big|^{i,j,k}\right)\left(\sigma_z^D\big|^{i,j,k}\right)\Delta t}{4\varepsilon_0^2}\right]$$

$$m_{Dy2}\Big|^{i,j,k} = \frac{c_0}{m_{Dy0}\big|^{i,j,k}} \qquad m_{Dy3}\Big|^{i,j,k} = \frac{1}{m_{Dy0}\big|^{i,j,k}}\frac{c_0\Delta t\,\sigma_y^D\big|^{i,j,k}}{\varepsilon_0} \qquad m_{Dy4}\Big|^{i,j,k} = -\frac{1}{m_{Dy0}\big|^{i,j,k}}\frac{\Delta t}{\varepsilon_0^2}\left(\sigma_x^D\big|^{i,j,k}\right)\left(\sigma_z^D\big|^{i,j,k}\right)$$

The integration terms are computed inside the main FDTD loop, but before the update equation.

$$I_{CHy}\Big|_{t-\frac{\Delta t}{2}}^{i,j,k} = \sum_{T=0}^{t-\frac{\Delta t}{2}} C_y^H\Big|_T^{i,j,k} \qquad I_{Dy}\Big|_{t-\Delta t}^{i,j,k} = \sum_{T=0}^{t-\Delta t} \tilde{D}_y\Big|_T^{i,j,k} \qquad C_y^H\Big|_{t+\frac{\Delta t}{2}}^{i,j,k} = \frac{H_x\big|_{t+\frac{\Delta t}{2}}^{i,j,k} - H_x\big|_{t+\frac{\Delta t}{2}}^{i,j,k-1}}{\Delta z} - \frac{H_z\big|_{t+\frac{\Delta t}{2}}^{i,j,k} - H_z\big|_{t+\frac{\Delta t}{2}}^{i-1,j,k}}{\Delta x}$$

The update equation is computed inside the main FDTD loop immediately after the integration terms are updated.

$$\tilde{D}_y\Big|_{t+\Delta t}^{i,j,k} = \left(m_{Dy1}\big|^{i,j,k}\right)\tilde{D}_y\Big|_t^{i,j,k} + \left(m_{Dy2}\big|^{i,j,k}\right)C_y^H\Big|_{t+\frac{\Delta t}{2}}^{i,j,k} + \left(m_{Dy3}\big|^{i,j,k}\right)I_{CHy}\Big|_{t-\frac{\Delta t}{2}}^{i,j,k} + \left(m_{Dy4}\big|^{i,j,k}\right)I_{Dy}\Big|_{t-\Delta t}^{i,j,k}$$

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Final Form of the Update Equation for D_z

The update coefficients are computed before the main FDTD loop.

$$m_{Dz0}\Big|^{i,j,k} = \frac{1}{\Delta t} + \frac{\sigma_x^D\big|^{i,j,k} + \sigma_y^D\big|^{i,j,k}}{2\varepsilon_0} + \frac{\left(\sigma_x^D\big|^{i,j,k}\right)\left(\sigma_y^D\big|^{i,j,k}\right)\Delta t}{4\varepsilon_0^2} \qquad m_{Dx1}\Big|^{i,j,k} = \frac{1}{m_{Dx0}\big|^{i,j,k}}\left[\frac{1}{\Delta t} - \frac{\sigma_x^D\big|^{i,j,k} + \sigma_y^D\big|^{i,j,k}}{2\varepsilon_0} - \frac{\left(\sigma_x^D\big|^{i,j,k}\right)\left(\sigma_y^D\big|^{i,j,k}\right)\Delta t}{4\varepsilon_0^2}\right]$$

$$m_{Dz2}\Big|^{i,j,k} = \frac{c_0}{m_{Dz0}\big|^{i,j,k}} \qquad m_{Dz3}\Big|^{i,j,k} = \frac{1}{m_{Dz0}\big|^{i,j,k}}\frac{c_0 \Delta t\, \sigma_z^D\big|^{i,j,k}}{\varepsilon_0} \qquad m_{Dz4}\Big|^{i,j,k} = -\frac{1}{m_{Dz0}\big|^{i,j,k}}\frac{\Delta t}{\varepsilon_0^2}\left(\sigma_x^D\big|^{i,j,k}\right)\left(\sigma_y^D\big|^{i,j,k}\right)$$

The integration terms are computed inside the main FDTD loop, but before the update equation.

$$I_{CHz}\Big|_{t-\frac{\Delta t}{2}}^{i,j,k} = \sum_{T=0}^{t-\frac{\Delta t}{2}} C_z^H\Big|_T^{i,j,k} \qquad I_{Dz}\Big|_{t-\Delta t}^{i,j,k} = \sum_{T=0}^{t-\Delta t} \tilde{D}_z\Big|_T^{i,j,k} \qquad C_z^H\Big|_{t+\frac{\Delta t}{2}}^{i,j,k} = \frac{H_y\big|_{t+\frac{\Delta t}{2}}^{i,j,k} - H_y\big|_{t+\frac{\Delta t}{2}}^{i-1,j,k}}{\Delta x} - \frac{H_x\big|_{t+\frac{\Delta t}{2}}^{i,j,k} - H_x\big|_{t+\frac{\Delta t}{2}}^{i,j-1,k}}{\Delta y}$$

The update equation is computed inside the main FDTD loop immediately after the integration terms are updated.

$$\tilde{D}_z\Big|_{t+\Delta t}^{i,j,k} = \left(m_{Dz1}\big|^{i,j,k}\right)\tilde{D}_z\Big|_t^{i,j,k} + \left(m_{Dz2}\big|^{i,j,k}\right)C_z^H\Big|_{t+\frac{\Delta t}{2}}^{i,j,k} + \left(m_{Dz3}\big|^{i,j,k}\right)I_{CHz}\Big|_{t-\frac{\Delta t}{2}}^{i,j,k} + \left(m_{Dz4}\big|^{i,j,k}\right)I_{Dz}\Big|_{t-\Delta t}^{i,j,k}$$

Final Update Equations for E_x, E_y, and E_z

The update coefficients are computed before the main FDTD loop.

$$m_{Ex1}\Big|^{i,j,k} = \frac{1}{\varepsilon_{xx}\big|^{i,j,k}} \qquad m_{Ey1}\Big|^{i,j,k} = \frac{1}{\varepsilon_{yy}\big|^{i,j,k}} \qquad m_{Ez1}\Big|^{i,j,k} = \frac{1}{\varepsilon_{zz}\big|^{i,j,k}}$$

The update equations are computed inside the main FDTD loop.

$$\tilde{E}_x\Big|_{t+\Delta t}^{i,j,k} = \left(m_{Ex1}\big|^{i,j,k}\right)\tilde{D}_x\Big|_{t+\Delta t}^{i,j,k}$$

$$\tilde{E}_y\Big|_{t+\Delta t}^{i,j,k} = \left(m_{Ey1}\big|^{i,j,k}\right)\tilde{D}_y\Big|_{t+\Delta t}^{i,j,k}$$

$$\tilde{E}_z\Big|_{t+\Delta t}^{i,j,k} = \left(m_{Ez1}\big|^{i,j,k}\right)\tilde{D}_z\Big|_{t+\Delta t}^{i,j,k}$$

Total-Field/Scattered-Field Source

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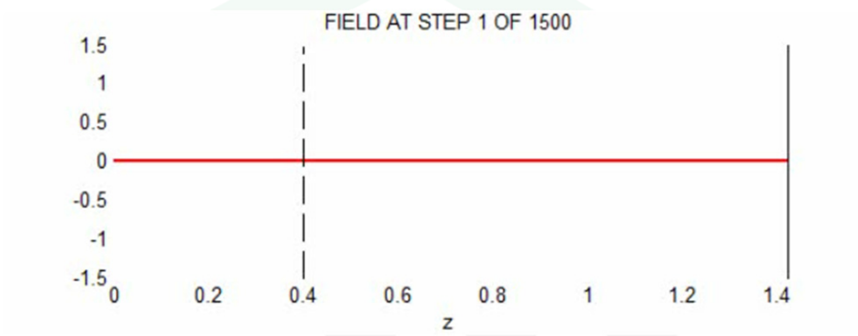
Source Types: *Simple Hard Source*



The simple hard source is the easiest to implement, but the location where the source is injected reflects waves 100%.

It is difficult to control the amplitude of this wave since power it injected in all directions.

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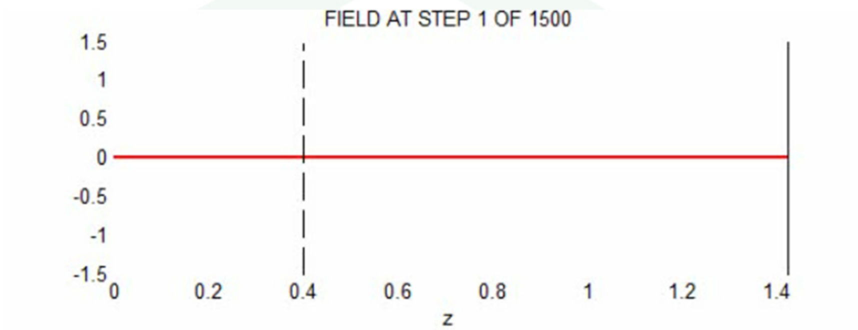
Source Types: *Simple Soft Source*



The simple soft source is almost as easy to implement and the waves pass completely through the location where the source is injected.

It is still difficult to control the amplitude of this wave since power it injected in all directions.

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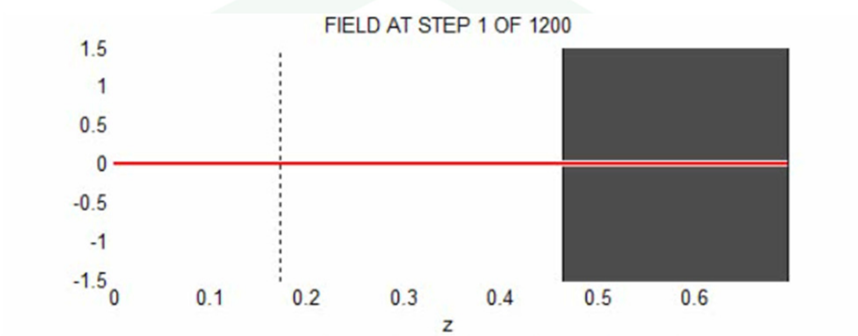
Source Types: *TF/SF Soft Source*



The TF/SF soft source is more difficult to implement, but is a soft source that is transparent to waves.

This method provides complete control over the amplitude of the source.

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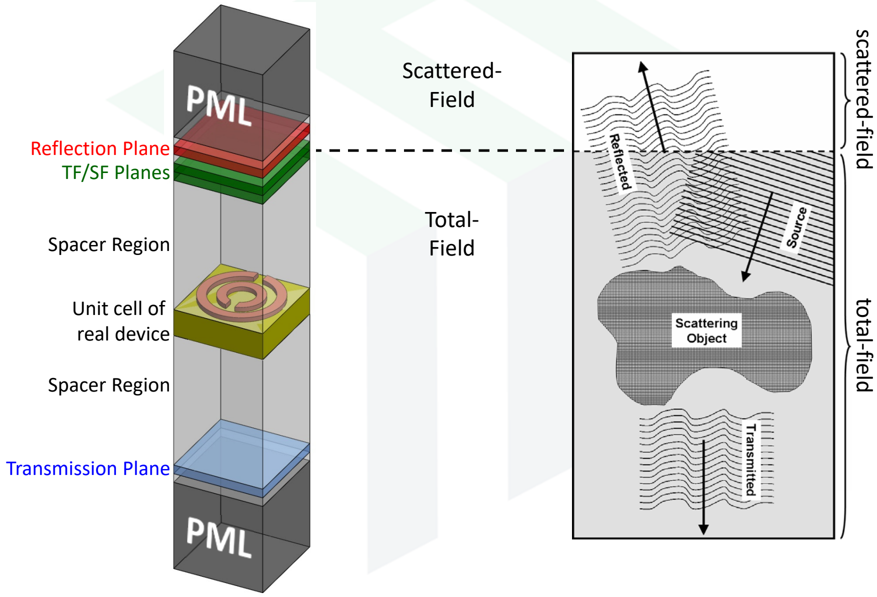
TF/SF Framework



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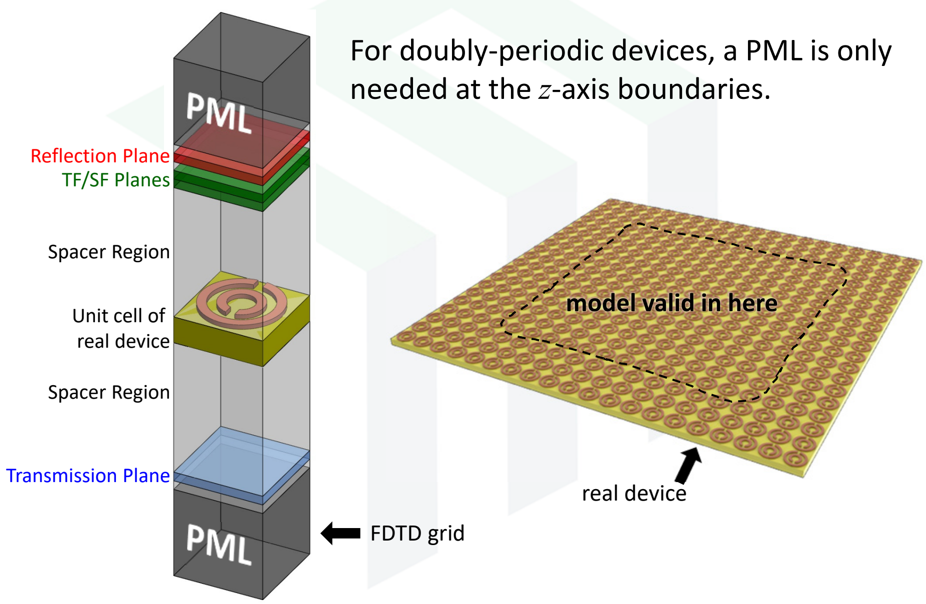
Grid Scheme for Doubly-Periodic Devices

For doubly-periodic devices, a PML is only needed at the z-axis boundaries.



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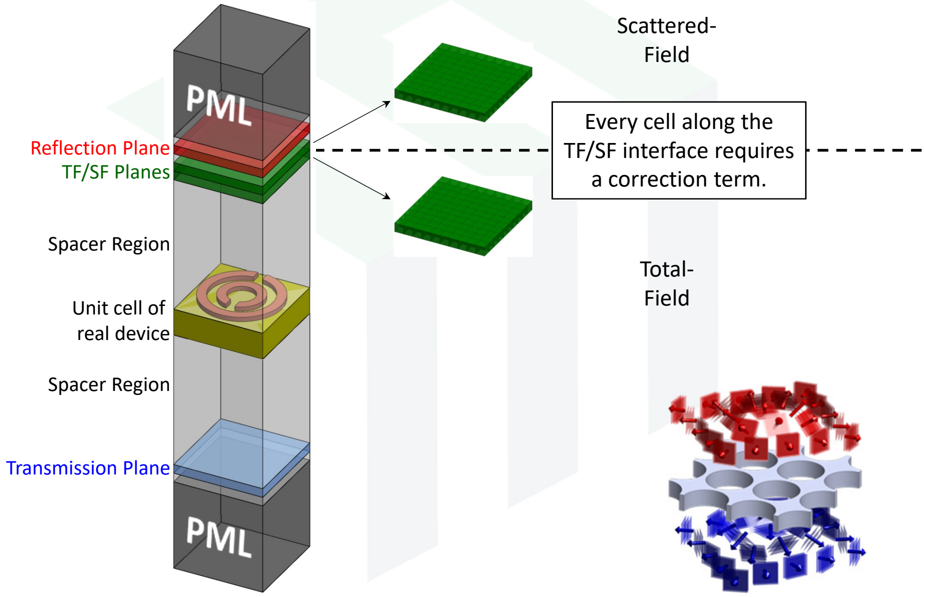
Corrections to Finite-Difference Eqs.



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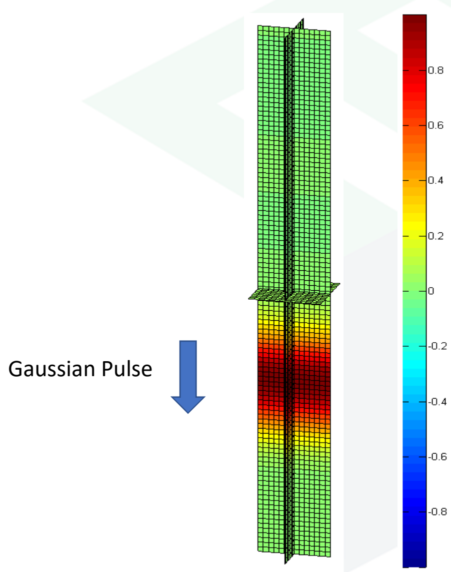
Typical View of 3D-FDTD with TF/SF



MATLAB's `slice()` function was used to generate this plot.

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Calculating Steady-State Fields

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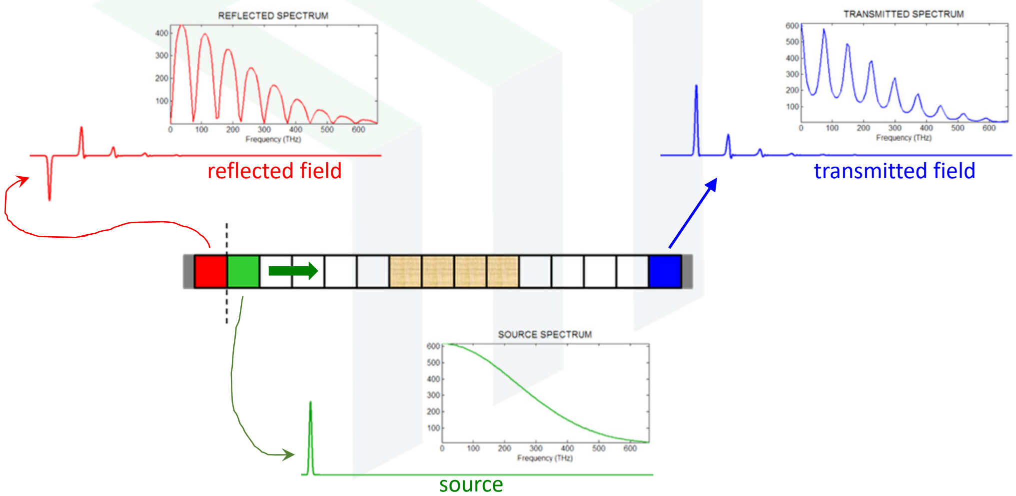
Brute Force Fourier Transforms

The easiest, but least memory efficient, method to compute a Fourier transform is to perform a simulation and record the desired field as a function of time. After the simulation is finished, these functions can be Fourier transformed using an FFT.



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Efficient Fourier Transform (1 of 2)

The standard Fourier transform is defined as

$$F(f)=\int_{-\infty}^{\infty} f(t)e^{-j2\pi ft}$$

If the function *f*(*t*) is only known at discrete points, the Fourier transform can be approximated numerically as

$$F(f)\cong\sum_{m=1}^{M} f(m\Delta t)e^{-j2\pi fm\Delta t}\Delta t$$

This can be written in a slightly different form.

$$F(f)\cong\Delta t\sum_{m=1}^{M}\left(e^{-j2\pi f\Delta t}\right)^{m}\cdot f(m\Delta t)$$

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Efficient Fourier Transform (2 of 2)

The final form on the previous slide suggests an efficient implementation. The Fourier transform is updated every iteration so by the end of the main loop:

$$F(f)\cong\Delta t\sum_{m=1}^{M}\left(e^{-j2\pi f\Delta t}\right)^{m}\cdot f(m)$$

This is simply the field value of interest at the current time step.

This multiplication can be done after the main FDTD loop in a post-processing step.

$e^{-j2\pi f\Delta t}$

This "kernel" can be computed prior to the main FDTD loop for each frequency of interest. The kernels can be stored in a 1D array.

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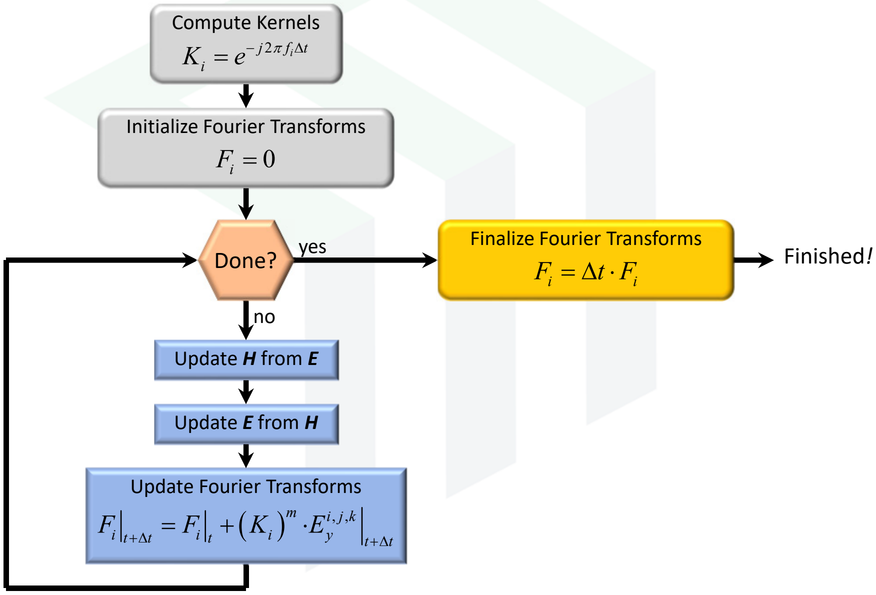
Efficient Fourier Transform Algorithm



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Calculating Transmission and Reflection

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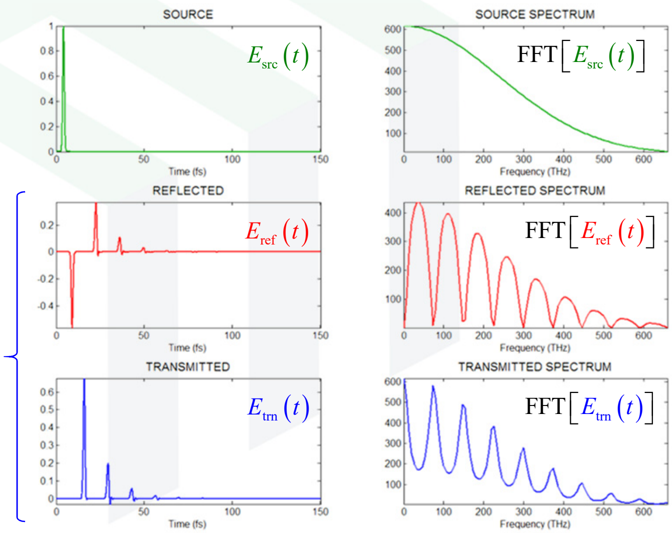
The Fourier Transforms

We typically start the computation of power by Fourier transforming the reflected and transmitted field using one of the methods described previously. Typical FDTD simulation results look like this…

A roll-off is observed in the frequency responses.

This occurs simply because there is less power in the source at the higher frequencies.

It does not mean the device is less reflective or transmissive.



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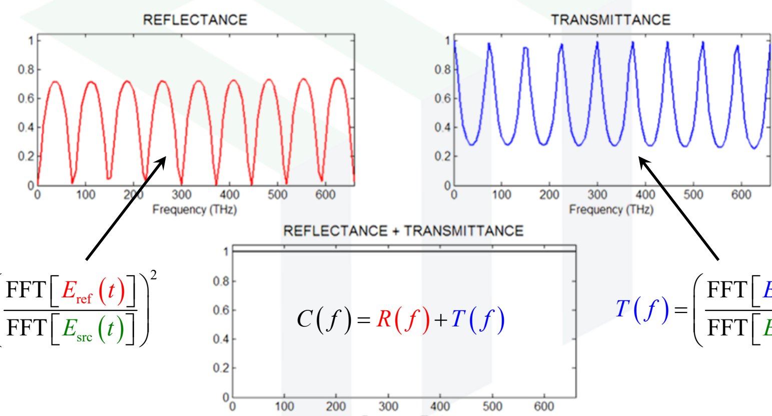
Normalize the Fourier Transforms

We must normalize the spectra to calculate transmittance and reflectance. We do this by dividing the reflection and transmission spectrum by the source spectrum.



$$R(f)=\left(\frac{\mathrm{FFT}[E_{ref}(t)]}{\mathrm{FFT}[E_{src}(t)]}\right)^2$$

$$C(f)=R(f)+T(f)$$

$$T(f)=\left(\frac{\mathrm{FFT}[E_{trn}(t)]}{\mathrm{FFT}[E_{src}(t)]}\right)^2$$

It is ALWAYS good practice to check for energy conservation by adding the reflectance and transmittance and ensuring the sum equals 100% (assuming no loss or gain in your device).

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Block Diagram of FDTD

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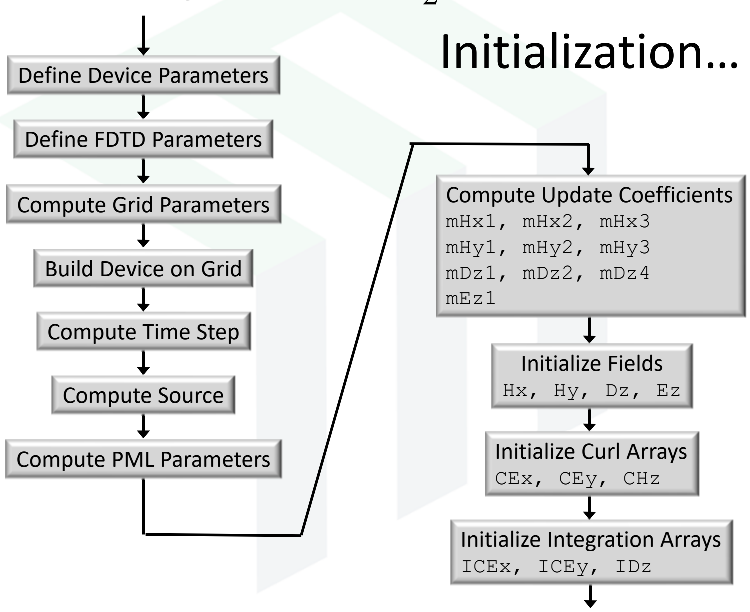
Block Diagram for E_z Mode (1 of 2)



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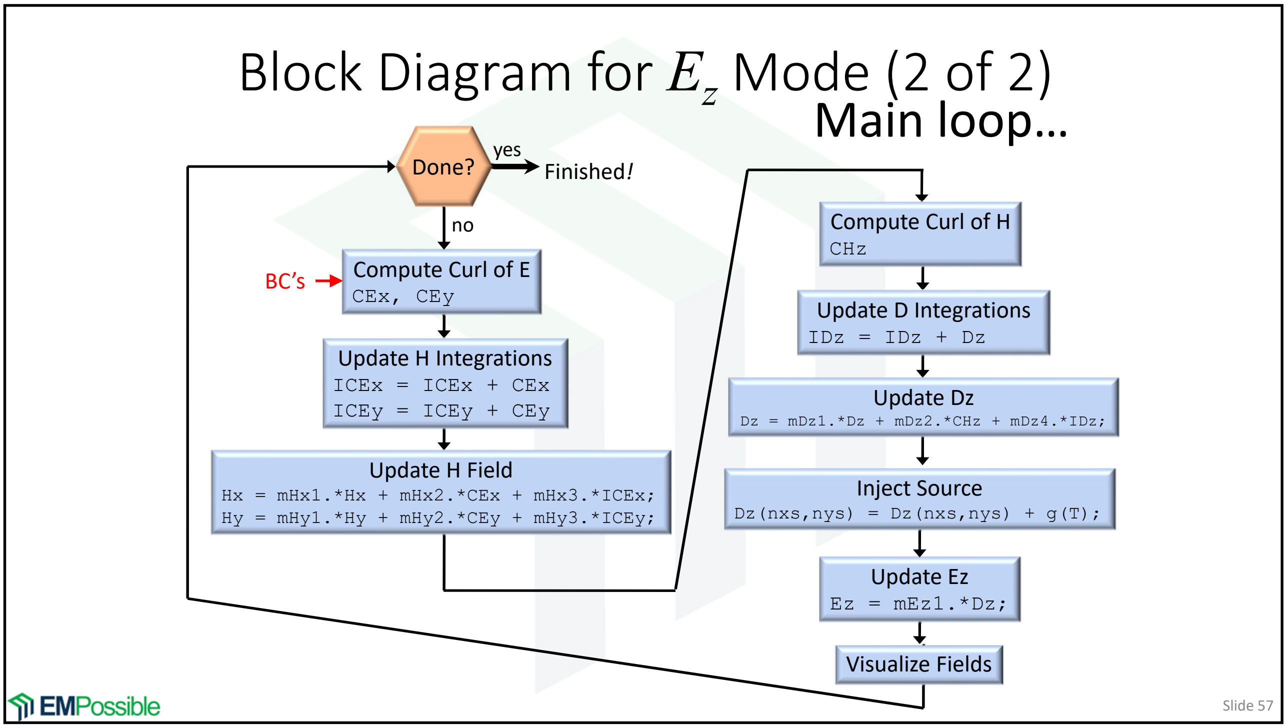
Block Diagram for E_z Mode (2 of 2)
Main loop…
Done?
yes
Finished!
no
BC's
Compute Curl of E
CEx, CEy
Update H Integrations
ICEx = ICEx + CEx
ICEy = ICEy + CEy
Update H Field
Hx = mHx1.*Hx + mHx2.*CEx + mHx3.*ICEx;
Hy = mHy1.*Hy + mHy2.*CEy + mHy3.*ICEy;
Compute Curl of H
CHz
Update D Integrations
IDz = IDz + Dz
Update Dz
Dz = mDz1.*Dz + mDz2.*CHz + mDz4.*IDz;
Inject Source
Dz(nxs,nys) = Dz(nxs,nys) + g(T);
Update Ez
Ez = mEz1.*Dz;
Visualize Fields
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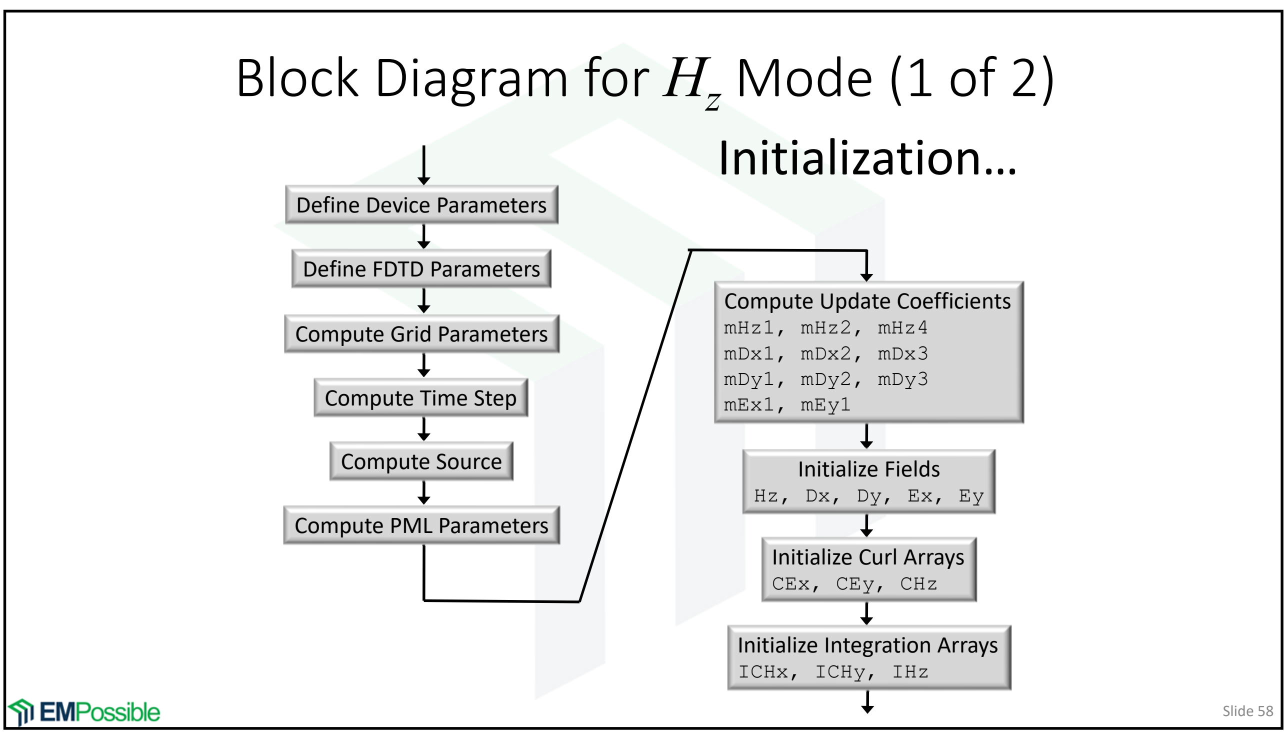
Block Diagram for H_z Mode (1 of 2)
Initialization…
Define Device Parameters
Define FDTD Parameters
Compute Grid Parameters
Compute Time Step
Compute Source
Compute PML Parameters
Compute Update Coefficients
mHz1, mHz2, mHz4
mDx1, mDx2, mDx3
mDy1, mDy2, mDy3
mEx1, mEy1
Initialize Fields
Hz, Dx, Dy, Ex, Ey
Initialize Curl Arrays
CEx, CEy, CHz
Initialize Integration Arrays
ICHx, ICHy, IHz
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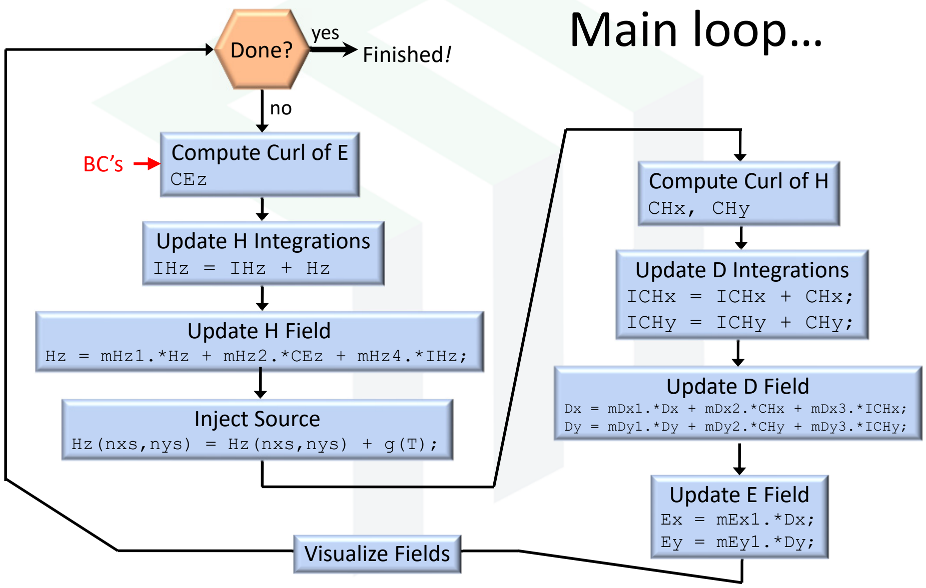
Block Diagram for H_z Mode (2 of 2)

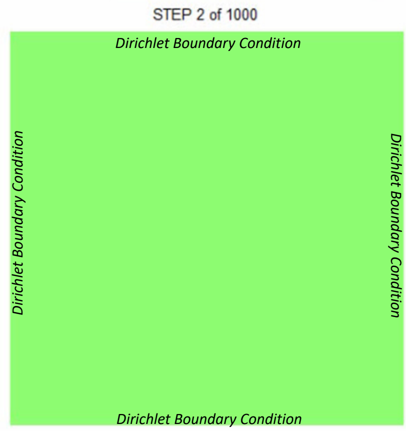
Main loop…



Sequence of Code Development

Step 1 – Basic Update Equations

The basic update equations are implemented along with simple Dirichlet boundary conditions.



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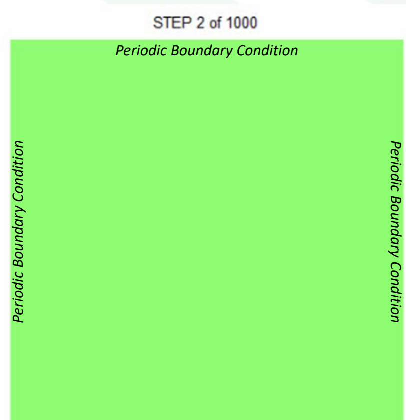
Step 2 – Incorporate Periodic Boundaries

Periodic boundary conditions are incorporated so that a wave leaving the grid reenters the grid at the other side.



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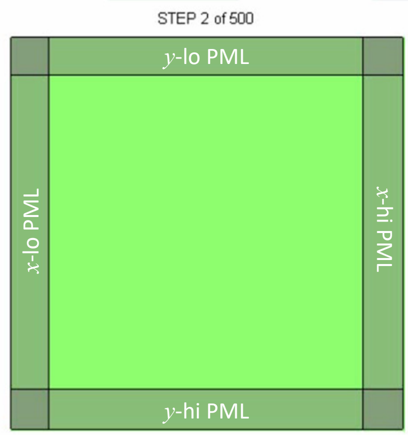
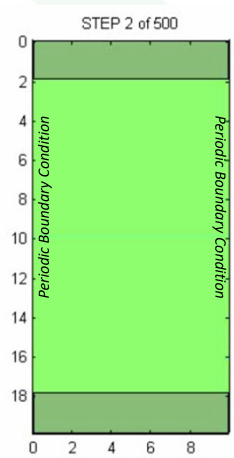
Step 3 – Incorporate a PML

The perfectly matched layer (PML) absorbing boundary condition is incorporated to absorb outgoing waves.



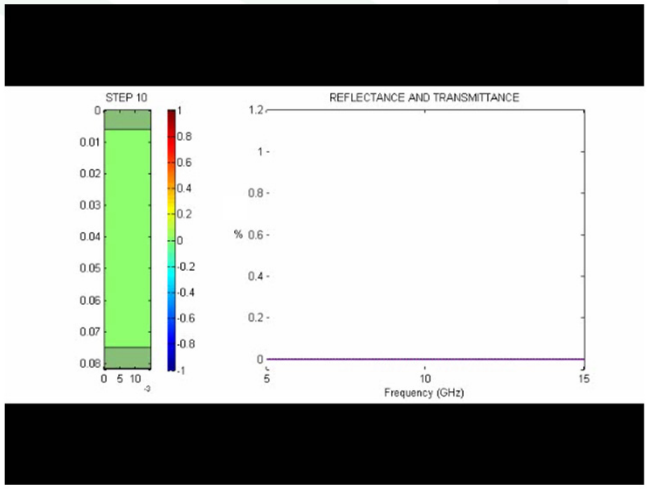
Step 4 – Total-Field/Scattered-Field

Most periodic electromagnetic devices are modeled by using periodic boundaries for the horizontal axis and a PML for the vertical axis. We then implement TF/SF at the vertical center of the grid for testing.



Step 5 – Calculate TRN, REF, and CON

We move the TF/SF interface to a unit cell or two outside of the top PML. We include code to calculate Fourier transforms and to calculate transmittance, reflectance, and conservation of power.



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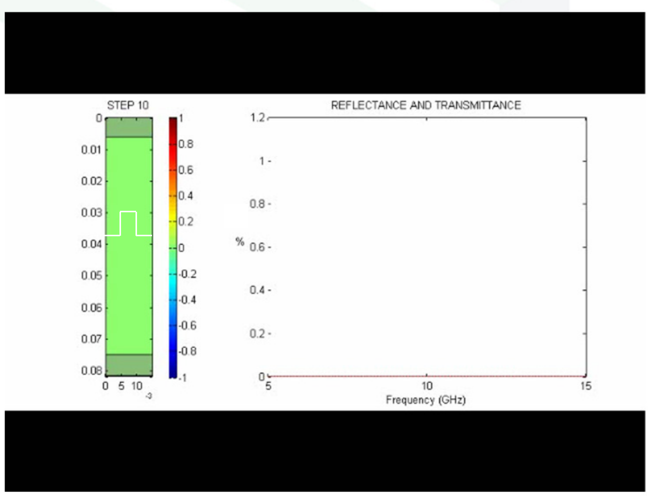
Step 6 – Model a Device to Benchmark

We build a device on the grid that has a known solution. We run the simulation and duplicate the known results to benchmark our new code.



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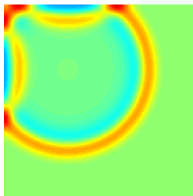
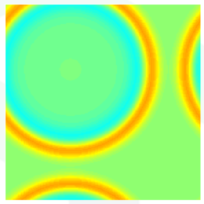
Summary of Code Development Sequence

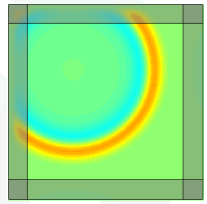
Step 1 – Basic Update + Dirichlet



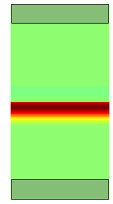
Step 2 – Basic Update + Periodic BC



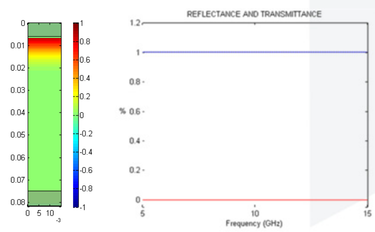
Step 3 – Add PML



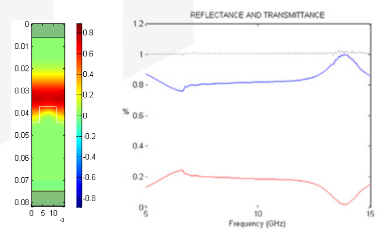
Step 4 – TF/SF



Step 5 – Calculate Response



Step 6 – Add a Device and Benchmark



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