Advanced Computation:
Computational Electromagnetics

The Method of Lines (MoL)

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Outline

- Formulation of 3D method of lines
- Formulation of 2D method of lines
- Multilayer devices using scattering matrices
- Comparison to RCWA

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Definition of Method of Lines

The method of lines was developed by mathematicians to solve partial differential equations (PDEs).

All but one independent variables are discretized.

This leads to a large set of coupled ordinary differential equations (ODEs).

The system of ODEs are solved analytically.

In electromagnetics, the independent variables are usually x, y, and z.

We typically discretize x and y and leave z analytical.

Any method can be used to discretize the independent variables. This includes Fourier transform, finite-differences, finite-elements, etc.

In electromagnetics, the "method of lines" implies that finite-differences are used to discretize x and y.

RCWA is the method of lines, but uses a Fourier transform instead of finite-differences to discretize x and y.

Sign Convention

This formulation of the method of lines uses the following sign convention for waves travelling in the + z direction.

$$e^{-jkz}$$

Formulation of 3D Method of Lines

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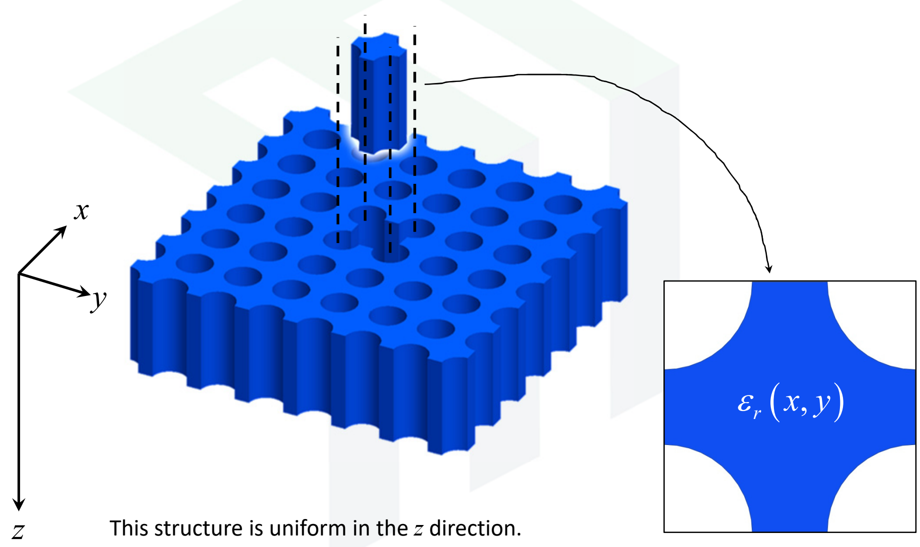
The 2D Unit Cell



This structure is uniform in the z direction.

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Starting Point for MOL

We start with Maxwell's equations in the following form…

$$\frac{\partial E_z}{\partial y'} - \frac{\partial E_y}{\partial z'} = \mu_r \tilde{H}_x \qquad \frac{\partial \tilde{H}_z}{\partial y'} - \frac{\partial \tilde{H}_y}{\partial z'} = \varepsilon_r E_x$$

$$\frac{\partial E_x}{\partial z'} - \frac{\partial E_z}{\partial x'} = \mu_r \tilde{H}_y \qquad \frac{\partial \tilde{H}_x}{\partial z'} - \frac{\partial \tilde{H}_z}{\partial x'} = \varepsilon_r E_y$$

$$\frac{\partial E_y}{\partial x'} - \frac{\partial E_x}{\partial y'} = \mu_r \tilde{H}_z \qquad \frac{\partial \tilde{H}_y}{\partial x'} - \frac{\partial \tilde{H}_x}{\partial y'} = \varepsilon_r E_z$$

Recall that we normalized the magnetic field and grid according to

$$\vec{\tilde{H}} = -j\sqrt{\frac{\mu_0}{\varepsilon_0}}\vec{H} \qquad x' = k_0 x \qquad y' = k_0 y \qquad z' = k_0 z$$

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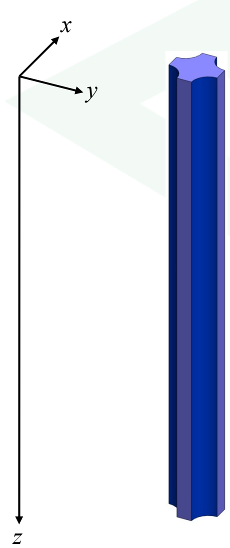
z-Uniform Media



We are going to consider Maxwell's equations inside a media that is uniform in the z direction.

The media may be inhomogeneous in the x-y plane, but it must be uniform in the z direction.

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Semi-Analytical Matrix Form of Maxwell's Equations

We can go straight to matrix form using the concept of matrix derivative operators. We keep the z direction analytical.

$$\frac{\partial E_z}{\partial y'} - \frac{\partial E_y}{\partial z'} = \mu_r \tilde{H}_x \qquad \frac{\partial \tilde{H}_z}{\partial y'} - \frac{\partial \tilde{H}_y}{\partial z'} = \varepsilon_r E_x$$

$$\frac{\partial E_x}{\partial z'} - \frac{\partial E_z}{\partial x'} = \mu_r \tilde{H}_y \qquad \frac{\partial \tilde{H}_x}{\partial z'} - \frac{\partial \tilde{H}_z}{\partial x'} = \varepsilon_r E_y$$

$$\frac{\partial E_y}{\partial x'} - \frac{\partial E_x}{\partial y'} = \mu_r \tilde{H}_z \qquad \frac{\partial \tilde{H}_y}{\partial x'} - \frac{\partial \tilde{H}_x}{\partial y'} = \varepsilon_r E_z$$

$$\mathbf{D}_{y'}^{e}\mathbf{e}_z - \frac{d}{dz'}\mathbf{e}_y = \boldsymbol{\mu}_{xx}\tilde{\mathbf{h}}_x \qquad \mathbf{D}_{y'}^{h}\tilde{\mathbf{h}}_z - \frac{d}{dz'}\tilde{\mathbf{h}}_y = \boldsymbol{\varepsilon}_{xx}\mathbf{e}_x$$

$$\frac{d}{dz'}\mathbf{e}_x - \mathbf{D}_{x'}^{e}\mathbf{e}_z = \boldsymbol{\mu}_{yy}\tilde{\mathbf{h}}_y \qquad \frac{d}{dz'}\tilde{\mathbf{h}}_x - \mathbf{D}_{x'}^{h}\tilde{\mathbf{h}}_z = \boldsymbol{\varepsilon}_{yy}\mathbf{e}_y$$

$$\mathbf{D}_{x'}^{e}\mathbf{e}_y - \mathbf{D}_{y'}^{e}\mathbf{e}_x = \boldsymbol{\mu}_{zz}\tilde{\mathbf{h}}_z \qquad \mathbf{D}_{x'}^{h}\tilde{\mathbf{h}}_y - \mathbf{D}_{y'}^{h}\tilde{\mathbf{h}}_x = \boldsymbol{\varepsilon}_{zz}\mathbf{e}_z$$

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Eliminate Longitudinal Field Components

We solve the third and sixth equation for $\mathbf{e}_z$ and $\mathbf{h}_z$ and substitute these back into the remaining four equations.

$$\mathbf{D}_{y'}^{e}\mathbf{e}_z - \frac{d}{dz'}\mathbf{e}_y = \boldsymbol{\mu}_{xx}\tilde{\mathbf{h}}_x$$

$$\frac{d}{dz'}\mathbf{e}_x - \mathbf{D}_{x'}^{e}\mathbf{e}_z = \boldsymbol{\mu}_{yy}\tilde{\mathbf{h}}_y$$

$$\mathbf{D}_{x'}^{e}\mathbf{e}_y - \mathbf{D}_{y'}^{e}\mathbf{e}_x = \boldsymbol{\mu}_{zz}\tilde{\mathbf{h}}_z$$

$$\tilde{\mathbf{h}}_z = \boldsymbol{\mu}_{zz}^{-1}\left(\mathbf{D}_{x'}^{e}\mathbf{e}_y - \mathbf{D}_{y'}^{e}\mathbf{e}_x\right)$$

$$\frac{d}{dz'}\mathbf{e}_x = -\mathbf{D}_{x'}^{e}\boldsymbol{\varepsilon}_{zz}^{-1}\mathbf{D}_{y'}^{h}\tilde{\mathbf{h}}_x + \left(\boldsymbol{\mu}_{yy} + \mathbf{D}_{x'}^{e}\boldsymbol{\varepsilon}_{zz}^{-1}\mathbf{D}_{x'}^{h}\right)\tilde{\mathbf{h}}_y$$

$$\frac{d}{dz'}\mathbf{e}_y = -\left(\boldsymbol{\mu}_{xx} + \mathbf{D}_{y'}^{e}\boldsymbol{\varepsilon}_{zz}^{-1}\mathbf{D}_{y'}^{h}\right)\tilde{\mathbf{h}}_x + \mathbf{D}_{y'}^{e}\boldsymbol{\varepsilon}_{zz}^{-1}\mathbf{D}_{x'}^{h}\tilde{\mathbf{h}}_y$$

$$\mathbf{D}_{y'}^{h}\tilde{\mathbf{h}}_z - \frac{d}{dz'}\tilde{\mathbf{h}}_y = \boldsymbol{\varepsilon}_{xx}\mathbf{e}_x$$

$$\frac{d}{dz'}\tilde{\mathbf{h}}_x - \mathbf{D}_{x'}^{h}\tilde{\mathbf{h}}_z = \boldsymbol{\varepsilon}_{yy}\mathbf{e}_y$$

$$\mathbf{D}_{x'}^{h}\tilde{\mathbf{h}}_y - \mathbf{D}_{y'}^{h}\tilde{\mathbf{h}}_x = \boldsymbol{\varepsilon}_{zz}\mathbf{e}_z$$

$$\mathbf{e}_z = \boldsymbol{\varepsilon}_{zz}^{-1}\left(\mathbf{D}_{x'}^{h}\tilde{\mathbf{h}}_y - \mathbf{D}_{y'}^{h}\tilde{\mathbf{h}}_x\right)$$

$$\frac{d}{dz'}\tilde{\mathbf{h}}_x = -\mathbf{D}_{x'}^{h}\boldsymbol{\mu}_{zz}^{-1}\mathbf{D}_{y'}^{e}\mathbf{e}_x + \left(\boldsymbol{\varepsilon}_{yy} + \mathbf{D}_{x'}^{h}\boldsymbol{\mu}_{zz}^{-1}\mathbf{D}_{x'}^{e}\right)\mathbf{e}_y$$

$$\frac{d}{dz'}\tilde{\mathbf{h}}_y = -\left(\boldsymbol{\varepsilon}_{xx} + \mathbf{D}_{y'}^{h}\boldsymbol{\mu}_{zz}^{-1}\mathbf{D}_{y'}^{e}\right)\mathbf{e}_x + \mathbf{D}_{y'}^{h}\boldsymbol{\mu}_{zz}^{-1}\mathbf{D}_{x'}^{e}\mathbf{e}_y$$

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Block Matrix Form

We write the remaining four equations in block matrix form as

$$\frac{d}{dz'}\mathbf{e}_y = -\left(\boldsymbol{\mu}_{xx} + \mathbf{D}_{y'}^{e}\boldsymbol{\varepsilon}_{zz}^{-1}\mathbf{D}_{y'}^{h}\right)\tilde{\mathbf{h}}_x + \mathbf{D}_{y'}^{e}\boldsymbol{\varepsilon}_{zz}^{-1}\mathbf{D}_{x'}^{h}\tilde{\mathbf{h}}_y$$

$$\frac{d}{dz'}\mathbf{e}_x = -\mathbf{D}_{x'}^{e}\boldsymbol{\varepsilon}_{zz}^{-1}\mathbf{D}_{y'}^{h}\tilde{\mathbf{h}}_x + \left(\boldsymbol{\mu}_{yy} + \mathbf{D}_{x'}^{e}\boldsymbol{\varepsilon}_{zz}^{-1}\mathbf{D}_{x'}^{h}\right)\tilde{\mathbf{h}}_y$$

$$\frac{d}{dz'}\begin{bmatrix}\mathbf{e}_x\\ \mathbf{e}_y\end{bmatrix} = \begin{bmatrix} -\mathbf{D}_{x'}^{e}\boldsymbol{\varepsilon}_{zz}^{-1}\mathbf{D}_{y'}^{h} & \left(\boldsymbol{\mu}_{yy} + \mathbf{D}_{x'}^{e}\boldsymbol{\varepsilon}_{zz}^{-1}\mathbf{D}_{x'}^{h}\right) \\ -\left(\boldsymbol{\mu}_{xx} + \mathbf{D}_{y'}^{e}\boldsymbol{\varepsilon}_{zz}^{-1}\mathbf{D}_{y'}^{h}\right) & \mathbf{D}_{y'}^{e}\boldsymbol{\varepsilon}_{zz}^{-1}\mathbf{D}_{x'}^{h}\end{bmatrix}\begin{bmatrix}\tilde{\mathbf{h}}_x\\ \tilde{\mathbf{h}}_y\end{bmatrix}$$

$$\frac{d}{dz'}\tilde{\mathbf{h}}_x = -\mathbf{D}_{x'}^{h}\boldsymbol{\mu}_{zz}^{-1}\mathbf{D}_{y'}^{e}\mathbf{e}_x + \left(\boldsymbol{\varepsilon}_{yy} + \mathbf{D}_{x'}^{h}\boldsymbol{\mu}_{zz}^{-1}\mathbf{D}_{x'}^{e}\right)\mathbf{e}_y$$

$$\frac{d}{dz'}\tilde{\mathbf{h}}_y = -\left(\boldsymbol{\varepsilon}_{xx} + \mathbf{D}_{y'}^{h}\boldsymbol{\mu}_{zz}^{-1}\mathbf{D}_{y'}^{e}\right)\mathbf{e}_x + \mathbf{D}_{y'}^{h}\boldsymbol{\mu}_{zz}^{-1}\mathbf{D}_{x'}^{e}\mathbf{e}_y$$

$$\frac{d}{dz'}\begin{bmatrix}\tilde{\mathbf{h}}_x\\ \tilde{\mathbf{h}}_y\end{bmatrix} = \begin{bmatrix} -\mathbf{D}_{x'}^{h}\boldsymbol{\mu}_{zz}^{-1}\mathbf{D}_{y'}^{e} & \left(\boldsymbol{\varepsilon}_{yy} + \mathbf{D}_{x'}^{h}\boldsymbol{\mu}_{zz}^{-1}\mathbf{D}_{x'}^{e}\right) \\ -\left(\boldsymbol{\varepsilon}_{xx} + \mathbf{D}_{y'}^{h}\boldsymbol{\mu}_{zz}^{-1}\mathbf{D}_{y'}^{e}\right) & \mathbf{D}_{y'}^{h}\boldsymbol{\mu}_{zz}^{-1}\mathbf{D}_{x'}^{e}\end{bmatrix}\begin{bmatrix}\mathbf{e}_x\\ \mathbf{e}_y\end{bmatrix}$$

Standard PQ Form

We can now write our two equations in the "standard" $\mathbf{P}$ and $\mathbf{Q}$ form.

$$\frac{d}{dz'}\begin{bmatrix}\mathbf{e}_x\\ \mathbf{e}_y\end{bmatrix} = \mathbf{P}\begin{bmatrix}\tilde{\mathbf{h}}_x\\ \tilde{\mathbf{h}}_y\end{bmatrix} \qquad \mathbf{P} = \begin{bmatrix} -\mathbf{D}_{x'}^{e}\boldsymbol{\varepsilon}_{zz}^{-1}\mathbf{D}_{y'}^{h} & \left(\boldsymbol{\mu}_{yy} + \mathbf{D}_{x'}^{e}\boldsymbol{\varepsilon}_{zz}^{-1}\mathbf{D}_{x'}^{h}\right) \\ -\left(\boldsymbol{\mu}_{xx} + \mathbf{D}_{y'}^{e}\boldsymbol{\varepsilon}_{zz}^{-1}\mathbf{D}_{y'}^{h}\right) & \mathbf{D}_{y'}^{e}\boldsymbol{\varepsilon}_{zz}^{-1}\mathbf{D}_{x'}^{h}\end{bmatrix}$$

$$\frac{d}{dz'}\begin{bmatrix}\tilde{\mathbf{h}}_x\\ \tilde{\mathbf{h}}_y\end{bmatrix} = \mathbf{Q}\begin{bmatrix}\mathbf{e}_x\\ \mathbf{e}_y\end{bmatrix} \qquad \mathbf{Q} = \begin{bmatrix} -\mathbf{D}_{x'}^{h}\boldsymbol{\mu}_{zz}^{-1}\mathbf{D}_{y'}^{e} & \left(\boldsymbol{\varepsilon}_{yy} + \mathbf{D}_{x'}^{h}\boldsymbol{\mu}_{zz}^{-1}\mathbf{D}_{x'}^{e}\right) \\ -\left(\boldsymbol{\varepsilon}_{xx} + \mathbf{D}_{y'}^{h}\boldsymbol{\mu}_{zz}^{-1}\mathbf{D}_{y'}^{e}\right) & \mathbf{D}_{y'}^{h}\boldsymbol{\mu}_{zz}^{-1}\mathbf{D}_{x'}^{e}\end{bmatrix}$$

Matrix Wave Equation

We differentiate the "P" equation with respect to z' and substitute the "Q" equation into the result to derive the matrix wave equation.

$$\frac{d}{dz'}\begin{bmatrix}\mathbf{e}_x\\ \mathbf{e}_y\end{bmatrix} = \mathbf{P}\begin{bmatrix}\tilde{\mathbf{h}}_x\\ \tilde{\mathbf{h}}_y\end{bmatrix} \qquad \frac{d}{dz'}\begin{bmatrix}\tilde{\mathbf{h}}_x\\ \tilde{\mathbf{h}}_y\end{bmatrix} = \mathbf{Q}\begin{bmatrix}\mathbf{e}_x\\ \mathbf{e}_y\end{bmatrix}$$

$$\frac{d^2}{dz'^2}\begin{bmatrix}\mathbf{e}_x\\ \mathbf{e}_y\end{bmatrix} = \mathbf{P}\cdot\frac{d}{dz'}\begin{bmatrix}\tilde{\mathbf{h}}_x\\ \tilde{\mathbf{h}}_y\end{bmatrix}$$

$$\frac{d^2}{dz'^2}\begin{bmatrix}\mathbf{e}_x\\ \mathbf{e}_y\end{bmatrix} = \mathbf{PQ}\begin{bmatrix}\mathbf{e}_x\\ \mathbf{e}_y\end{bmatrix}$$

$$\frac{d^2}{dz'^2}\begin{bmatrix}\mathbf{e}_x\\ \mathbf{e}_y\end{bmatrix} - \mathbf{PQ}\begin{bmatrix}\mathbf{e}_x\\ \mathbf{e}_y\end{bmatrix} = \begin{bmatrix}\mathbf{0}\\ \mathbf{0}\end{bmatrix}$$

$$\frac{d^2}{dz'^2}\begin{bmatrix}\mathbf{e}_x\\ \mathbf{e}_y\end{bmatrix} - \mathbf{\Omega}^2\begin{bmatrix}\mathbf{e}_x\\ \mathbf{e}_y\end{bmatrix} = \mathbf{0}$$

$$\mathbf{\Omega}^2 = \mathbf{PQ}$$

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Solution to the Wave Equation

The solution to the wave equation is written the same as in RCWA and TMM.

$$\begin{bmatrix}\mathbf{e}_x\\ \mathbf{e}_y\end{bmatrix} = \mathbf{W}e^{-\boldsymbol{\lambda}z'}\mathbf{c}^+ + \mathbf{W}e^{\boldsymbol{\lambda}z'}\mathbf{c}^-$$

where $\mathbf{W}$ and $\boldsymbol{\lambda}^2$ are the eigen-vectors and eigen-values of $\mathbf{\Omega}^2$

The overall solution is then

$$\boldsymbol{\psi}(\tilde{z}) = \begin{bmatrix}\mathbf{e}_x(\tilde{z})\\ \mathbf{e}_y(\tilde{z})\\ \tilde{\mathbf{h}}_x(\tilde{z})\\ \tilde{\mathbf{h}}_y(\tilde{z})\end{bmatrix} = \begin{bmatrix}\mathbf{W} & \mathbf{W}\\ -\mathbf{V} & \mathbf{V}\end{bmatrix}\begin{bmatrix}e^{-\boldsymbol{\lambda}\tilde{z}} & \mathbf{0}\\ \mathbf{0} & e^{\boldsymbol{\lambda}\tilde{z}}\end{bmatrix}\begin{bmatrix}\mathbf{c}^+\\ \mathbf{c}^-\end{bmatrix} \qquad \mathbf{V} = \mathbf{QW}\boldsymbol{\lambda}^{-1}$$

Based on previous lectures, you know the rest of this story!

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Interpretation of the Solution

$$\boldsymbol{\psi}(\tilde{z}) = \mathbf{W} e^{\boldsymbol{\lambda}\tilde{z}} \mathbf{c}$$

$\psi(z)$ – Overall solution which is the sum of all the modes at plane z'.

c – Column vector containing the amplitude coefficient of each of the modes. This quantities how much energy is in each mode.

W – Square matrix who's column vectors describe the "modes" that can exist in the material. These are essentially pictures of the modes which quantify the relative amplitudes of E_x, E_y, H_x, and H_y.

$e^{\lambda z'}$ – Diagonal matrix describing how the modes propagate. This includes accumulation of phase as well as decaying (loss) or growing (gain) amplitude.

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Solution in a Homogeneous Layer

In a homogeneous layer, the $\mathbf{P}$ and $\mathbf{Q}$ matrices are

$$\mathbf{P} = \frac{1}{\varepsilon_\mathrm{r}} \begin{bmatrix} -\mathbf{D}_{x'}^{e}\mathbf{D}_{y'}^{h} & \left(\mu_\mathrm{r}\varepsilon_\mathrm{r}\mathbf{I} + \mathbf{D}_{x'}^{e}\mathbf{D}_{x'}^{h}\right) \\ -\left(\mu_\mathrm{r}\varepsilon_\mathrm{r}\mathbf{I} + \mathbf{D}_{y'}^{e}\mathbf{D}_{y'}^{h}\right) & \mathbf{D}_{y'}^{e}\mathbf{D}_{x'}^{h} \end{bmatrix} \qquad \mathbf{Q} = \frac{1}{\mu_\mathrm{r}} \begin{bmatrix} -\mathbf{D}_{x'}^{h}\mathbf{D}_{y'}^{e} & \left(\mu_\mathrm{r}\varepsilon_\mathrm{r}\mathbf{I} + \mathbf{D}_{x'}^{h}\mathbf{D}_{x'}^{e}\right) \\ -\left(\mu_\mathrm{r}\varepsilon_\mathrm{r}\mathbf{I} + \mathbf{D}_{y'}^{h}\mathbf{D}_{y'}^{e}\right) & \mathbf{D}_{y'}^{h}\mathbf{D}_{x'}^{e} \end{bmatrix}$$

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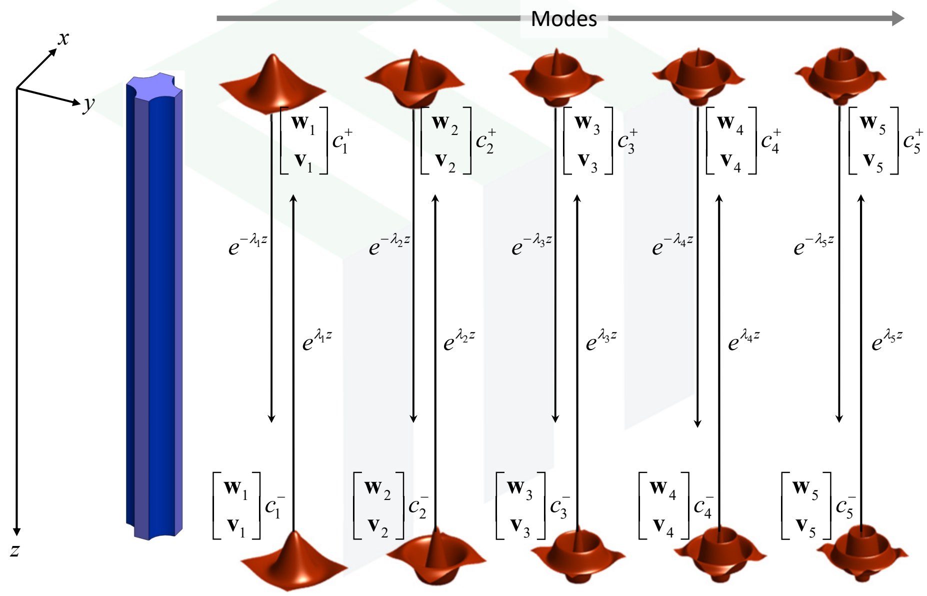
Visualization of this Solution



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Formulation of 2D Method of Lines

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Starting Point

We will start with Maxwell's equations in real-space and in matrix form as derived earlier this lecture.

$$\frac{\partial E_z}{\partial y'} - \frac{\partial E_y}{\partial z'} = \mu_r \tilde{H}_x \qquad \frac{\partial \tilde{H}_z}{\partial y'} - \frac{\partial \tilde{H}_y}{\partial z'} = \varepsilon_r E_x$$

$$\frac{\partial E_x}{\partial z'} - \frac{\partial E_z}{\partial x'} = \mu_r \tilde{H}_y \qquad \frac{\partial \tilde{H}_x}{\partial z'} - \frac{\partial \tilde{H}_z}{\partial x'} = \varepsilon_r E_y$$

$$\frac{\partial E_y}{\partial x'} - \frac{\partial E_x}{\partial y'} = \mu_r \tilde{H}_z \qquad \frac{\partial \tilde{H}_y}{\partial x'} - \frac{\partial \tilde{H}_x}{\partial y'} = \varepsilon_r E_z$$

$$\mathbf{D}_{y'}^{e}\mathbf{e}_z - \frac{d}{dz'}\mathbf{e}_y = \boldsymbol{\mu}_{xx}\tilde{\mathbf{h}}_x \qquad \mathbf{D}_{y'}^{h}\tilde{\mathbf{h}}_z - \frac{d}{dz'}\tilde{\mathbf{h}}_y = \boldsymbol{\varepsilon}_{xx}\mathbf{e}_x$$

$$\frac{d}{dz'}\mathbf{e}_x - \mathbf{D}_{x'}^{e}\mathbf{e}_z = \boldsymbol{\mu}_{yy}\tilde{\mathbf{h}}_y \qquad \frac{d}{dz'}\tilde{\mathbf{h}}_x - \mathbf{D}_{x'}^{h}\tilde{\mathbf{h}}_z = \boldsymbol{\varepsilon}_{yy}\mathbf{e}_y$$

$$\mathbf{D}_{x'}^{e}\mathbf{e}_y - \mathbf{D}_{y'}^{e}\mathbf{e}_x = \boldsymbol{\mu}_{zz}\tilde{\mathbf{h}}_z \qquad \mathbf{D}_{x'}^{h}\tilde{\mathbf{h}}_y - \mathbf{D}_{y'}^{h}\tilde{\mathbf{h}}_x = \boldsymbol{\varepsilon}_{zz}\mathbf{e}_z$$

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Reduction to Two Dimensions

For diagonally anisotropic devices that are uniform in the y direction and when there is no wave propagation in the y direction, we have

$$\mathbf{D}_{y'}^{e} = \mathbf{D}_{y'}^{h} = \mathbf{0}$$

Maxwell's equations reduce to

$$-\frac{d}{dz'}\mathbf{e}_y = \boldsymbol{\mu}_{xx}\tilde{\mathbf{h}}_x \qquad -\frac{d}{dz'}\tilde{\mathbf{h}}_y = \boldsymbol{\varepsilon}_{xx}\mathbf{e}_x$$

$$\frac{d}{dz'}\mathbf{e}_x - \mathbf{D}_{x'}^{e}\mathbf{e}_z = \boldsymbol{\mu}_{yy}\tilde{\mathbf{h}}_y \qquad \frac{d}{dz'}\tilde{\mathbf{h}}_x - \mathbf{D}_{x'}^{h}\tilde{\mathbf{h}}_z = \boldsymbol{\varepsilon}_{yy}\mathbf{e}_y$$

$$\mathbf{D}_{x'}^{e}\mathbf{e}_y = \boldsymbol{\mu}_{zz}\tilde{\mathbf{h}}_z \qquad \mathbf{D}_{x'}^{h}\tilde{\mathbf{h}}_y = \boldsymbol{\varepsilon}_{zz}\mathbf{e}_z$$

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Two Independent Modes

Maxwell's equations have decoupled into two independent modes.

$$-\frac{d}{dz'}\mathbf{e}_y = \boldsymbol{\mu}_{xx}\tilde{\mathbf{h}}_x \qquad -\frac{d}{dz'}\tilde{\mathbf{h}}_y = \boldsymbol{\varepsilon}_{xx}\mathbf{e}_x$$

$$\frac{d}{dz'}\mathbf{e}_x - \mathbf{D}_{x'}^{e}\mathbf{e}_z = \boldsymbol{\mu}_{yy}\tilde{\mathbf{h}}_y \qquad \frac{d}{dz'}\tilde{\mathbf{h}}_x - \mathbf{D}_{x'}^{h}\tilde{\mathbf{h}}_z = \boldsymbol{\varepsilon}_{yy}\mathbf{e}_y$$

$$\mathbf{D}_{x'}^{e}\mathbf{e}_y = \boldsymbol{\mu}_{zz}\tilde{\mathbf{h}}_z \qquad \mathbf{D}_{x'}^{h}\tilde{\mathbf{h}}_y = \boldsymbol{\varepsilon}_{zz}\mathbf{e}_z$$

E Mode

$$\frac{d}{dz'}\tilde{\mathbf{h}}_x - \mathbf{D}_{x'}^{h}\tilde{\mathbf{h}}_z = \boldsymbol{\varepsilon}_{yy}\mathbf{e}_y$$

$$-\frac{d}{dz'}\mathbf{e}_y = \boldsymbol{\mu}_{xx}\tilde{\mathbf{h}}_x$$

$$\mathbf{D}_{x'}^{e}\mathbf{e}_y = \boldsymbol{\mu}_{zz}\tilde{\mathbf{h}}_z$$

H Mode

$$\frac{d}{dz'}\mathbf{e}_x - \mathbf{D}_{x'}^{e}\mathbf{e}_z = \boldsymbol{\mu}_{yy}\tilde{\mathbf{h}}_y$$

$$-\frac{d}{dz'}\tilde{\mathbf{h}}_y = \boldsymbol{\varepsilon}_{xx}\mathbf{e}_x$$

$$\mathbf{D}_{x'}^{h}\tilde{\mathbf{h}}_y = \boldsymbol{\varepsilon}_{zz}\mathbf{e}_z$$

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Apply Dielectric Smoothing

To improve convergence, we can incorporate dielectric smoothing as follows:

E Mode

$$\frac{d}{dz'}\tilde{\mathbf{h}}_x - \mathbf{D}_{x'}^{h}\tilde{\mathbf{h}}_z = \langle\boldsymbol{\varepsilon}_{yy}\rangle\mathbf{e}_y$$

$$-\frac{d}{dz'}\mathbf{e}_y = \langle\boldsymbol{\mu}_{xx}^{-1}\rangle^{-1}\tilde{\mathbf{h}}_x$$

$$\mathbf{D}_{x'}^{e}\mathbf{e}_y = \langle\boldsymbol{\mu}_{zz}\rangle\tilde{\mathbf{h}}_z$$

H Mode

$$\frac{d}{dz'}\mathbf{e}_x - \mathbf{D}_{x'}^{e}\mathbf{e}_z = \langle\boldsymbol{\mu}_{yy}\rangle\tilde{\mathbf{h}}_y$$

$$-\frac{d}{dz'}\tilde{\mathbf{h}}_y = \langle\boldsymbol{\varepsilon}_{xx}^{-1}\rangle^{-1}\mathbf{e}_x$$

$$\mathbf{D}_{x'}^{h}\tilde{\mathbf{h}}_y = \langle\boldsymbol{\varepsilon}_{zz}\rangle\mathbf{e}_z$$

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Eliminate Longitudinal Components

We solve for the longitudinal components

$$\tilde{\mathbf{h}}_z = \langle \boldsymbol{\mu}_{zz} \rangle^{-1} \mathbf{D}_{x'}^{e} \mathbf{e}_y \qquad \mathbf{e}_z = \langle \boldsymbol{\varepsilon}_{zz} \rangle^{-1} \mathbf{D}_{x'}^{h} \tilde{\mathbf{h}}_y$$

and substitute these expressions into the remaining equations.

E Mode

$$\frac{d}{dz'}\tilde{\mathbf{h}}_x = \langle \boldsymbol{\varepsilon}_{yy} \rangle \mathbf{e}_y + \mathbf{D}_{x'}^{h} \langle \boldsymbol{\mu}_{zz} \rangle^{-1} \mathbf{D}_{x'}^{e} \mathbf{e}_y$$

$$\frac{d}{dz'}\mathbf{e}_y = -\langle \boldsymbol{\mu}_{xx}^{-1} \rangle^{-1} \tilde{\mathbf{h}}_x$$

H Mode

$$\frac{d}{dz'}\mathbf{e}_x = \langle \boldsymbol{\mu}_{yy} \rangle \tilde{\mathbf{h}}_y + \mathbf{D}_{x'}^{e} \langle \boldsymbol{\varepsilon}_{zz} \rangle^{-1} \mathbf{D}_{x'}^{h} \tilde{\mathbf{h}}_y$$

$$\frac{d}{dz'}\tilde{\mathbf{h}}_y = -\langle \boldsymbol{\varepsilon}_{xx}^{-1} \rangle^{-1} \mathbf{e}_x$$

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Standard P and Q Form

We write our matrix equations in the standard **PQ** form.

E Mode

$$\frac{d}{dz'}\mathbf{e}_y = \mathbf{P}\tilde{\mathbf{h}}_x$$

$$\frac{d}{dz'}\tilde{\mathbf{h}}_x = \mathbf{Q}\mathbf{e}_y$$

$$\mathbf{P} = -\langle \boldsymbol{\mu}_{xx}^{-1} \rangle^{-1}$$

$$\mathbf{Q} = \langle \boldsymbol{\varepsilon}_{yy} \rangle + \mathbf{D}_{x'}^{h} \langle \boldsymbol{\mu}_{zz} \rangle^{-1} \mathbf{D}_{x'}^{e}$$

H Mode

$$\frac{d}{dz'}\tilde{\mathbf{h}}_y = \mathbf{P}\mathbf{e}_x$$

$$\frac{d}{dz'}\mathbf{e}_x = \mathbf{Q}\tilde{\mathbf{h}}_y$$

$$\mathbf{P} = -\langle \boldsymbol{\varepsilon}_{xx}^{-1} \rangle^{-1}$$

$$\mathbf{Q} = \langle \boldsymbol{\mu}_{yy} \rangle + \mathbf{D}_{x'}^{e} \langle \boldsymbol{\varepsilon}_{zz} \rangle^{-1} \mathbf{D}_{x'}^{h}$$

…and now you know the rest of the story.

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Multilayer Devices Using Scattering Matrices

R. C. Rumpf, "Improved formulation of scattering matrices for semi-analytical methods that is consistent with convention," PIERS B, Vol. 35, 241-261, 2011.

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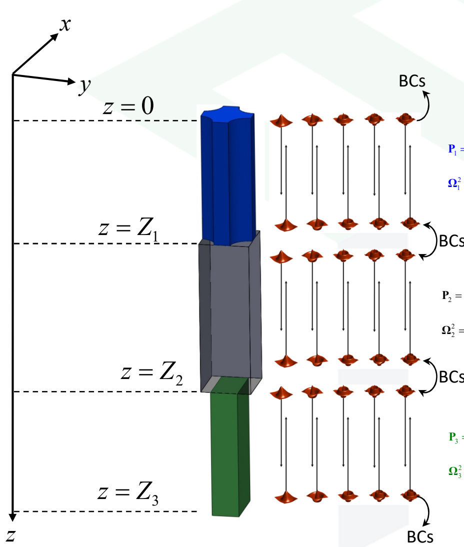
Eigen System in Each Layer

Boundary conditions required that all layers have the same $\mathbf{K}_x$ and $\mathbf{K}_y$ matrices.



$$\mathbf{P}_1 = \begin{bmatrix} -\mathbf{D}_{x'}^{e}\boldsymbol{\varepsilon}_{1,zz}^{-1}\mathbf{D}_{y'}^{h} & \left(\boldsymbol{\mu}_{1,yy} + \mathbf{D}_{x'}^{e}\boldsymbol{\varepsilon}_{1,zz}^{-1}\mathbf{D}_{x'}^{h}\right) \\ -\left(\boldsymbol{\mu}_{1,xx} + \mathbf{D}_{y'}^{e}\boldsymbol{\varepsilon}_{1,zz}^{-1}\mathbf{D}_{y'}^{h}\right) & \mathbf{D}_{y'}^{e}\boldsymbol{\varepsilon}_{1,zz}^{-1}\mathbf{D}_{x'}^{h} \end{bmatrix} \qquad \mathbf{Q}_1 = \begin{bmatrix} -\mathbf{D}_{x'}^{h}\boldsymbol{\mu}_{1,zz}^{-1}\mathbf{D}_{y'}^{e} & \left(\boldsymbol{\varepsilon}_{1,yy} + \mathbf{D}_{x'}^{h}\boldsymbol{\mu}_{1,zz}^{-1}\mathbf{D}_{x'}^{e}\right) \\ -\left(\boldsymbol{\varepsilon}_{1,xx} + \mathbf{D}_{y'}^{h}\boldsymbol{\mu}_{1,zz}^{-1}\mathbf{D}_{y'}^{e}\right) & \mathbf{D}_{y'}^{h}\boldsymbol{\mu}_{1,zz}^{-1}\mathbf{D}_{x'}^{e} \end{bmatrix}$$

$$\boldsymbol{\Omega}_1^2 = \mathbf{P}_1\mathbf{Q}_1 \;\rightarrow\; \mathbf{W}_1, \boldsymbol{\lambda}_1 \;\rightarrow\; \mathbf{V}_1 \qquad \mathbf{c}_1^+, \mathbf{c}_1^-$$

$$\mathbf{P}_2 = \begin{bmatrix} -\mathbf{D}_{x'}^{e}\boldsymbol{\varepsilon}_{2,zz}^{-1}\mathbf{D}_{y'}^{h} & \left(\boldsymbol{\mu}_{2,yy} + \mathbf{D}_{x'}^{e}\boldsymbol{\varepsilon}_{2,zz}^{-1}\mathbf{D}_{x'}^{h}\right) \\ -\left(\boldsymbol{\mu}_{2,xx} + \mathbf{D}_{y'}^{e}\boldsymbol{\varepsilon}_{2,zz}^{-1}\mathbf{D}_{y'}^{h}\right) & \mathbf{D}_{y'}^{e}\boldsymbol{\varepsilon}_{2,zz}^{-1}\mathbf{D}_{x'}^{h} \end{bmatrix} \qquad \mathbf{Q}_2 = \begin{bmatrix} -\mathbf{D}_{x'}^{h}\boldsymbol{\mu}_{2,zz}^{-1}\mathbf{D}_{y'}^{e} & \left(\boldsymbol{\varepsilon}_{2,yy} + \mathbf{D}_{x'}^{h}\boldsymbol{\mu}_{2,zz}^{-1}\mathbf{D}_{x'}^{e}\right) \\ -\left(\boldsymbol{\varepsilon}_{2,xx} + \mathbf{D}_{y'}^{h}\boldsymbol{\mu}_{2,zz}^{-1}\mathbf{D}_{y'}^{e}\right) & \mathbf{D}_{y'}^{h}\boldsymbol{\mu}_{2,zz}^{-1}\mathbf{D}_{x'}^{e} \end{bmatrix}$$

$$\boldsymbol{\Omega}_2^2 = \mathbf{P}_2\mathbf{Q}_2 \;\rightarrow\; \mathbf{W}_2, \boldsymbol{\lambda}_2 \;\rightarrow\; \mathbf{V}_2 \qquad \mathbf{c}_2^+, \mathbf{c}_2^-$$

$$\mathbf{P}_3 = \begin{bmatrix} -\mathbf{D}_{x'}^{e}\boldsymbol{\varepsilon}_{3,zz}^{-1}\mathbf{D}_{y'}^{h} & \left(\boldsymbol{\mu}_{3,yy} + \mathbf{D}_{x'}^{e}\boldsymbol{\varepsilon}_{3,zz}^{-1}\mathbf{D}_{x'}^{h}\right) \\ -\left(\boldsymbol{\mu}_{3,xx} + \mathbf{D}_{y'}^{e}\boldsymbol{\varepsilon}_{3,zz}^{-1}\mathbf{D}_{y'}^{h}\right) & \mathbf{D}_{y'}^{e}\boldsymbol{\varepsilon}_{3,zz}^{-1}\mathbf{D}_{x'}^{h} \end{bmatrix} \qquad \mathbf{Q}_3 = \begin{bmatrix} -\mathbf{D}_{x'}^{h}\boldsymbol{\mu}_{3,zz}^{-1}\mathbf{D}_{y'}^{e} & \left(\boldsymbol{\varepsilon}_{3,yy} + \mathbf{D}_{x'}^{h}\boldsymbol{\mu}_{3,zz}^{-1}\mathbf{D}_{x'}^{e}\right) \\ -\left(\boldsymbol{\varepsilon}_{3,xx} + \mathbf{D}_{y'}^{h}\boldsymbol{\mu}_{3,zz}^{-1}\mathbf{D}_{y'}^{e}\right) & \mathbf{D}_{y'}^{h}\boldsymbol{\mu}_{3,zz}^{-1}\mathbf{D}_{x'}^{e} \end{bmatrix}$$

$$\boldsymbol{\Omega}_3^2 = \mathbf{P}_3\mathbf{Q}_3 \;\rightarrow\; \mathbf{W}_2, \boldsymbol{\lambda}_2 \;\rightarrow\; \mathbf{V}_3 \qquad \mathbf{c}_3^+, \mathbf{c}_3^-$$

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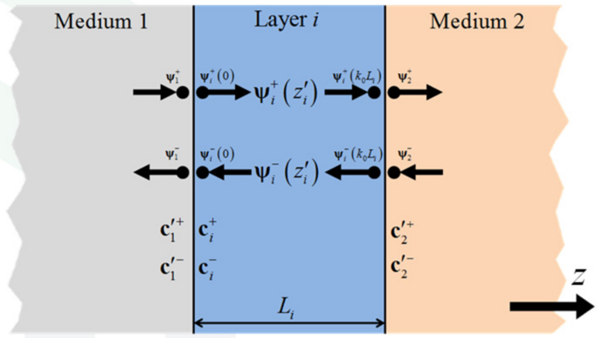
Field Relations & Boundary Conditions

Field inside the i^{th} layer:

$$\boldsymbol{\psi}_i(\tilde{z}) = \begin{bmatrix} \mathbf{e}_x(\tilde{z}) \\ \mathbf{e}_y(\tilde{z}) \\ \tilde{\mathbf{h}}_x(\tilde{z}) \\ \tilde{\mathbf{h}}_y(\tilde{z}) \end{bmatrix} = \begin{bmatrix} \mathbf{W}_i & \mathbf{W}_i \\ -\mathbf{V}_i & \mathbf{V}_i \end{bmatrix} \begin{bmatrix} e^{-\boldsymbol{\lambda}_i \tilde{z}} & \mathbf{0} \\ \mathbf{0} & e^{\boldsymbol{\lambda}_i \tilde{z}} \end{bmatrix} \begin{bmatrix} \mathbf{c}_i^+ \\ \mathbf{c}_i^- \end{bmatrix}$$



Boundary conditions at the first interface:

$$\boldsymbol{\psi}_1 = \boldsymbol{\psi}_i(0)$$

$$\begin{bmatrix} \mathbf{W}_1 & \mathbf{W}_1 \\ -\mathbf{V}_1 & \mathbf{V}_1 \end{bmatrix} \begin{bmatrix} \mathbf{c}_1'^+ \\ \mathbf{c}_1'^- \end{bmatrix} = \begin{bmatrix} \mathbf{W}_i & \mathbf{W}_i \\ -\mathbf{V}_i & \mathbf{V}_i \end{bmatrix} \begin{bmatrix} \mathbf{c}_i^+ \\ \mathbf{c}_i^- \end{bmatrix}$$

Boundary conditions at the second interface:

$$\boldsymbol{\psi}_i(k_0 L_i) = \boldsymbol{\psi}_2$$

$$\begin{bmatrix} \mathbf{W}_i & \mathbf{W}_i \\ -\mathbf{V}_i & \mathbf{V}_i \end{bmatrix} \begin{bmatrix} e^{-\boldsymbol{\lambda}_i k_0 L_i} & \mathbf{0} \\ \mathbf{0} & e^{\boldsymbol{\lambda}_i k_0 L_i} \end{bmatrix} \begin{bmatrix} \mathbf{c}_i^+ \\ \mathbf{c}_i^- \end{bmatrix} = \begin{bmatrix} \mathbf{W}_2 & \mathbf{W}_2 \\ -\mathbf{V}_2 & \mathbf{V}_2 \end{bmatrix} \begin{bmatrix} \mathbf{c}_2'^+ \\ \mathbf{c}_2'^- \end{bmatrix}$$

Note: k_0 has been incorporated to normalize L_i.

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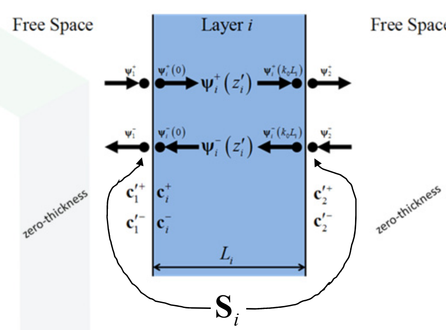
Adopt the Symmetric S-Matrix Approach

The scattering matrix $\mathbf{S}_i$ of the i^{th} layer is still defined as:

$$\begin{bmatrix} \mathbf{c}_1'^- \\ \mathbf{c}_2'^+ \end{bmatrix} = \mathbf{S}^{(i)} \begin{bmatrix} \mathbf{c}_1'^+ \\ \mathbf{c}_2'^- \end{bmatrix} \qquad \mathbf{S}^{(i)} = \begin{bmatrix} \mathbf{S}_{11}^{(i)} & \mathbf{S}_{12}^{(i)} \\ \mathbf{S}_{21}^{(i)} & \mathbf{S}_{22}^{(i)} \end{bmatrix}$$



But the elements are calculated as

$$\mathbf{S}_{11}^{(i)} = \left(\mathbf{A}_i - \mathbf{X}_i\mathbf{B}_i\mathbf{A}_i^{-1}\mathbf{X}_i\mathbf{B}_i\right)^{-1}\left(\mathbf{X}_i\mathbf{B}_i\mathbf{A}_i^{-1}\mathbf{X}_i\mathbf{A}_i - \mathbf{B}_i\right)$$

$$\mathbf{S}_{12}^{(i)} = \left(\mathbf{A}_i - \mathbf{X}_i\mathbf{B}_i\mathbf{A}_i^{-1}\mathbf{X}_i\mathbf{B}_i\right)^{-1}\mathbf{X}_i\left(\mathbf{A}_i - \mathbf{B}_i\mathbf{A}_i^{-1}\mathbf{B}_i\right)$$

$$\left.\begin{aligned} \mathbf{S}_{21}^{(i)} &= \mathbf{S}_{12}^{(i)} \\ \mathbf{S}_{22}^{(i)} &= \mathbf{S}_{11}^{(i)} \end{aligned}\right\}$$

- Layers are symmetric so the scattering matrix elements have redundancy.
- Scattering matrix equations are simplified.
- Fewer calculations.
- Less memory storage.

$$\mathbf{A}_i = \mathbf{W}_i^{-1}\mathbf{W}_0 + \mathbf{V}_i^{-1}\mathbf{V}_0$$

$$\mathbf{B}_i = \mathbf{W}_i^{-1}\mathbf{W}_0 - \mathbf{V}_i^{-1}\mathbf{V}_0$$

$$\mathbf{X}_i = e^{-\boldsymbol{\lambda}_i k_0 L_i}$$

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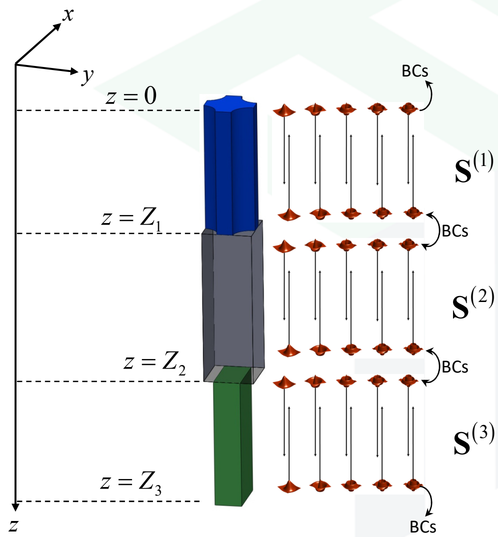
Global Scattering Matrix



Scattering matrix for all layers.

$$\mathbf{S}^{(\text{device})} = \mathbf{S}^{(1)} \otimes \mathbf{S}^{(2)} \otimes \mathbf{S}^{(3)}$$

Connection to outside regions

$$\mathbf{S}^{(\text{global})} = \mathbf{S}^{(\text{ref})} \otimes \mathbf{S}^{(\text{device})} \otimes \mathbf{S}^{(\text{trn})}$$

Recall this procedure from Lecture 5.

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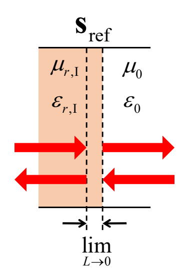
Reflection/Transmission Side Scattering Matrices

The reflection-side scattering matrix is

$$\mathbf{S}_{11}^{(\text{ref})} = -\mathbf{A}_{\text{ref}}^{-1}\mathbf{B}_{\text{ref}}$$
$$\mathbf{S}_{12}^{(\text{ref})} = 2\mathbf{A}_{\text{ref}}^{-1}$$
$$\mathbf{S}_{21}^{(\text{ref})} = 0.5\left(\mathbf{A}_{\text{ref}} - \mathbf{B}_{\text{ref}}\mathbf{A}_{\text{ref}}^{-1}\mathbf{B}_{\text{ref}}\right)$$
$$\mathbf{S}_{22}^{(\text{ref})} = \mathbf{B}_{\text{ref}}\mathbf{A}_{\text{ref}}^{-1}$$

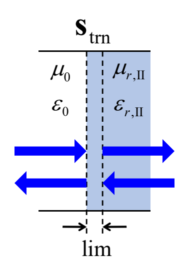
$$\mathbf{A}_{\text{ref}} = \mathbf{W}_0^{-1}\mathbf{W}_{\text{ref}} + \mathbf{V}_0^{-1}\mathbf{V}_{\text{ref}}$$
$$\mathbf{B}_{\text{ref}} = \mathbf{W}_0^{-1}\mathbf{W}_{\text{ref}} - \mathbf{V}_0^{-1}\mathbf{V}_{\text{ref}}$$



The transmission-side scattering matrix is

$$\mathbf{S}_{11}^{(\text{trn})} = \mathbf{B}_{\text{trn}}\mathbf{A}_{\text{trn}}^{-1}$$
$$\mathbf{S}_{12}^{(\text{trn})} = 0.5\left(\mathbf{A}_{\text{trn}} - \mathbf{B}_{\text{trn}}\mathbf{A}_{\text{trn}}^{-1}\mathbf{B}_{\text{trn}}\right)$$
$$\mathbf{S}_{21}^{(\text{trn})} = 2\mathbf{A}_{\text{trn}}^{-1}$$
$$\mathbf{S}_{22}^{(\text{trn})} = -\mathbf{A}_{\text{trn}}^{-1}\mathbf{B}_{\text{trn}}$$

$$\mathbf{A}_{\text{trn}} = \mathbf{W}_0^{-1}\mathbf{W}_{\text{trn}} + \mathbf{V}_0^{-1}\mathbf{V}_{\text{trn}}$$
$$\mathbf{B}_{\text{trn}} = \mathbf{W}_0^{-1}\mathbf{W}_{\text{trn}} - \mathbf{V}_0^{-1}\mathbf{V}_{\text{trn}}$$



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Calculating the Transmitted and Reflected Fields

The electric field source is calculated as

$$\mathbf{c}_{\text{inc}} = \mathbf{W}_{\text{ref}}^{-1}\vec{\mathbf{e}}_T^{\text{inc}} \qquad \vec{\mathbf{e}}_{xy}^{\text{inc}} = \begin{bmatrix} p_x e^{-j\left(k_{x,\text{inc}}\mathbf{x}+k_{y,\text{inc}}\mathbf{y}\right)} \\ p_y e^{-j\left(k_{x,\text{inc}}\mathbf{x}+k_{y,\text{inc}}\mathbf{y}\right)} \end{bmatrix}$$

$\mathbf{x}$ and $\mathbf{y}$ are column vectors containing the coordinates of the points in the cross section.

Recall calculating the polarization components p_x and p_y in TMM.

$|\vec{p}| = 1$

Given the global scattering matrix, the coefficients for the reflected and transmitted fields are

$$\mathbf{c}_{\text{ref}} = \mathbf{S}_{11}\mathbf{c}_{\text{inc}} \qquad \mathbf{c}_{\text{trn}} = \mathbf{S}_{21}\mathbf{c}_{\text{inc}}$$

The transverse components of the reflected and transmitted fields are then

$$\vec{\mathbf{e}}_{xy}^{\text{ref}} = \mathbf{W}_{\text{ref}}\mathbf{c}_{\text{ref}} = \mathbf{W}_{\text{ref}}\mathbf{S}_{11}\mathbf{c}_{\text{inc}} = \begin{bmatrix} \mathbf{r}_x \\ \mathbf{r}_y \end{bmatrix}$$

$$\vec{\mathbf{e}}_{xy}^{\text{trn}} = \mathbf{W}_{\text{trn}}\mathbf{c}_{\text{trn}} = \mathbf{W}_{\text{trn}}\mathbf{S}_{21}\mathbf{c}_{\text{inc}} = \begin{bmatrix} \mathbf{t}_x \\ \mathbf{t}_y \end{bmatrix}$$

Calculate the Amplitudes of the Spatial Harmonics

Remove the phase tilt

$$A_{x,\text{ref}}(x,y) = r_x(x,y) \div \exp\left[-j\left(k_{x,\text{inc}}x + k_{y,\text{inc}}y\right)\right]$$

$$A_{y,\text{ref}}(x,y) = r_y(x,y) \div \exp\left[-j\left(k_{x,\text{inc}}x + k_{y,\text{inc}}y\right)\right]$$

$$A_{x,\text{trn}}(x,y) = t_x(x,y) \div \exp\left[-j\left(k_{x,\text{inc}}x + k_{y,\text{inc}}y\right)\right]$$

$$A_{y,\text{trn}}(x,y) = t_y(x,y) \div \exp\left[-j\left(k_{x,\text{inc}}x + k_{y,\text{inc}}y\right)\right]$$

Calculate amplitudes of the spatial harmonics

$$\mathbf{r}_x = \text{FFT}_{2D}\left\{A_{x,\text{ref}}(x,y)\right\}$$

$$\mathbf{r}_y = \text{FFT}_{2D}\left\{A_{y,\text{ref}}(x,y)\right\}$$

$$\mathbf{t}_x = \text{FFT}_{2D}\left\{A_{x,\text{trn}}(x,y)\right\}$$

$$\mathbf{t}_y = \text{FFT}_{2D}\left\{A_{y,\text{trn}}(x,y)\right\}$$

Calculating the Longitudinal Components

The longitudinal field components are calculated from the transverse components using the divergence equation.

$$\mathbf{r}_z = -\mathbf{K}_{z,\text{ref}}^{-1}\left(\mathbf{K}_x\mathbf{r}_x + \mathbf{K}_y\mathbf{r}_y\right)$$

$$\mathbf{t}_z = -\mathbf{K}_{z,\text{trn}}^{-1}\left(\mathbf{K}_x\mathbf{t}_x + \mathbf{K}_y\mathbf{t}_y\right)$$

Note, the $\mathbf{K}$ matrices are not normalized.

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Calculating the Diffraction Efficiencies

The diffraction efficiencies $\mathbf{R}$ and $\mathbf{T}$ are calculated as

$$\left|\vec{\mathbf{r}}\right|^2 = \left|\mathbf{r}_x\right|^2 + \left|\mathbf{r}_y\right|^2 + \left|\mathbf{r}_z\right|^2$$

$$\left|\vec{\mathbf{t}}\right|^2 = \left|\mathbf{t}_x\right|^2 + \left|\mathbf{t}_y\right|^2 + \left|\mathbf{t}_z\right|^2$$

$$\mathbf{R} = \frac{\text{Re}\left[-\mathbf{K}_{z,\text{ref}}/\mu_{\text{r,inc}}\right]}{\text{Re}\left[k_z^{\text{inc}}/\mu_{\text{r,inc}}\right]}\left|\vec{\mathbf{r}}\right|^2$$

$$\mathbf{T} = \frac{\text{Re}\left[\mathbf{K}_{z,\text{trn}}/\mu_{\text{r,trn}}\right]}{\text{Re}\left[k_z^{\text{inc}}/\mu_{\text{r,inc}}\right]}\left|\vec{\mathbf{t}}\right|^2$$

Remember that these equations assume the source was given unit amplitude.

Note, the $\mathbf{K}$ matrices are not normalized here.

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Comparison to RCWA

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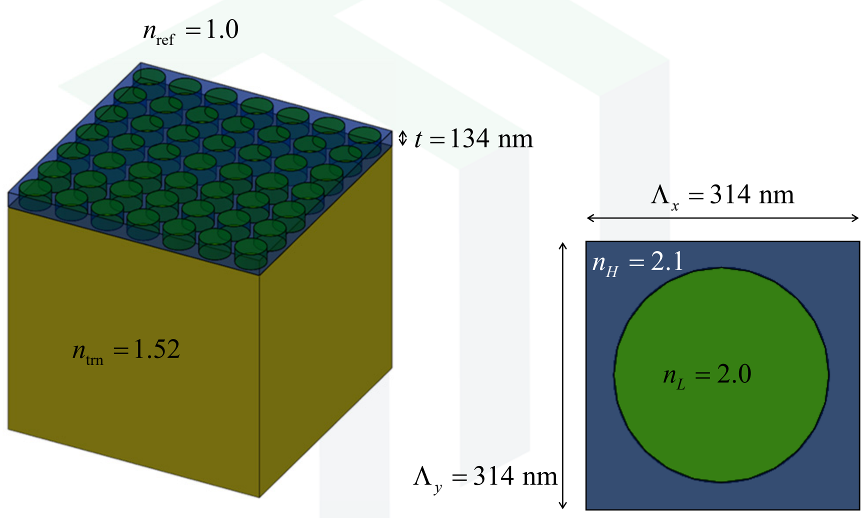
Dielectric Device Comparison



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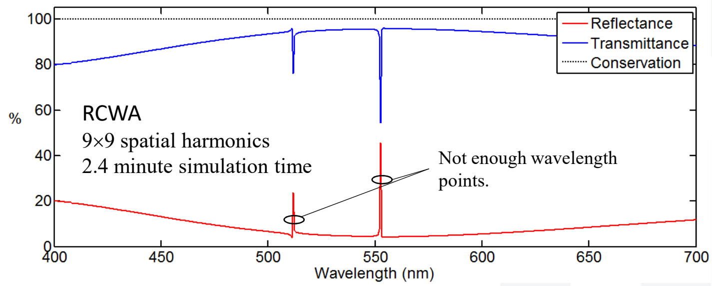
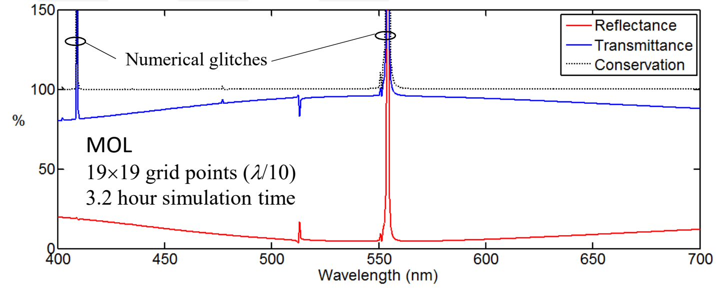
Comparison of Model Results





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