



Magnetostatics: *Boundary Conditions*

EE3321

Electromagnetic Field Theory



Outline



- Boundary conditions for tangential fields
- Boundary conditions for normal fields
- Law of refraction for magnetic flux

Deriving Boundary Conditions



Just like we did for electrostatic fields, we will derive our boundary conditions using Maxwell's equations in integral form.


$$I = \oint_L \vec{H} \cdot d\vec{\ell} \implies \text{Boundary conditions for tangential magnetic fields.}$$

$$0 = \oiint_S \vec{B} \cdot d\vec{s} \implies \text{Boundary conditions for normal magnetic fields.}$$

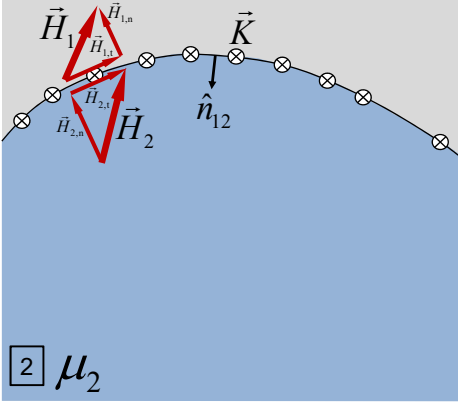
Boundary Conditions for Tangential Fields



Analysis Setup



1 μ_1



2 μ_2

Let's examine the interface between two materials with a surface current at the interface.


We wish to examine the relation between magnetic fields on either side of the interface, so that if one is known the other can be calculated.

It will be useful to separate the field on either side of the interface into tangential and normal components.

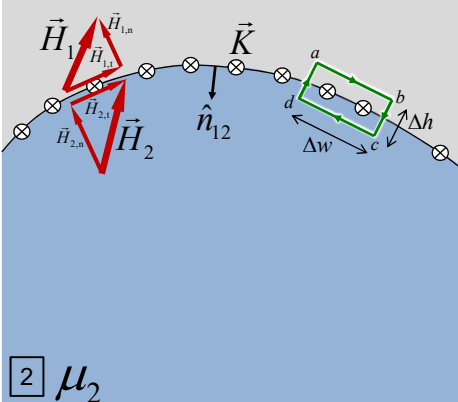
Magnetostatics -- Boundary Conditions

Slide 5

Derivation of Tangential BCs



1 μ_1



2 μ_2

Apply the following integral to a closed path spanning some section of the interface.

$$I = \oint_L \vec{H} \cdot d\vec{\ell}$$


$$K\Delta w = \int_a^b \vec{H} \cdot d\vec{\ell} + \int_b^0 \vec{H} \cdot d\vec{\ell} + \int_0^c \vec{H} \cdot d\vec{\ell} + \int_c^d \vec{H} \cdot d\vec{\ell} + \int_d^0 \vec{H} \cdot d\vec{\ell} + \int_0^a \vec{H} \cdot d\vec{\ell}$$

$$= H_{1,t}\Delta w - H_{1,n} \frac{\Delta h}{2} - H_{2,n} \frac{\Delta h}{2} - H_{2,t}\Delta w + H_{2,n} \frac{\Delta h}{2} + H_{1,n} \frac{\Delta h}{2}$$

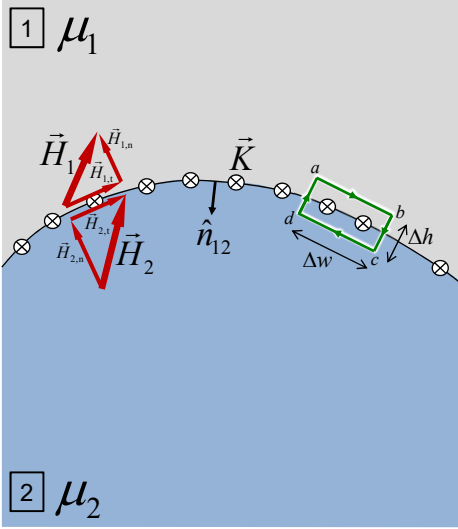
Magnetostatics -- Boundary Conditions

Slide 6

Derivation of Tangential BCs



1 μ_1



2 μ_2

Cancel like terms with opposite sign.

$$K \Delta w = H_{1,t} \Delta w - H_{1,n} \frac{\Delta h}{2} - H_{2,n} \frac{\Delta h}{2}$$

$$-H_{2,t} \Delta w + H_{2,n} \frac{\Delta h}{2} + H_{1,n} \frac{\Delta h}{2}$$

$$= H_{1,t} \Delta w - H_{2,t} \Delta w$$


$$K = H_{1,t} - H_{2,t}$$

From this, we conclude that

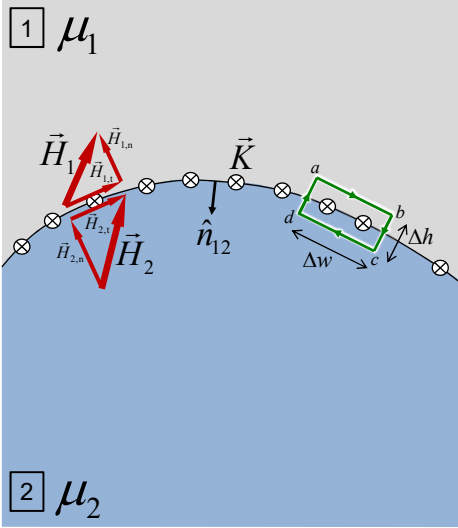
$$(\vec{H}_1 - \vec{H}_2) \times \hat{n}_{12} = \vec{K}$$

Magnetostatics -- Boundary Conditions Slide 7

Derivation of Tangential BCs



1 μ_1



2 μ_2

Apply the constitutive relation to get the boundary condition for B .

$$(\vec{H}_1 - \vec{H}_2) \times \hat{n}_{12} = \vec{K}$$

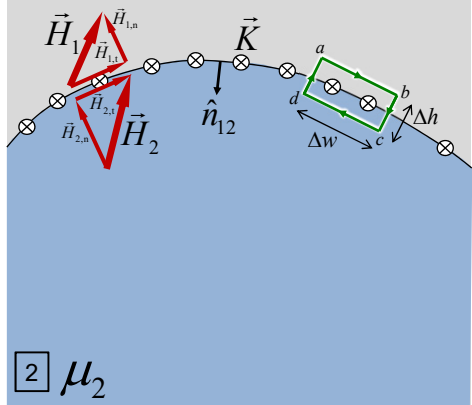
$$\left(\frac{\vec{B}_1}{\mu_1} - \frac{\vec{B}_2}{\mu_2} \right) \times \hat{n}_{12} = \vec{K}$$

Magnetostatics -- Boundary Conditions Slide 8

Derivation of Tangential BCs



1 μ_1



In the absence of a surface current K , the boundary conditions reduce to

$$\vec{H}_{1,t} = \vec{H}_{2,t}$$


$$\frac{\vec{B}_{1,t}}{\mu_1} = \frac{\vec{B}_{2,t}}{\mu_2}$$

2 μ_2

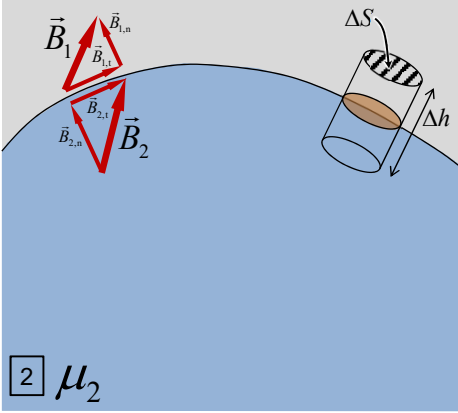
Boundary Conditions for Normal Fields



Derivation of Normal BCs



1 μ_1



2 μ_2

Apply the following surface integral to a pillbox spanning the interface.


$$0 = \oiint_s \vec{B} \cdot d\vec{s}$$

Separate the closed-surface integral into three separate surface integrals.

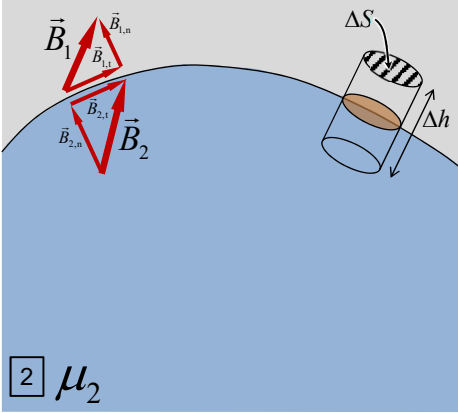
$$0 = \iint_{\text{top}} \vec{B} \cdot d\vec{s} + \iint_{\text{bottom}} \vec{B} \cdot d\vec{s} + \iint_{\text{sides}} \vec{B} \cdot d\vec{s}$$

Magnetostatics -- Boundary Conditions Slide 11

Derivation of Normal BCs



1



2 μ_2

In the limit as $\Delta h \rightarrow 0$

$$0 = \iint_{\text{top}} \vec{B} \cdot d\vec{s} + \iint_{\text{bottom}} \vec{B} \cdot d\vec{s} + \cancel{\iint_{\text{sides}} \vec{B} \cdot d\vec{s}}$$

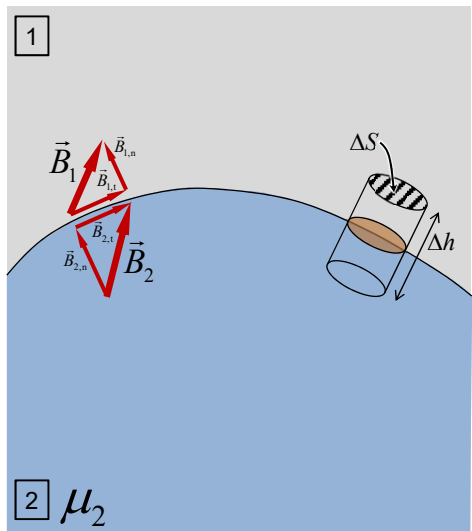
$$= B_{1,n} \Delta S - B_{2,n} \Delta S$$

The boundary condition is that the normal component of B is continuous across the interface.

$$\vec{B}_{1,n} = \vec{B}_{2,n}$$

Magnetostatics -- Boundary Conditions Slide 12

Derivation of Normal BCs



Apply the constitutive relation to get the boundary condition for H .

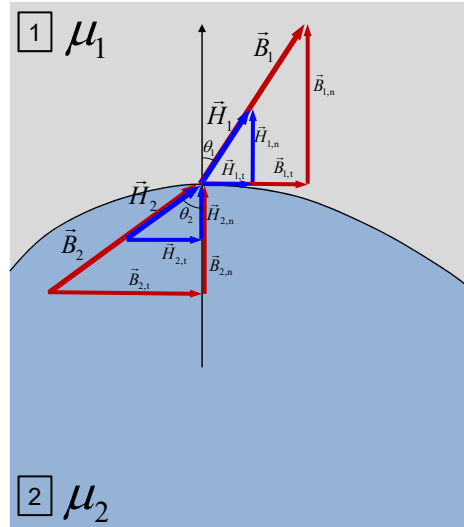
$$\vec{B}_{1,n} = \vec{B}_{2,n}$$

$$\mu_1 \vec{H}_{1,n} = \mu_2 \vec{H}_{2,n}$$

The normal component of H is NOT continuous across the interface, but the product of μH is.

Law of Refraction for Magnetic Flux

Analysis Setup



We want a single equation that relates θ_1 , θ_2 , μ_1 , and μ_2 without any field quantities in the equation.

Given the angles θ_1 and θ_2 , the field components can be written as

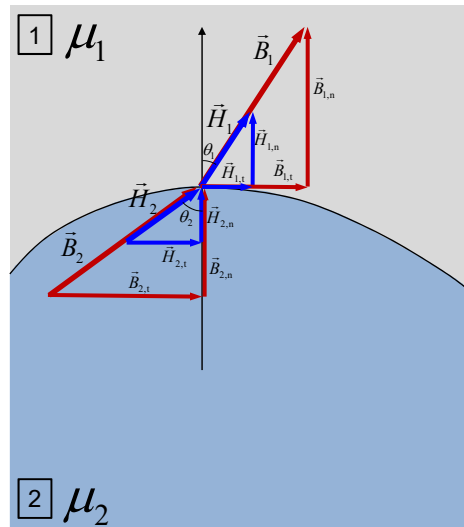
$$\begin{aligned}\vec{H}_1 &= H_{1,t}\hat{a}_t + H_{1,n}\hat{a}_n \\ &= (H_1 \sin \theta_1)\hat{a}_t + (H_1 \cos \theta_1)\hat{a}_n\end{aligned}$$

$$\begin{aligned}\vec{H}_2 &= H_{2,t}\hat{a}_t + H_{2,n}\hat{a}_n \\ &= (H_2 \sin \theta_2)\hat{a}_t + (H_2 \cos \theta_2)\hat{a}_n\end{aligned}$$

Magnetostatics -- Boundary Conditions

Slide 15

Derivation of Refraction Law



Apply the boundary conditions for tangential components.

$$H_{1,t} = H_{2,t}$$

$$H_1 \sin \theta_1 = H_2 \sin \theta_2$$

Apply the boundary conditions for normal components.

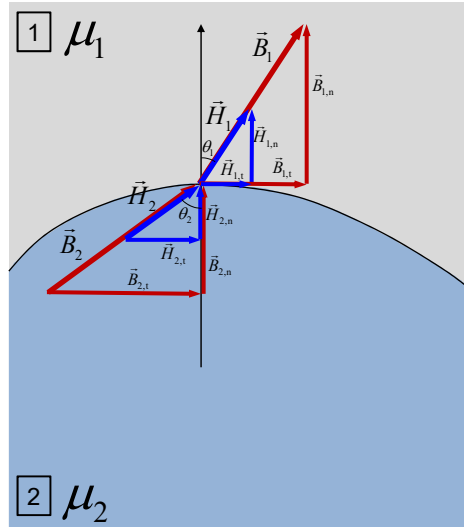
$$\mu_1 H_{1,n} = \mu_2 H_{2,n}$$

$$\mu_1 H_1 \cos \theta_1 = \mu_2 H_2 \cos \theta_2$$

Magnetostatics -- Boundary Conditions

Slide 16

Derivation of Refraction Law



We now have

$$H_1 \sin \theta_1 = H_2 \sin \theta_2$$

$$\mu_1 H_1 \cos \theta_1 = \mu_2 H_2 \cos \theta_2$$

Divide these equations to get

$$\frac{H_1 \sin \theta_1}{\mu_1 H_1 \cos \theta_1} = \frac{H_2 \sin \theta_2}{\mu_2 H_2 \cos \theta_2}$$

Simplify

$$\frac{\tan \theta_1}{\mu_1} = \frac{\tan \theta_2}{\mu_2}$$

This is NOT Snell's law.