Advanced Computation:
Computational Electromagnetics

PWEM Extras

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Outline

- Using 3D PWEM for 2D and 1D Analysis, and using 2D PWEM for 1D Analysis
- Visualizing the fields
- Formulation of efficient 3D PWEM
- Formulation of efficient 1D PWEM
- PWEM with anisotropic materials
- Incorporating dispersion
- Reduced Bloch mode expansion (RBME) technique
- Generalized symmetry

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Using 3D PWEM for 2D and 1D Analysis, and Using 2D PWEM for 1D Analysis

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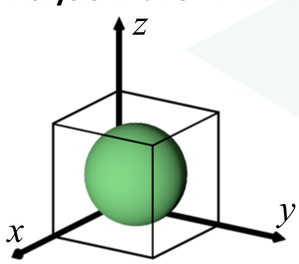
Using 3D PWEM for 3D, 2D & 1D Analysis

3D Analysis with 3D PWEM



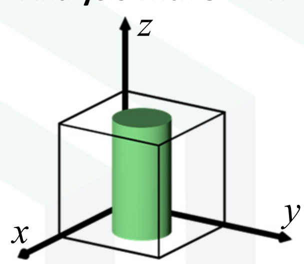
$P > 1$ along x

$Q > 1$ along y

$R > 1$ along z

Conventional 3D PWEM

2D Analysis with 3D PWEM



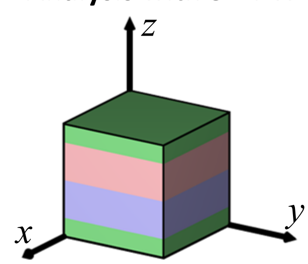
$P > 1$ along x

$Q > 1$ along y

$R = 1$ along z

The device is uniform along the z-axis so R can be set to 1.

1D Analysis with 3D PWEM



$P = 1$ along x

$Q = 1$ along y

$R > 1$ along z

The device is uniform along the x- and y-axes so P and Q can both be set to 1.

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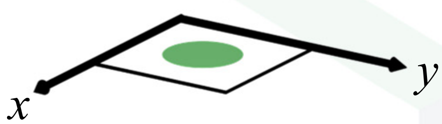
Using 2D PWEM for 2D & 1D Analysis

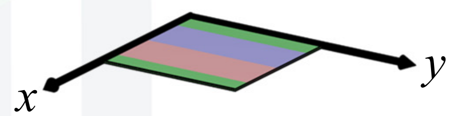
2D Analysis with 2D PWEM



$P > 1$ along x
$Q > 1$ along y

Conventional 2D PWEM

1D Analysis with 2D PWEM



$P > 1$ along x
$Q = 1$ along y

The device is uniform along the y-axis so Q can be set to 1.

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Visualizing the Fields

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Eigen-Vectors Contain Field Information

For 2D PWEM, we solved the following eigen-value problem.

$$\left(\mathbf{K}_x [\![\mu_r]\!]^{-1} \mathbf{K}_x + \mathbf{K}_y [\![\mu_r]\!]^{-1} \mathbf{K}_y\right)\mathbf{s}_z = k_0^2 [\![\varepsilon_r]\!]\mathbf{s}_z \quad \rightarrow \quad \begin{array}{l}\mathbf{V} \equiv \text{eigen vectors}\\ \boldsymbol{\lambda} \equiv \text{eigen values}\end{array}$$

Recall the field expansion into a plane wave basis

$$\vec{E}(\vec{r}) = \sum_{p=-\infty}^{\infty}\sum_{q=-\infty}^{\infty} \vec{S}(p,q) e^{-j\vec{k}(p,q)\bullet\vec{r}} \qquad \vec{k}(p,q) = \vec{\beta} - \frac{2\pi p}{\Lambda_x}\hat{x} - \frac{2\pi q}{\Lambda_y}\hat{y}$$

$\mathbf{s}_z$ — The complex amplitudes of the plane waves are stored in the elements of the eigen-vectors.

Rearrange Terms

The field expansion is

$$\vec{E}(\vec{r}) = \sum_{p=-\infty}^{\infty}\sum_{q=-\infty}^{\infty} \vec{S}(p,q) e^{-j\vec{k}(p,q)\bullet\vec{r}} \qquad \vec{k}(p,q) = \vec{\beta} - \frac{2\pi p}{\Lambda_x}\hat{x} - \frac{2\pi q}{\Lambda_y}\hat{y}$$

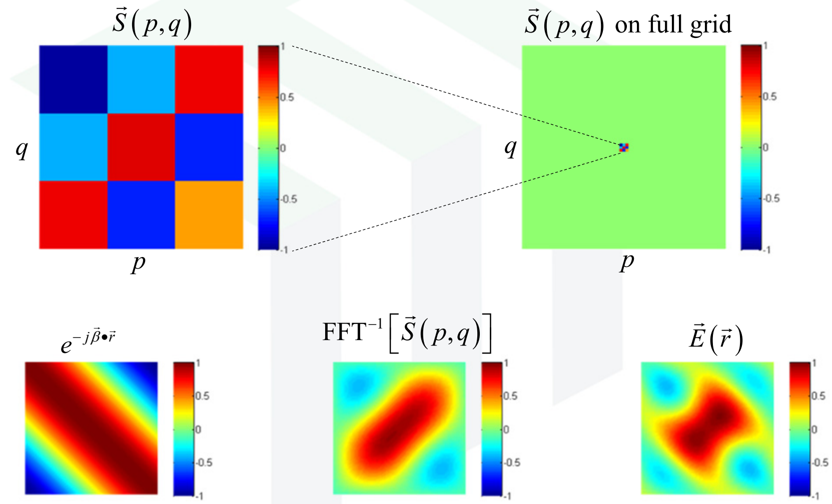
We substitute in the expression for $\vec{k}(p,q)$ and factor out $e^{-j\vec{\beta}\bullet\vec{r}}$.

$$\vec{E}(\vec{r}) = e^{-j\vec{\beta}\bullet\vec{r}} \underbrace{\sum_{p=-\infty}^{\infty}\sum_{q=-\infty}^{\infty} \vec{S}(p,q) e^{j\left(\frac{2\pi p}{\Lambda_x}\hat{x}+\frac{2\pi q}{\Lambda_y}\hat{y}\right)\bullet\vec{r}}}_{\text{This is just an inverse 2D FFT.}}$$

We can now write the electric field expression as

$$\vec{E}(\vec{r}) = e^{-j\vec{\beta}\bullet\vec{r}}\ \text{FFT}^{-1}\left[\vec{S}(p,q)\right]$$

Visualizing the Data



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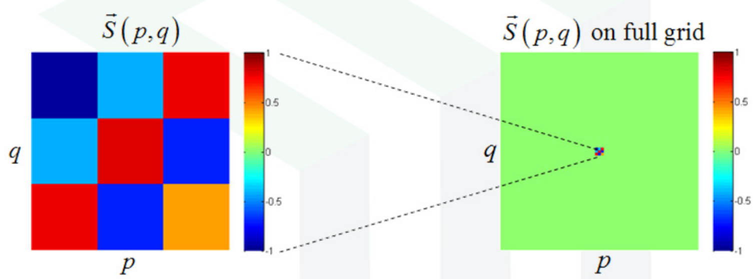
Procedure (1 of 2)

Step 1 – Extract eigen-vector and reshape

Step 2 – Insert Fourier coefficients into large grid.




```
s = V(:,m);
s = reshape(s,P,Q);
```

```
nxc = ceil(Nx/2);
nx1 = nxc - floor(P/2);
nx2 = nxc + floor(P/2);
nyc = ceil(Ny/2);
ny1 = nyc - floor(Q/2);
ny2 = nyc + floor(Q/2);

sf = zeros(Nx,Ny);
sf(nx1:nx2,ny1:ny2) = s;
```

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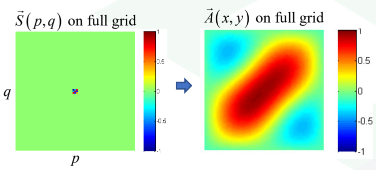
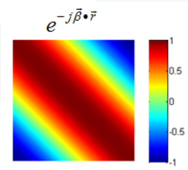
Procedure (2 of 2)

Step 3 – Calculate inverse FFT of $\vec{S}(p,q)$.



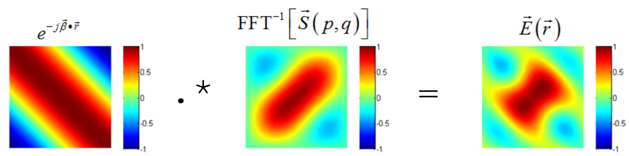

```
az = ifft2(ifftshift(sf));
az = az/max(abs(az(:)));
```

Step 4 – Calculate phase term.




```
phase = exp(-1i*(beta(1)*X + beta(2)*Y));
```

Step 5 – Calculate overall field.




```
Ez = phase.*az;
```

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Formulation of Efficient 3D Plane Wave Expansion Method

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Notation

Unbolded letters are scalar quantities. $a = 1.678$

Bold capital letters are matrices, usually square matrices.

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & & a_{1N} \\ a_{21} & a_{22} & \cdots & a_{2N} \\ & \vdots & \ddots & \\ a_{M1} & a_{M2} & & a_{MN} \end{bmatrix}$$

Bold lower-case letters are column vectors.

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_M \end{bmatrix}$$

Bold lower-case letters with a vector sign are column vectors of column vectors.

$$\vec{\mathbf{e}} = \begin{bmatrix} \mathbf{e}_x \\ \mathbf{e}_y \\ \mathbf{e}_z \end{bmatrix} \qquad \vec{\mathbf{a}} = \begin{bmatrix} \mathbf{a}_x \\ \mathbf{a}_y \end{bmatrix}$$

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Vector-Matrix Operations

Definitions

$\mathbf{A}_x \equiv$ diagonal matrix containing x components of $\vec{\mathbf{A}}$

$\mathbf{A}_y \equiv$ diagonal matrix containing y components of $\vec{\mathbf{A}}$

$\mathbf{A}_z \equiv$ diagonal matrix containing z components of $\vec{\mathbf{A}}$

$$\mathbf{A}_i = \begin{bmatrix} A_{i1} & & & 0 \\ & A_{i2} & & \\ & & \ddots & \\ 0 & & & A_{iN} \end{bmatrix} \qquad \vec{\mathbf{A}} = \begin{bmatrix} \mathbf{A}_x \\ \mathbf{A}_y \\ \mathbf{A}_z \end{bmatrix}$$

Vector-Matrix Operations

$$\left|\vec{\mathbf{A}}\right| = \sqrt{\mathbf{A}_x^2 + \mathbf{A}_y^2 + \mathbf{A}_z^2}$$

$$\left[\vec{\mathbf{A}}\bullet\right] = \left[\mathbf{A}_x \quad \mathbf{A}_y \quad \mathbf{A}_z\right]$$

$$\left[\vec{\mathbf{A}}\times\right] = \begin{bmatrix} \mathbf{0} & -\mathbf{A}_z & \mathbf{A}_y \\ \mathbf{A}_z & \mathbf{0} & -\mathbf{A}_x \\ -\mathbf{A}_y & \mathbf{A}_x & \mathbf{0} \end{bmatrix}$$

$\left[\vec{\mathbf{A}}\bullet\right]\vec{\mathbf{B}} \neq \left[\vec{\mathbf{B}}\bullet\right]\vec{\mathbf{A}}$ Unless $\mathbf{A}_x$, $\mathbf{A}_y$, $\mathbf{A}_z$, $\mathbf{B}_x$, $\mathbf{B}_y$, and $\mathbf{B}_z$ are all diagonal.

$\left[\vec{\mathbf{A}}\times\right]\vec{\mathbf{B}} \neq -\left[\vec{\mathbf{B}}\times\right]\vec{\mathbf{A}}$ Unless $\mathbf{A}_x$, $\mathbf{A}_y$, $\mathbf{A}_z$, $\mathbf{B}_x$, $\mathbf{B}_y$, and $\mathbf{B}_z$ are all diagonal.

In our formulation, only $[\varepsilon_r]$ and $[\mu_r]$ are not diagonal matrices.

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Expand the Field Into Two Orthogonal Polarizations

We write the field as the sum of two orthogonal polarizations.

$$\vec{\mathbf{s}} = \hat{\mathbf{P}}_1 \mathbf{s}_1 + \hat{\mathbf{P}}_2 \mathbf{s}_2$$

Two polarization vectors are chosen for each spatial harmonic in the plane wave expansion. These must be orthogonal to the direction of the harmonic. If these are chosen correctly, the following equations will be satisfied.

$$\hat{P}_1 \times \hat{P}_2 = \frac{\vec{k}}{|\vec{k}|} \quad \rightarrow \quad \left[\hat{\mathbf{P}}_1 \times\right]\hat{\mathbf{P}}_2 = \vec{\mathbf{K}}\left|\vec{\mathbf{K}}\right|^{-1}$$

$$\frac{\vec{k}}{|\vec{k}|} \times \hat{P}_1 = \hat{P}_2 \quad \rightarrow \quad \left[\vec{\mathbf{K}} \times\right]\hat{\mathbf{P}}_1 = \hat{\mathbf{P}}_2\left|\vec{\mathbf{K}}\right|$$

$$\frac{\vec{k}}{|\vec{k}|} \times \hat{P}_2 = -\hat{P}_1 \quad \rightarrow \quad \left[\vec{\mathbf{K}} \times\right]\hat{\mathbf{P}}_2 = -\hat{\mathbf{P}}_1\left|\vec{\mathbf{K}}\right|$$

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Recipe for Calculating Orthogonal Polarization Vectors

Given the wave vector of the i^{th} spatial harmonic

$$\vec{k}_i = k_{x,i}\hat{a}_x + k_{y,i}\hat{a}_y + k_{z,i}\hat{a}_z$$

We construct another vector that is in a different direction than $\vec{k}_i$.

$$\vec{v}_i = 4k_{y,i}\hat{a}_x + 2k_{z,i}\hat{a}_y + 3k_{x,i}\hat{a}_z$$

We can now calculate two vectors that are perpendicular to the wave vector.

$$\hat{p}_{1,i} = \frac{\vec{k}_i \times \vec{v}_i}{\left|\vec{k}_i \times \vec{v}_i\right|}$$

$$\hat{p}_{2,i} = \frac{\vec{k}_i \times \hat{p}_{1,i}}{\left|\vec{k}_i \times \hat{p}_{1,i}\right|}$$

Any choice of $\hat{p}_{1,i}$ and $\hat{p}_{2,i}$ can be used as long as $\hat{p}_{1,i} \perp \hat{p}_{2,i} \perp \vec{k}_i$.

Perhaps there exists a better choice than described here.

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Wave Equation with Polarization Expansion

We substitute the polarization expansion into our matrix wave equation.

$$\left[\vec{\mathbf{K}}\times\right]\left[\left[\mu_r\right]\right]^{-1}\left[\vec{\mathbf{K}}\times\right]\vec{\mathbf{s}} = -k_0^2\left[\left[\varepsilon_r\right]\right]\vec{\mathbf{s}}$$

$$\left[\vec{\mathbf{K}}\times\right]\left[\left[\mu_r\right]\right]^{-1}\left[\vec{\mathbf{K}}\times\right]\left(\hat{\mathbf{P}}_1\mathbf{s}_1+\hat{\mathbf{P}}_2\mathbf{s}_2\right) = -k_0^2\left[\left[\varepsilon_r\right]\right]\left(\hat{\mathbf{P}}_1\mathbf{s}_1+\hat{\mathbf{P}}_2\mathbf{s}_2\right)$$

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Projecting a Vector-Matrix Equation Onto Two Orthogonal Polarizations

We can project one vector-matrix onto another using

$$\text{Proj } \vec{a} \text{ onto } \vec{b} = \frac{\vec{a}\bullet\vec{b}}{\left|\vec{b}\right|^2}\vec{b} \qquad \left|\text{Proj } \vec{a} \text{ onto } \vec{b}\right| = \frac{\vec{a}\bullet\vec{b}}{\left|\vec{b}\right|} \quad\rightarrow\quad \left\langle\vec{\mathbf{a}}\right\rangle_{\vec{\mathbf{b}}} = \left|\vec{\mathbf{b}}\right|^{-1}\left[\vec{\mathbf{b}}\bullet\right]\vec{\mathbf{a}}$$

Using this equation, we can project both sides of a vector-matrix equation onto our two orthogonal polarizations to get two separate equations.

$$\mathbf{A}\vec{\mathbf{x}} = \mathbf{B}\vec{\mathbf{y}}$$

One 3×3 matrix equation

$$\left\langle\mathbf{A}\vec{\mathbf{x}}\right\rangle_{\hat{\mathbf{P}}_1} = \left\langle\mathbf{B}\vec{\mathbf{y}}\right\rangle_{\hat{\mathbf{P}}_1} \quad\rightarrow\quad \left[\hat{\mathbf{P}}_1\bullet\right]\mathbf{A}\vec{\mathbf{x}} = \left[\hat{\mathbf{P}}_1\bullet\right]\mathbf{B}\vec{\mathbf{y}}$$

$$\left\langle\mathbf{A}\vec{\mathbf{x}}\right\rangle_{\hat{\mathbf{P}}_2} = \left\langle\mathbf{B}\vec{\mathbf{y}}\right\rangle_{\hat{\mathbf{P}}_2} \quad\rightarrow\quad \left[\hat{\mathbf{P}}_2\bullet\right]\mathbf{A}\vec{\mathbf{x}} = \left[\hat{\mathbf{P}}_2\bullet\right]\mathbf{B}\vec{\mathbf{y}}$$

Two 1×1 matrix equations

Note: $\left|\hat{\mathbf{P}}_1\right|^{-1} = \left|\hat{\mathbf{P}}_2\right|^{-1} = \mathbf{I}$

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Project Our Vector-Matrix Equation Onto Two Orthogonal Polarizations

We apply the results from the last slide to our vector-matrix equation.

$$\left[\vec{\mathbf{K}}\times\right]\left[\left[\mu_r\right]\right]^{-1}\left[\vec{\mathbf{K}}\times\right]\left(\hat{\mathbf{P}}_1\mathbf{s}_1+\hat{\mathbf{P}}_2\mathbf{s}_2\right)=-k_0^2\left[\left[\varepsilon_r\right]\right]\left(\hat{\mathbf{P}}_1\mathbf{s}_1+\hat{\mathbf{P}}_2\mathbf{s}_2\right)$$

Project onto $\hat{\mathbf{P}}_1$

$$\left[\hat{\mathbf{P}}_1\bullet\right]\left[\vec{\mathbf{K}}\times\right]\left[\left[\mu_r\right]\right]^{-1}\left[\vec{\mathbf{K}}\times\right]\left(\hat{\mathbf{P}}_1\mathbf{s}_1+\hat{\mathbf{P}}_2\mathbf{s}_2\right)=-k_0^2\left[\hat{\mathbf{P}}_1\bullet\right]\left[\left[\varepsilon_r\right]\right]\left(\hat{\mathbf{P}}_1\mathbf{s}_1+\hat{\mathbf{P}}_2\mathbf{s}_2\right)$$

Project onto $\hat{\mathbf{P}}_2$

$$\left[\hat{\mathbf{P}}_2\bullet\right]\left[\vec{\mathbf{K}}\times\right]\left[\left[\mu_r\right]\right]^{-1}\left[\vec{\mathbf{K}}\times\right]\left(\hat{\mathbf{P}}_1\mathbf{s}_1+\hat{\mathbf{P}}_2\mathbf{s}_2\right)=-k_0^2\left[\hat{\mathbf{P}}_2\bullet\right]\left[\left[\varepsilon_r\right]\right]\left(\hat{\mathbf{P}}_1\mathbf{s}_1+\hat{\mathbf{P}}_2\mathbf{s}_2\right)$$

We now have two matrix equations.

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2×2 Block Matrix Form

We can put our two previous equations into block matrix form as follows.

$$\left[\hat{\mathbf{P}}_1\bullet\right]\left[\vec{\mathbf{K}}\times\right]\left[\left[\mu_r\right]\right]^{-1}\left[\vec{\mathbf{K}}\times\right]\left(\hat{\mathbf{P}}_1\mathbf{s}_1+\hat{\mathbf{P}}_2\mathbf{s}_2\right)=-k_0^2\left[\hat{\mathbf{P}}_1\bullet\right]\left[\left[\varepsilon_r\right]\right]\left(\hat{\mathbf{P}}_1\mathbf{s}_1+\hat{\mathbf{P}}_2\mathbf{s}_2\right)$$
$$\left[\hat{\mathbf{P}}_2\bullet\right]\left[\vec{\mathbf{K}}\times\right]\left[\left[\mu_r\right]\right]^{-1}\left[\vec{\mathbf{K}}\times\right]\left(\hat{\mathbf{P}}_1\mathbf{s}_1+\hat{\mathbf{P}}_2\mathbf{s}_2\right)=-k_0^2\left[\hat{\mathbf{P}}_2\bullet\right]\left[\left[\varepsilon_r\right]\right]\left(\hat{\mathbf{P}}_1\mathbf{s}_1+\hat{\mathbf{P}}_2\mathbf{s}_2\right)$$

$$\begin{bmatrix}\left[\hat{\mathbf{P}}_1\bullet\right]\left[\vec{\mathbf{K}}\times\right]\left[\left[\mu_r\right]\right]^{-1}\left[\vec{\mathbf{K}}\times\right]\hat{\mathbf{P}}_1 & \left[\hat{\mathbf{P}}_1\bullet\right]\left[\vec{\mathbf{K}}\times\right]\left[\left[\mu_r\right]\right]^{-1}\left[\vec{\mathbf{K}}\times\right]\hat{\mathbf{P}}_2\\ \left[\hat{\mathbf{P}}_2\bullet\right]\left[\vec{\mathbf{K}}\times\right]\left[\left[\mu_r\right]\right]^{-1}\left[\vec{\mathbf{K}}\times\right]\hat{\mathbf{P}}_1 & \left[\hat{\mathbf{P}}_2\bullet\right]\left[\vec{\mathbf{K}}\times\right]\left[\left[\mu_r\right]\right]^{-1}\left[\vec{\mathbf{K}}\times\right]\hat{\mathbf{P}}_2\end{bmatrix}\begin{bmatrix}\mathbf{s}_1\\ \mathbf{s}_2\end{bmatrix}$$
$$=-k_0^2\begin{bmatrix}\left[\hat{\mathbf{P}}_1\bullet\right]\left[\left[\varepsilon_r\right]\right]\hat{\mathbf{P}}_1 & \left[\hat{\mathbf{P}}_1\bullet\right]\left[\left[\varepsilon_r\right]\right]\hat{\mathbf{P}}_2\\ \left[\hat{\mathbf{P}}_2\bullet\right]\left[\left[\varepsilon_r\right]\right]\hat{\mathbf{P}}_1 & \left[\hat{\mathbf{P}}_2\bullet\right]\left[\left[\varepsilon_r\right]\right]\hat{\mathbf{P}}_2\end{bmatrix}\begin{bmatrix}\mathbf{s}_1\\ \mathbf{s}_2\end{bmatrix}$$

This is a 2×2 generalized eigen-value problem. It is smaller than the 3×3 generalized eigen-value problem that we derived previously.

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Final Form of Generalized Eigen-Value Problems

Reduced Eigen-Value Problem for Electric Fields

$$\mathbf{A}\begin{bmatrix}\mathbf{s}_1\\ \mathbf{s}_2\end{bmatrix} = -k_0^2\mathbf{B}\begin{bmatrix}\mathbf{s}_1\\ \mathbf{s}_2\end{bmatrix}$$

$$\mathbf{A} = \begin{bmatrix}\left[\hat{\mathbf{P}}_1\bullet\right]\left[\vec{\mathbf{K}}\times\right]\left[\left[\mu_r\right]\right]^{-1}\left[\vec{\mathbf{K}}\times\right]\hat{\mathbf{P}}_1 & \left[\hat{\mathbf{P}}_1\bullet\right]\left[\vec{\mathbf{K}}\times\right]\left[\left[\mu_r\right]\right]^{-1}\left[\vec{\mathbf{K}}\times\right]\hat{\mathbf{P}}_2\\ \left[\hat{\mathbf{P}}_2\bullet\right]\left[\vec{\mathbf{K}}\times\right]\left[\left[\mu_r\right]\right]^{-1}\left[\vec{\mathbf{K}}\times\right]\hat{\mathbf{P}}_1 & \left[\hat{\mathbf{P}}_2\bullet\right]\left[\vec{\mathbf{K}}\times\right]\left[\left[\mu_r\right]\right]^{-1}\left[\vec{\mathbf{K}}\times\right]\hat{\mathbf{P}}_2\end{bmatrix}$$

$$\mathbf{B} = \begin{bmatrix}\left[\hat{\mathbf{P}}_1\bullet\right]\left[\left[\varepsilon_r\right]\right]\hat{\mathbf{P}}_1 & \left[\hat{\mathbf{P}}_1\bullet\right]\left[\left[\varepsilon_r\right]\right]\hat{\mathbf{P}}_2\\ \left[\hat{\mathbf{P}}_2\bullet\right]\left[\left[\varepsilon_r\right]\right]\hat{\mathbf{P}}_1 & \left[\hat{\mathbf{P}}_2\bullet\right]\left[\left[\varepsilon_r\right]\right]\hat{\mathbf{P}}_2\end{bmatrix}$$

Reduced Eigen-Value Problem for Magnetic Fields

$$\mathbf{A}\begin{bmatrix}\mathbf{u}_1\\ \mathbf{u}_2\end{bmatrix} = -k_0^2\mathbf{B}\begin{bmatrix}\mathbf{u}_1\\ \mathbf{u}_2\end{bmatrix}$$

$$\mathbf{A} = \begin{bmatrix}\left[\hat{\mathbf{P}}_1\bullet\right]\left[\vec{\mathbf{K}}\times\right]\left[\left[\varepsilon_r\right]\right]^{-1}\left[\vec{\mathbf{K}}\times\right]\hat{\mathbf{P}}_1 & \left[\hat{\mathbf{P}}_1\bullet\right]\left[\vec{\mathbf{K}}\times\right]\left[\left[\varepsilon_r\right]\right]^{-1}\left[\vec{\mathbf{K}}\times\right]\hat{\mathbf{P}}_2\\ \left[\hat{\mathbf{P}}_2\bullet\right]\left[\vec{\mathbf{K}}\times\right]\left[\left[\varepsilon_r\right]\right]^{-1}\left[\vec{\mathbf{K}}\times\right]\hat{\mathbf{P}}_1 & \left[\hat{\mathbf{P}}_2\bullet\right]\left[\vec{\mathbf{K}}\times\right]\left[\left[\varepsilon_r\right]\right]^{-1}\left[\vec{\mathbf{K}}\times\right]\hat{\mathbf{P}}_2\end{bmatrix}$$

$$\mathbf{B} = \begin{bmatrix}\left[\hat{\mathbf{P}}_1\bullet\right]\left[\left[\mu_r\right]\right]\hat{\mathbf{P}}_1 & \left[\hat{\mathbf{P}}_1\bullet\right]\left[\left[\mu_r\right]\right]\hat{\mathbf{P}}_2\\ \left[\hat{\mathbf{P}}_2\bullet\right]\left[\left[\mu_r\right]\right]\hat{\mathbf{P}}_1 & \left[\hat{\mathbf{P}}_2\bullet\right]\left[\left[\mu_r\right]\right]\hat{\mathbf{P}}_2\end{bmatrix}$$

Note that both of these eigen-value problems are valid for general anisotropic media.

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Generalized Eigen-Value Problem with NO MAGNETIC RESPONSE

Generalized Eigen-Value Problem for Electric Fields

$$\mathbf{A}\begin{bmatrix}\mathbf{s}_1\\ \mathbf{s}_2\end{bmatrix} = -k_0^2\mathbf{B}\begin{bmatrix}\mathbf{s}_1\\ \mathbf{s}_2\end{bmatrix}$$

$$\mathbf{A} = \begin{bmatrix}\left[\hat{\mathbf{P}}_1\bullet\right]\left[\vec{\mathbf{K}}\times\right]^2\hat{\mathbf{P}}_1 & \left[\hat{\mathbf{P}}_1\bullet\right]\left[\vec{\mathbf{K}}\times\right]^2\hat{\mathbf{P}}_2\\ \left[\hat{\mathbf{P}}_2\bullet\right]\left[\vec{\mathbf{K}}\times\right]^2\hat{\mathbf{P}}_1 & \left[\hat{\mathbf{P}}_2\bullet\right]\left[\vec{\mathbf{K}}\times\right]^2\hat{\mathbf{P}}_2\end{bmatrix}$$

$$\mathbf{B} = \begin{bmatrix}\left[\hat{\mathbf{P}}_1\bullet\right]\left[\left[\varepsilon_r\right]\right]\hat{\mathbf{P}}_1 & \left[\hat{\mathbf{P}}_1\bullet\right]\left[\left[\varepsilon_r\right]\right]\hat{\mathbf{P}}_2\\ \left[\hat{\mathbf{P}}_2\bullet\right]\left[\left[\varepsilon_r\right]\right]\hat{\mathbf{P}}_1 & \left[\hat{\mathbf{P}}_2\bullet\right]\left[\left[\varepsilon_r\right]\right]\hat{\mathbf{P}}_2\end{bmatrix}$$

Ordinary Eigen-Value Problem for Magnetic Fields

$$\mathbf{A}\begin{bmatrix}\mathbf{u}_1\\ \mathbf{u}_2\end{bmatrix} = -k_0^2\begin{bmatrix}\mathbf{u}_1\\ \mathbf{u}_2\end{bmatrix}$$

$$\mathbf{A} = \begin{bmatrix}\left[\hat{\mathbf{P}}_1\bullet\right]\left[\vec{\mathbf{K}}\times\right]\left[\left[\varepsilon_r\right]\right]^{-1}\left[\vec{\mathbf{K}}\times\right]\hat{\mathbf{P}}_1 & \left[\hat{\mathbf{P}}_1\bullet\right]\left[\vec{\mathbf{K}}\times\right]\left[\left[\varepsilon_r\right]\right]^{-1}\left[\vec{\mathbf{K}}\times\right]\hat{\mathbf{P}}_2\\ \left[\hat{\mathbf{P}}_2\bullet\right]\left[\vec{\mathbf{K}}\times\right]\left[\left[\varepsilon_r\right]\right]^{-1}\left[\vec{\mathbf{K}}\times\right]\hat{\mathbf{P}}_1 & \left[\hat{\mathbf{P}}_2\bullet\right]\left[\vec{\mathbf{K}}\times\right]\left[\left[\varepsilon_r\right]\right]^{-1}\left[\vec{\mathbf{K}}\times\right]\hat{\mathbf{P}}_2\end{bmatrix}$$

Note that both of these eigen-value problems are valid for general anisotropic media.

This is the most common formulation because it is an ordinary eigen-value problem.

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Recovering $\vec{\mathbf{s}}$ from s_1 and s_2

The generalized eigen-value problem for the electric fields is

$$\mathbf{A}\begin{bmatrix}\mathbf{s}_1\\ \mathbf{s}_2\end{bmatrix} = -k_0^2\mathbf{B}\begin{bmatrix}\mathbf{s}_1\\ \mathbf{s}_2\end{bmatrix}$$

Let the eigen-vector and eigen-value matrices be

$$\mathbf{A}\begin{bmatrix}\mathbf{s}_1\\ \mathbf{s}_2\end{bmatrix} = -k_0^2\mathbf{B}\begin{bmatrix}\mathbf{s}_1\\ \mathbf{s}_2\end{bmatrix} \quad \rightarrow \quad \begin{array}{l}\mathbf{W} \equiv \text{eigen-vector matrix}\\ \boldsymbol{\lambda} \equiv \text{eigen-value matrix}\end{array}$$

If we extract a single eigen-mode, $\mathbf{w}_i$ and λ_i, we have

$$\mathbf{w}_i = \begin{bmatrix}\mathbf{s}_{i1}\\ \mathbf{s}_{i2}\end{bmatrix}$$

Recall that $\vec{\mathbf{s}} = \hat{\mathbf{P}}_1\mathbf{s}_1 + \hat{\mathbf{P}}_2\mathbf{s}_2$. Therefore,

$$\vec{\mathbf{s}}_i = \hat{\mathbf{P}}_1\mathbf{s}_{i1} + \hat{\mathbf{P}}_2\mathbf{s}_{i2} = \begin{bmatrix}\hat{\mathbf{P}}_1 & \hat{\mathbf{P}}_2\end{bmatrix}\begin{bmatrix}\mathbf{s}_{i1}\\ \mathbf{s}_{i2}\end{bmatrix} = \begin{bmatrix}\hat{\mathbf{P}}_1 & \hat{\mathbf{P}}_2\end{bmatrix}\mathbf{w}_i = \begin{bmatrix}\mathbf{s}_x\\ \mathbf{s}_y\\ \mathbf{s}_z\end{bmatrix}$$

These terms are now independent of the chosen polarization vectors.

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Formulation of Efficient 1D Plane Wave Expansion Method

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Reduction to One Dimension

For the 1D devices where propagation is only in the z-direction, the wave has no wave vector components in the x and y directions.

$$\mathbf{K}_x = \mathbf{0} \quad \text{and} \quad \mathbf{K}_y = \mathbf{0}$$

Maxwell's equations reduce to

$$-\mathbf{K}_z \mathbf{u}_y = jk_0 \left[\!\left[\varepsilon_{xx} \right]\!\right] \mathbf{s}_x \qquad -\mathbf{K}_z \mathbf{s}_y = jk_0 \left[\!\left[\mu_{xx} \right]\!\right] \mathbf{u}_x$$

$$\mathbf{K}_z \mathbf{u}_x = jk_0 \left[\!\left[\varepsilon_{yy} \right]\!\right] \mathbf{s}_y \qquad \mathbf{K}_z \mathbf{s}_x = jk_0 \left[\!\left[\mu_{yy} \right]\!\right] \mathbf{u}_y$$

$$\mathbf{0} = jk_0 \left[\!\left[\varepsilon_{zz} \right]\!\right] \mathbf{s}_z \qquad \mathbf{0} = jk_0 \left[\!\left[\mu_{zz} \right]\!\right] \mathbf{u}_z$$

Immediately, we see that

$$\mathbf{s}_z = \mathbf{0} \qquad \mathbf{u}_z = \mathbf{0}$$

The field will have no components in the longitudinal direction z.

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Two Modes

Maxwell's equations have again decoupled into two distinct sets of equations.

$$-\mathbf{K}_z \mathbf{u}_y = jk_0 \left[\!\left[\varepsilon_{xx} \right]\!\right] \mathbf{s}_x \qquad -\mathbf{K}_z \mathbf{s}_y = jk_0 \left[\!\left[\mu_{xx} \right]\!\right] \mathbf{u}_x$$

$$\mathbf{K}_z \mathbf{u}_x = jk_0 \left[\!\left[\varepsilon_{yy} \right]\!\right] \mathbf{s}_y \qquad \mathbf{K}_z \mathbf{s}_x = jk_0 \left[\!\left[\mu_{yy} \right]\!\right] \mathbf{u}_y$$

$$\mathbf{0} = jk_0 \left[\!\left[\varepsilon_{zz} \right]\!\right] \mathbf{s}_z \qquad \mathbf{0} = jk_0 \left[\!\left[\mu_{zz} \right]\!\right] \mathbf{u}_z$$

E_x Mode

$$-\mathbf{K}_z \mathbf{u}_y = jk_0 \left[\!\left[\varepsilon_{xx} \right]\!\right] \mathbf{s}_x$$

$$\mathbf{K}_z \mathbf{s}_x = jk_0 \left[\!\left[\mu_{yy} \right]\!\right] \mathbf{u}_y$$

E_y Mode

$$\mathbf{K}_z \mathbf{u}_x = jk_0 \left[\!\left[\varepsilon_{yy} \right]\!\right] \mathbf{s}_y$$

$$-\mathbf{K}_z \mathbf{s}_y = jk_0 \left[\!\left[\mu_{xx} \right]\!\right] \mathbf{u}_x$$

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Final Eigen-Value Problems

We can substitute one equation into the other to derive two eigen-value problems.

Ex Mode

$$\mathbf{K}_z \boldsymbol{\mu}_{yy}^{-1} \mathbf{K}_z \mathbf{s}_x = k_0^2 [\![\varepsilon_{xx}]\!] \mathbf{s}_x$$

$$\mathbf{u}_y = \frac{1}{jk_0} [\![\mu_{yy}]\!]^{-1} \mathbf{K}_z \mathbf{s}_x$$

or

$$\mathbf{K}_z \boldsymbol{\varepsilon}_{xx}^{-1} \mathbf{K}_z \mathbf{u}_y = k_0^2 [\![\mu_{yy}]\!] \mathbf{u}_y$$

$$\mathbf{s}_x = -\frac{1}{jk_0} [\![\varepsilon_{xx}]\!]^{-1} \mathbf{K}_z \mathbf{u}_y$$

Ey Mode

$$\mathbf{K}_z \boldsymbol{\mu}_{xx}^{-1} \mathbf{K}_z \mathbf{s}_y = k_0^2 [\![\varepsilon_{yy}]\!] \mathbf{s}_y$$

$$\mathbf{u}_x = -\frac{1}{jk_0} [\![\mu_{xx}]\!]^{-1} \mathbf{K}_z \mathbf{s}_y$$

or

$$\mathbf{K}_z \boldsymbol{\varepsilon}_{yy}^{-1} \mathbf{K}_z \mathbf{u}_x = k_0^2 [\![\mu_{xx}]\!] \mathbf{u}_x$$

$$\mathbf{s}_y = \frac{1}{jk_0} [\![\varepsilon_{yy}]\!]^{-1} \mathbf{K}_z \mathbf{u}_x$$

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PWEM with Fully Anisotropic Materials

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Anisotropic 3D PWEM

The matrix equations derived for 3D PWEM are valid for fully anisotropic materials. In the anisotropic case, however, the material tensors become block matrices composed of convolution matrices for each tensor element individually.

$$[\![\varepsilon_{\mathrm{r}}]\!]=\begin{bmatrix}[\![\varepsilon_{xx}]\!] & [\![\varepsilon_{xy}]\!] & [\![\varepsilon_{xz}]\!]\\ [\![\varepsilon_{yx}]\!] & [\![\varepsilon_{yy}]\!] & [\![\varepsilon_{yz}]\!]\\ [\![\varepsilon_{zx}]\!] & [\![\varepsilon_{zy}]\!] & [\![\varepsilon_{zz}]\!]\end{bmatrix}$$

$$[\![\mu_{\mathrm{r}}]\!]=\begin{bmatrix}[\![\mu_{xx}]\!] & [\![\mu_{xy}]\!] & [\![\mu_{xz}]\!]\\ [\![\mu_{yx}]\!] & [\![\mu_{yy}]\!] & [\![\mu_{yz}]\!]\\ [\![\mu_{zx}]\!] & [\![\mu_{zy}]\!] & [\![\mu_{zz}]\!]\end{bmatrix}$$

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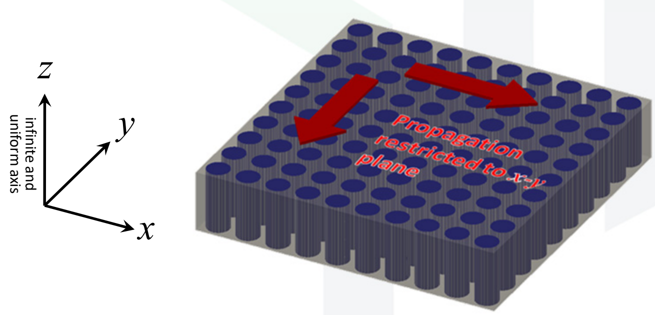
Anisotropic 2D PWEM

In general, all field components remain coupled in anisotropic media. Since Maxwell's equations will not simplify in this case, there is almost no numerical advantage to developing a 2D PWEM for anisotropic media. Instead, use your anisotropic 3D PWEM for 2D problems by using only on spatial harmonic in the uniform direction.



$P \equiv$ # harmonics along x

$Q \equiv$ # harmonics along y

$R \equiv$ # harmonics along z

In this case, you would set $R = 1$.

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Incorporating Dispersion

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Techniques for Incorporating Dispersion

1. Iterative PWEM
   - Can only handle one frequency at a time.

2. Do not use k_0 as the eigen-value

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Alternate 2D-PWEM Formulation (1 of 5)

Recall our starting point for E and H modes

E-Mode

$$\mathbf{K}_x\mathbf{u}_y - \mathbf{K}_y\mathbf{u}_x = jk_0[\![\varepsilon_{zz}]\!]\mathbf{s}_z$$
$$\mathbf{K}_y\mathbf{s}_z = jk_0[\![\mu_{xx}]\!]\mathbf{u}_x$$
$$-\mathbf{K}_x\mathbf{s}_z = jk_0[\![\mu_{yy}]\!]\mathbf{u}_y$$

H-Mode

$$\mathbf{K}_x\mathbf{s}_y - \mathbf{K}_y\mathbf{s}_x = jk_0[\![\mu_{zz}]\!]\mathbf{u}_z$$
$$\mathbf{K}_y\mathbf{u}_z = jk_0[\![\varepsilon_{xx}]\!]\mathbf{s}_x$$
$$-\mathbf{K}_x\mathbf{u}_z = jk_0[\![\varepsilon_{yy}]\!]\mathbf{s}_y$$

We will choose β_y as our eigen-value instead of k_0.

E-Mode

$$(\beta_x\mathbf{I} - \mathbf{G}_x)\mathbf{u}_y - (\beta_y\mathbf{I} - \mathbf{G}_y)\mathbf{u}_x = jk_0[\![\varepsilon_{zz}]\!]\mathbf{s}_z$$
$$(\beta_y\mathbf{I} - \mathbf{G}_y)\mathbf{s}_z = jk_0[\![\mu_{xx}]\!]\mathbf{u}_x$$
$$-(\beta_x\mathbf{I} - \mathbf{G}_x)\mathbf{s}_z = jk_0[\![\mu_{yy}]\!]\mathbf{u}_y$$

H-Mode

$$(\beta_x\mathbf{I} - \mathbf{G}_x)\mathbf{s}_y - (\beta_y\mathbf{I} - \mathbf{G}_y)\mathbf{s}_x = jk_0[\![\mu_{zz}]\!]\mathbf{u}_z$$
$$(\beta_y\mathbf{I} - \mathbf{G}_y)\mathbf{u}_z = jk_0[\![\varepsilon_{xx}]\!]\mathbf{s}_x$$
$$-(\beta_x\mathbf{I} - \mathbf{G}_x)\mathbf{u}_z = jk_0[\![\varepsilon_{yy}]\!]\mathbf{s}_y$$

$\mathbf{G}_x = \mathbf{G}_x \equiv$ components of wave vector expansion

$\mathbf{G}_y = \mathbf{G}_y \equiv$ components of wave vector expansion

Alternate 2D-PWEM Formulation (2 of 5)

Next we normalize our wave vector terms by dividing by k_0.

$\tilde{\mathbf{K}}_x = k_0^{-1}\mathbf{K}_x \quad \tilde{\mathbf{K}}_y = k_0^{-1}\mathbf{K}_y \quad \tilde{\mathbf{G}}_x = k_0^{-1}\mathbf{G}_x \quad \tilde{\mathbf{G}}_y = k_0^{-1}\mathbf{G}_y \quad \tilde{\beta}_x = k_0^{-1}\beta_x \quad \tilde{\beta}_y = k_0^{-1}\beta_y$

Our governing equations can now be written as

E-Mode

$$(\tilde{\beta}_x\mathbf{I} - \tilde{\mathbf{G}}_x)\mathbf{u}_y - (\tilde{\beta}_y\mathbf{I} - \tilde{\mathbf{G}}_y)\mathbf{u}_x = j[\![\varepsilon_{zz}]\!]\mathbf{s}_z$$
$$(\tilde{\beta}_y\mathbf{I} - \tilde{\mathbf{G}}_y)\mathbf{s}_z = j[\![\mu_{xx}]\!]\mathbf{u}_x$$
$$-(\tilde{\beta}_x\mathbf{I} - \tilde{\mathbf{G}}_x)\mathbf{s}_z = j[\![\mu_{yy}]\!]\mathbf{u}_y$$

H-Mode

$$(\tilde{\beta}_x\mathbf{I} - \tilde{\mathbf{G}}_x)\mathbf{s}_y - (\tilde{\beta}_y\mathbf{I} - \tilde{\mathbf{G}}_y)\mathbf{s}_x = j[\![\mu_{zz}]\!]\mathbf{u}_z$$
$$(\tilde{\beta}_y\mathbf{I} - \tilde{\mathbf{G}}_y)\mathbf{u}_z = j[\![\varepsilon_{xx}]\!]\mathbf{s}_x$$
$$-(\tilde{\beta}_x\mathbf{I} - \tilde{\mathbf{G}}_x)\mathbf{u}_z = j[\![\varepsilon_{yy}]\!]\mathbf{s}_y$$

Alternate 2D-PWEM Formulation (3 of 5)

Solving for $\mathbf{s}_y$ and $\mathbf{u}_y$, we get

E-Mode

$$\mathbf{u}_y = j\left[\!\left[\mu_{yy}\right]\!\right]^{-1}\left(\tilde{\beta}_x\mathbf{I} - \tilde{\mathbf{G}}_x\right)\mathbf{s}_z$$

H-Mode

$$\mathbf{s}_y = j\left[\!\left[\varepsilon_{yy}\right]\!\right]^{-1}\left(\tilde{\beta}_x\mathbf{I} - \tilde{\mathbf{G}}_x\right)\mathbf{u}_z$$

We eliminate these terms by substituting these expressions into our governing equations.

E-Mode

$$j\left(\tilde{\beta}_x\mathbf{I} - \tilde{\mathbf{G}}_x\right)\left[\!\left[\mu_{yy}\right]\!\right]^{-1}\left(\tilde{\beta}_x\mathbf{I} - \tilde{\mathbf{G}}_x\right)\mathbf{s}_z - \left(\tilde{\beta}_y\mathbf{I} - \tilde{\mathbf{G}}_y\right)\mathbf{u}_x = j\left[\!\left[\varepsilon_{zz}\right]\!\right]\mathbf{s}_z$$

$$\left(\tilde{\beta}_y\mathbf{I} - \tilde{\mathbf{G}}_y\right)\mathbf{s}_z = j\left[\!\left[\mu_{xx}\right]\!\right]\mathbf{u}_x$$

H-Mode

$$j\left(\tilde{\beta}_x\mathbf{I} - \tilde{\mathbf{G}}_x\right)\left[\!\left[\varepsilon_{yy}\right]\!\right]^{-1}\left(\tilde{\beta}_x\mathbf{I} - \tilde{\mathbf{G}}_x\right)\mathbf{u}_z - \left(\tilde{\beta}_y\mathbf{I} - \tilde{\mathbf{G}}_y\right)\mathbf{s}_x = j\left[\!\left[\mu_{zz}\right]\!\right]\mathbf{u}_z$$

$$\left(\tilde{\beta}_y\mathbf{I} - \tilde{\mathbf{G}}_y\right)\mathbf{u}_z = j\left[\!\left[\varepsilon_{xx}\right]\!\right]\mathbf{s}_x$$

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Alternate 2D-PWEM Formulation (4 of 5)

We rearrange these equations to bring β_y to the right-hand side.

E-Mode

$$\tilde{\mathbf{G}}_y\mathbf{u}_x + j\left[\left(\tilde{\beta}_x\mathbf{I} - \tilde{\mathbf{G}}_x\right)\left[\!\left[\mu_{yy}\right]\!\right]^{-1}\left(\tilde{\beta}_x\mathbf{I} - \tilde{\mathbf{G}}_x\right) - \left[\!\left[\varepsilon_{zz}\right]\!\right]\right]\mathbf{s}_z = \tilde{\beta}_y\mathbf{u}_x$$

$$j\left[\!\left[\mu_{xx}\right]\!\right]\mathbf{u}_x + \tilde{\mathbf{G}}_y\mathbf{s}_z = \tilde{\beta}_y\mathbf{s}_z$$

H-Mode

$$\tilde{\mathbf{G}}_y\mathbf{s}_x + j\left[\left(\tilde{\beta}_x\mathbf{I} - \tilde{\mathbf{G}}_x\right)\left[\!\left[\varepsilon_{yy}\right]\!\right]^{-1}\left(\tilde{\beta}_x\mathbf{I} - \tilde{\mathbf{G}}_x\right) - \left[\!\left[\mu_{zz}\right]\!\right]\right]\mathbf{u}_z = \tilde{\beta}_y\mathbf{s}_x$$

$$j\left[\!\left[\varepsilon_{xx}\right]\!\right]\mathbf{s}_x + \tilde{\mathbf{G}}_y\mathbf{u}_z = \tilde{\beta}_y\mathbf{u}_z$$

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Alternate 2D-PWEM Formulation (5 of 5)

Finally, we write our equations in block matrix form.

E-Mode

$$\begin{bmatrix} \tilde{\mathbf{G}}_y & j\left[\left(\tilde{\beta}_x\mathbf{I}-\tilde{\mathbf{G}}_x\right)[\![\mu_{yy}]\!]^{-1}\left(\tilde{\beta}_x\mathbf{I}-\tilde{\mathbf{G}}_x\right)-[\![\varepsilon_{zz}]\!]\right] \\ j[\![\mu_{xx}]\!] & \tilde{\mathbf{G}}_y \end{bmatrix}\begin{bmatrix}\mathbf{u}_x \\ \mathbf{s}_z\end{bmatrix}=\tilde{\beta}_y\begin{bmatrix}\mathbf{u}_x \\ \mathbf{s}_z\end{bmatrix}$$

H-Mode

$$\begin{bmatrix} \tilde{\mathbf{G}}_y & j\left[\left(\tilde{\beta}_x\mathbf{I}-\tilde{\mathbf{G}}_x\right)[\![\varepsilon_{yy}]\!]^{-1}\left(\tilde{\beta}_x\mathbf{I}-\tilde{\mathbf{G}}_x\right)-[\![\mu_{zz}]\!]\right] \\ j[\![\varepsilon_{xx}]\!] & \tilde{\mathbf{G}}_y \end{bmatrix}\begin{bmatrix}\mathbf{s}_x \\ \mathbf{u}_z\end{bmatrix}=\tilde{\beta}_y\begin{bmatrix}\mathbf{s}_x \\ \mathbf{u}_z\end{bmatrix}$$

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Alternate 3D-PWEM Formulation (1 of 2)

We choose β_z to be the eigen-value.

$$\tilde{\mathbf{K}}_y\mathbf{u}_z-\left(\tilde{\beta}_z\mathbf{I}-\tilde{\mathbf{G}}_z\right)\mathbf{u}_y=j[\![\varepsilon_{xx}]\!]\mathbf{s}_x$$

$$\left(\tilde{\beta}_z\mathbf{I}-\tilde{\mathbf{G}}_z\right)\mathbf{u}_x-\tilde{\mathbf{K}}_x\mathbf{u}_z=j[\![\varepsilon_{yy}]\!]\mathbf{s}_y$$

$$\tilde{\mathbf{K}}_x\mathbf{u}_y-\tilde{\mathbf{K}}_y\mathbf{u}_x=j[\![\varepsilon_{zz}]\!]\mathbf{s}_z$$

$$\tilde{\mathbf{K}}_y\mathbf{s}_z-\left(\tilde{\beta}_z\mathbf{I}-\tilde{\mathbf{G}}_z\right)\mathbf{s}_y=j[\![\mu_{xx}]\!]\mathbf{u}_x$$

$$\left(\tilde{\beta}_z\mathbf{I}-\tilde{\mathbf{G}}_z\right)\mathbf{s}_x-\tilde{\mathbf{K}}_x\mathbf{s}_z=j[\![\mu_{yy}]\!]\mathbf{u}_y$$

$$\tilde{\mathbf{K}}_x\mathbf{s}_y-\tilde{\mathbf{K}}_y\mathbf{s}_x=j[\![\mu_{zz}]\!]\mathbf{u}_z$$

Solve for longitudinal components.

$$\mathbf{u}_z=-j[\![\mu_{zz}]\!]^{-1}\left(\tilde{\mathbf{K}}_x\mathbf{s}_y-\tilde{\mathbf{K}}_y\mathbf{s}_x\right)$$

$$\mathbf{s}_z=-j[\![\varepsilon_{zz}]\!]^{-1}\left(\tilde{\mathbf{K}}_x\mathbf{u}_y-\tilde{\mathbf{K}}_y\mathbf{u}_x\right)$$

Eliminate longitudinal components.

$$j\tilde{\mathbf{K}}_x[\![\mu_{zz}]\!]^{-1}\tilde{\mathbf{K}}_y\mathbf{s}_x+j\left([\![\varepsilon_{yy}]\!]-\tilde{\mathbf{K}}_x[\![\mu_{zz}]\!]^{-1}\tilde{\mathbf{K}}_x\right)\mathbf{s}_y+\tilde{\mathbf{G}}_z\mathbf{u}_x=\tilde{\beta}_z\mathbf{u}_x$$

$$j\left(\tilde{\mathbf{K}}_y[\![\mu_{zz}]\!]^{-1}\tilde{\mathbf{K}}_y-[\![\varepsilon_{xx}]\!]\right)\mathbf{s}_x-j\tilde{\mathbf{K}}_y[\![\mu_{zz}]\!]^{-1}\tilde{\mathbf{K}}_x\mathbf{s}_y+\tilde{\mathbf{G}}_z\mathbf{u}_y=\tilde{\beta}_z\mathbf{u}_y$$

$$\tilde{\mathbf{G}}_z\mathbf{s}_x+j\tilde{\mathbf{K}}_x[\![\varepsilon_{zz}]\!]^{-1}\tilde{\mathbf{K}}_y\mathbf{u}_x+j\left([\![\mu_{yy}]\!]-\tilde{\mathbf{K}}_x[\![\varepsilon_{zz}]\!]^{-1}\tilde{\mathbf{K}}_x\right)\mathbf{u}_y=\tilde{\beta}_z\mathbf{s}_x$$

$$\tilde{\mathbf{G}}_z\mathbf{s}_y+\left(j\tilde{\mathbf{K}}_y[\![\varepsilon_{zz}]\!]^{-1}\tilde{\mathbf{K}}_y-j[\![\mu_{xx}]\!]\right)\mathbf{u}_x-j\tilde{\mathbf{K}}_y[\![\varepsilon_{zz}]\!]^{-1}\tilde{\mathbf{K}}_x\mathbf{u}_y=\tilde{\beta}_z\mathbf{s}_y$$

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Alternate 3D-PWEM Formulation (2 of 2)

In block matrix form, we have

$$\begin{bmatrix} \tilde{\mathbf{G}}_z & \mathbf{0} & j\tilde{\mathbf{K}}_x [\![\varepsilon_{zz}]\!]^{-1}\tilde{\mathbf{K}}_y & j\left([\![\mu_{yy}]\!] - \tilde{\mathbf{K}}_x [\![\varepsilon_{zz}]\!]^{-1}\tilde{\mathbf{K}}_x\right) \\ \mathbf{0} & \tilde{\mathbf{G}}_z & j\left(\tilde{\mathbf{K}}_y [\![\varepsilon_{zz}]\!]^{-1}\tilde{\mathbf{K}}_y - [\![\mu_{xx}]\!]\right) & -j\tilde{\mathbf{K}}_y [\![\varepsilon_{zz}]\!]^{-1}\tilde{\mathbf{K}}_x \\ j\tilde{\mathbf{K}}_x [\![\mu_{zz}]\!]^{-1}\tilde{\mathbf{K}}_y & j\left([\![\varepsilon_{yy}]\!] - \tilde{\mathbf{K}}_x [\![\mu_{zz}]\!]^{-1}\tilde{\mathbf{K}}_x\right) & \tilde{\mathbf{G}}_z & \mathbf{0} \\ j\left(\tilde{\mathbf{K}}_y [\![\mu_{zz}]\!]^{-1}\tilde{\mathbf{K}}_y - [\![\varepsilon_{xx}]\!]\right) & -j\tilde{\mathbf{K}}_y [\![\mu_{zz}]\!]^{-1}\tilde{\mathbf{K}}_x & \mathbf{0} & \tilde{\mathbf{G}}_z \end{bmatrix} \begin{bmatrix} \mathbf{s}_x \\ \mathbf{s}_y \\ \mathbf{u}_x \\ \mathbf{u}_y \end{bmatrix} = \tilde{\beta}_z \begin{bmatrix} \mathbf{s}_x \\ \mathbf{s}_y \\ \mathbf{u}_x \\ \mathbf{u}_y \end{bmatrix}$$

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Alternate 1D-PWEM Formulation

For 1D, our eigen-value problem is

***Ex* Mode**

$$-\mathbf{K}_z\mathbf{u}_y = jk_0 [\![\varepsilon_{xx}]\!]\mathbf{s}_x$$
$$\mathbf{K}_z\mathbf{s}_x = jk_0 [\![\mu_{yy}]\!]\mathbf{u}_y$$
$$\downarrow$$
$$-\left(\tilde{\beta}_z\mathbf{I} - \tilde{\mathbf{G}}_z\right)\mathbf{u}_y = j[\![\varepsilon_{xx}]\!]\mathbf{s}_x$$
$$\left(\tilde{\beta}_z\mathbf{I} - \tilde{\mathbf{G}}_z\right)\mathbf{s}_x = j[\![\mu_{yy}]\!]\mathbf{u}_y$$
$$\downarrow$$
$$\tilde{\mathbf{G}}_z\mathbf{s}_x + j[\![\mu_{yy}]\!]\mathbf{u}_y = \tilde{\beta}_z\mathbf{s}_x$$
$$-j[\![\varepsilon_{xx}]\!]\mathbf{s}_x + \tilde{\mathbf{G}}_z\mathbf{u}_y = \tilde{\beta}_z\mathbf{u}_y$$
$$\downarrow$$
$$\begin{bmatrix} \tilde{\mathbf{G}}_z & j[\![\mu_{yy}]\!] \\ -j[\![\varepsilon_{xx}]\!] & \tilde{\mathbf{G}}_z \end{bmatrix}\begin{bmatrix} \mathbf{s}_x \\ \mathbf{u}_y \end{bmatrix} = \tilde{\beta}_z \begin{bmatrix} \mathbf{s}_x \\ \mathbf{u}_y \end{bmatrix}$$

***Ey* Mode**

$$\mathbf{K}_z\mathbf{u}_x = jk_0 [\![\varepsilon_{yy}]\!]\mathbf{s}_y$$
$$-\mathbf{K}_z\mathbf{s}_y = jk_0 [\![\mu_{xx}]\!]\mathbf{u}_x$$
$$\downarrow$$
$$\left(\tilde{\beta}_z\mathbf{I} - \tilde{\mathbf{G}}_z\right)\mathbf{u}_x = j[\![\varepsilon_{yy}]\!]\mathbf{s}_y$$
$$-\left(\tilde{\beta}_z\mathbf{I} - \tilde{\mathbf{G}}_z\right)\mathbf{s}_y = j[\![\mu_{xx}]\!]\mathbf{u}_x$$
$$\downarrow$$
$$\tilde{\mathbf{G}}_z\mathbf{s}_y - j[\![\mu_{xx}]\!]\mathbf{u}_x = \tilde{\beta}_z\mathbf{s}_y$$
$$j[\![\varepsilon_{yy}]\!]\mathbf{s}_y + \tilde{\mathbf{G}}_z\mathbf{u}_x = \tilde{\beta}_z\mathbf{u}_x$$
$$\downarrow$$
$$\begin{bmatrix} \tilde{\mathbf{G}}_z & -j[\![\mu_{xx}]\!] \\ j[\![\varepsilon_{yy}]\!] & \tilde{\mathbf{G}}_z \end{bmatrix}\begin{bmatrix} \mathbf{s}_y \\ \mathbf{u}_x \end{bmatrix} = \tilde{\beta}_z \begin{bmatrix} \mathbf{s}_y \\ \mathbf{u}_x \end{bmatrix}$$

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Reduced Bloch Mode Expansion (RBME) Technique

Mahmoud I. Hussein, "Reduced Bloch mode expansion for periodic media band structure calculations," Proc. R. Soc. A **465**, pp. 2825-2848, 2009.

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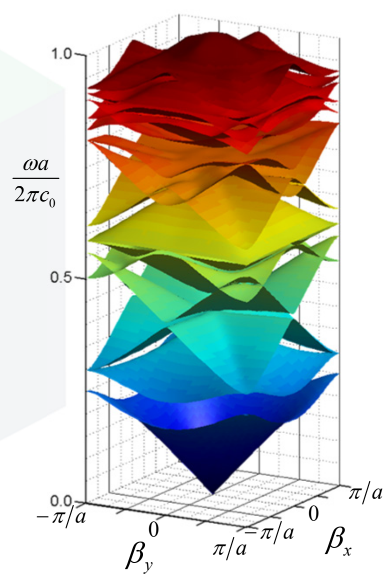
The Problem

Suppose we wish to calculate a photonic band diagram or calculate the full bands throughout the entire Brillouin zone.

For 2D simulations, typical matrix size is 400×400. For 3D simulations, typical matrix size is 12,000×12,000. These large matrices must be solved for each Bloch wave vector of interest. This is typically 100's of vectors.

Run time is long and computation is bogged down.



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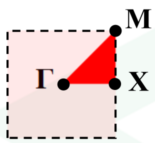
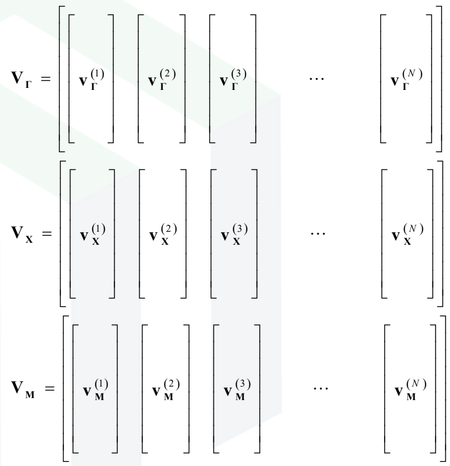
Step 1: Calculate the Full Solution at Only the Key Points of Symmetry





$$\mathbf{V}_\Gamma = \begin{bmatrix} \mathbf{v}_\Gamma^{(1)} & \mathbf{v}_\Gamma^{(2)} & \mathbf{v}_\Gamma^{(3)} & \cdots & \mathbf{v}_\Gamma^{(N)} \end{bmatrix}$$

$$\mathbf{V}_X = \begin{bmatrix} \mathbf{v}_X^{(1)} & \mathbf{v}_X^{(2)} & \mathbf{v}_X^{(3)} & \cdots & \mathbf{v}_X^{(N)} \end{bmatrix}$$

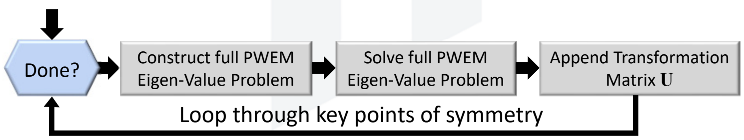
$$\mathbf{V}_M = \begin{bmatrix} \mathbf{v}_M^{(1)} & \mathbf{v}_M^{(2)} & \mathbf{v}_M^{(3)} & \cdots & \mathbf{v}_M^{(N)} \end{bmatrix}$$

Notes

Ensure that no redundant points are used or RBME will fail.

An asymmetric distribution of key points will lead to asymmetry in the data calculated from them.

You can improve accuracy by adding more key points, but this will be slower and less efficient.



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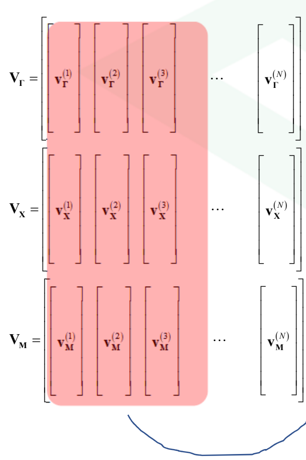
Step 2: Combine Eigen-Vector Matrices Using Lowest Order Modes

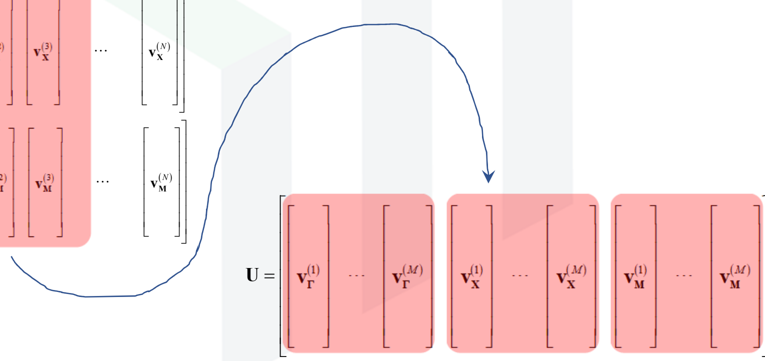


$$\mathbf{V}_\Gamma = \begin{bmatrix} \mathbf{v}_\Gamma^{(1)} & \mathbf{v}_\Gamma^{(2)} & \mathbf{v}_\Gamma^{(3)} & \cdots & \mathbf{v}_\Gamma^{(N)} \end{bmatrix}$$

$$\mathbf{V}_X = \begin{bmatrix} \mathbf{v}_X^{(1)} & \mathbf{v}_X^{(2)} & \mathbf{v}_X^{(3)} & \cdots & \mathbf{v}_X^{(N)} \end{bmatrix}$$

$$\mathbf{V}_M = \begin{bmatrix} \mathbf{v}_M^{(1)} & \mathbf{v}_M^{(2)} & \mathbf{v}_M^{(3)} & \cdots & \mathbf{v}_M^{(N)} \end{bmatrix}$$

Take the M lowest order modes from each eigen-vector matrix and construct a new eigen-vector matrix $\mathbf{U}$.



$$\mathbf{U} = \begin{bmatrix} \mathbf{v}_\Gamma^{(1)} & \cdots & \mathbf{v}_\Gamma^{(M)} & \mathbf{v}_X^{(1)} & \cdots & \mathbf{v}_X^{(M)} & \mathbf{v}_M^{(1)} & \cdots & \mathbf{v}_M^{(M)} \end{bmatrix}$$

$\mathbf{U}$ does not have to be a square matrix.

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Step 3: Perform Gram-Schmidt Orthonormalization on the Matrix $\mathbf{U}$

$$\mathbf{U}=\begin{bmatrix}\begin{bmatrix}\\ \mathbf{v}_{\Gamma}^{(1)}\\ \\ \end{bmatrix} & \cdots & \begin{bmatrix}\\ \mathbf{v}_{\Gamma}^{(M)}\\ \\ \end{bmatrix} & \begin{bmatrix}\\ \mathbf{v}_{\mathbf{X}}^{(1)}\\ \\ \end{bmatrix} & \cdots & \begin{bmatrix}\\ \mathbf{v}_{\mathbf{X}}^{(M)}\\ \\ \end{bmatrix} & \begin{bmatrix}\\ \mathbf{v}_{\mathbf{M}}^{(1)}\\ \\ \end{bmatrix} & \cdots & \begin{bmatrix}\\ \mathbf{v}_{\mathbf{M}}^{(M)}\\ \\ \end{bmatrix}\end{bmatrix}$$

$$\tilde{\mathbf{U}}=\begin{bmatrix}\begin{bmatrix}\\ \tilde{\mathbf{v}}_{\Gamma}^{(1)}\\ \\ \end{bmatrix} & \cdots & \begin{bmatrix}\\ \tilde{\mathbf{v}}_{\Gamma}^{(M)}\\ \\ \end{bmatrix} & \begin{bmatrix}\\ \tilde{\mathbf{v}}_{\mathbf{X}}^{(1)}\\ \\ \end{bmatrix} & \cdots & \begin{bmatrix}\\ \tilde{\mathbf{v}}_{\mathbf{X}}^{(M)}\\ \\ \end{bmatrix} & \begin{bmatrix}\\ \tilde{\mathbf{v}}_{\mathbf{M}}^{(1)}\\ \\ \end{bmatrix} & \cdots & \begin{bmatrix}\\ \tilde{\mathbf{v}}_{\mathbf{M}}^{(M)}\\ \\ \end{bmatrix}\end{bmatrix}$$

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Gram-Schmidt Orthonormalization

$$\mathbf{U}=\begin{bmatrix}\begin{bmatrix}\\ \mathbf{v}_1\\ \\ \end{bmatrix} & \begin{bmatrix}\\ \mathbf{v}_2\\ \\ \end{bmatrix} & \begin{bmatrix}\\ \mathbf{v}_3\\ \\ \end{bmatrix} & \cdots & \begin{bmatrix}\\ \mathbf{v}_{M-1}\\ \\ \end{bmatrix} & \begin{bmatrix}\\ \mathbf{v}_{M-1}\\ \\ \end{bmatrix} & \begin{bmatrix}\\ \mathbf{v}_{M}\\ \\ \end{bmatrix}\end{bmatrix}$$

$$\text{Step 1: } \tilde{\mathbf{v}}_1=\frac{\mathbf{v}_1}{|\mathbf{v}_1|}$$

$$\text{Step 2: } \tilde{\mathbf{v}}_2=\frac{\mathbf{v}_2-(\mathbf{v}_2\bullet\tilde{\mathbf{v}}_1)\tilde{\mathbf{v}}_1}{|\mathbf{v}_2-(\mathbf{v}_2\bullet\tilde{\mathbf{v}}_1)\tilde{\mathbf{v}}_1|}$$

$$\text{Step 3: } \tilde{\mathbf{v}}_3=\frac{\mathbf{v}_3-(\mathbf{v}_3\bullet\tilde{\mathbf{v}}_2)\tilde{\mathbf{v}}_2-(\mathbf{v}_3\bullet\tilde{\mathbf{v}}_1)\tilde{\mathbf{v}}_1}{|\mathbf{v}_3-(\mathbf{v}_3\bullet\tilde{\mathbf{v}}_2)\tilde{\mathbf{v}}_2-(\mathbf{v}_3\bullet\tilde{\mathbf{v}}_1)\tilde{\mathbf{v}}_1|}$$

$$\vdots$$

This algorithm essentially just removes the components of the previous vectors and then normalizes the amplitude.

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Step 4: Construct Eigen-Value Problem for the Next β

Construct the standard PWEM eigen-value problem for the next β, but do not solve it yet.

$$\mathbf{Ax} = \lambda\mathbf{Bx}$$

$$\mathbf{A} = \begin{cases} \mathbf{K}_x [\![\mu_r]\!]^{-1} \mathbf{K}_x + \mathbf{K}_y [\![\mu_r]\!]^{-1} \mathbf{K}_y & \text{E Mode} \\ \mathbf{K}_x [\![\varepsilon_r]\!]^{-1} \mathbf{K}_x + \mathbf{K}_y [\![\varepsilon_r]\!]^{-1} \mathbf{K}_y & \text{H Mode} \end{cases}$$

$$\mathbf{B} = \begin{cases} [\![\varepsilon_r]\!] & \text{E Mode} \\ [\![\mu_r]\!] & \text{H Mode} \end{cases}$$

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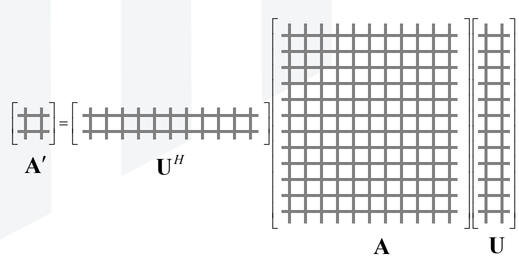
Step 5: Reduce Matrix Size by Expanding into Bloch Modes as the Basis

Perform a linear transformation that expresses the field in terms of the new basis formed from the reduced set of Bloch modes.

This dramatically reduces the size of the matrices while retaining excellent accuracy because those Bloch modes more efficiently describe the fields in the structure.

$$\mathbf{A}' = \tilde{\mathbf{U}}^H \mathbf{A} \tilde{\mathbf{U}}$$

$$\mathbf{B}' = \tilde{\mathbf{U}}^H \mathbf{B} \tilde{\mathbf{U}}$$



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Step 6: Solve the Reduced Eigen-Value Problem

The reduced eigen-value problem is solved according to

$$\mathbf{A}'\mathbf{x} = \lambda \mathbf{B}'\mathbf{x} \quad \rightarrow \quad \mathbf{V}', \lambda$$

The eigen-values are interpreted the same as in the standard PWEM.

If they are needed, the eigen-vectors must be transformed back into a plane wave basis to be compatible with the standard PWEM.

$$\mathbf{V} = \tilde{\mathbf{U}}\mathbf{V}'\tilde{\mathbf{U}}^{H}$$

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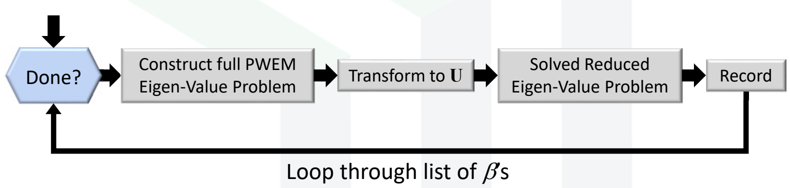
Step 7: Repeat Procedure for Each Bloch Wave Vector in IBZ



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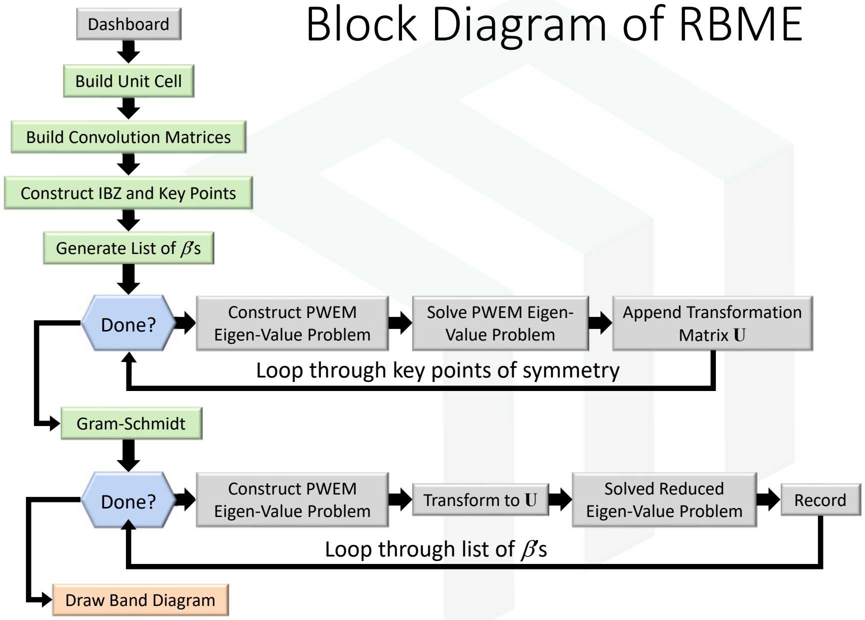
Block Diagram of RBME



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Generalized Symmetry

Slide 52

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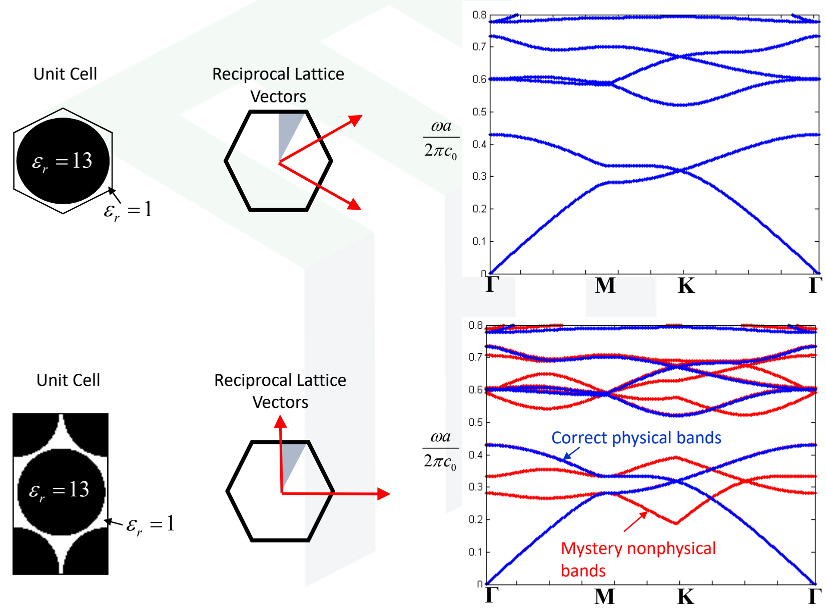
"Mystery Bands" When Modeling Hexagonal Arrays with Rectangular Symmetry



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Cause of "Mystery Bands"

- Lattice symmetry imposes phase conditions between adjacent unit cells expressed by the Bloch theorem.
- The higher the symmetry, the tighter the phase conditions that are imposed.
- There are fewer phase conditions with a rectangular unit cell so there are additional allowed states.

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The Fix

The fix is to construct the convolution matrices for the general symmetry case.

The standard FFT can no longer be used and this process is more numerically intensive.

For 2D periodic functions, the expansion is

$$f(x,y)=\sum_{p=-\infty}^{\infty}\sum_{q=-\infty}^{\infty}a_{p,q}e^{j\left(p\vec{T}_1+q\vec{T}_2\right)\bullet\vec{r}} \qquad a_{p,q}=\frac{1}{A}\iint_A f(x,y)e^{-j\left(p\vec{T}_1+q\vec{T}_2\right)\bullet\vec{r}}dA$$

For 3D periodic functions, this is

$$f(\vec{r})=\sum_{p=-\infty}^{\infty}\sum_{q=-\infty}^{\infty}\sum_{r=-\infty}^{\infty}a_{p,q,r}e^{j\left(p\vec{T}_1+q\vec{T}_2+r\vec{T}_3\right)\bullet\vec{r}} \qquad a_{p,q,r}=\frac{1}{V}\iiint_V f(\vec{r})e^{-j\left(p\vec{T}_1+q\vec{T}_2+r\vec{T}_3\right)\bullet\vec{r}}dV$$

⇧ These must be numerically evaluated.