



Advanced Computation:
Computational Electromagnetics

Introduction to Variational Methods

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Outline

- Overview
- Galerkin Method
- Example
- Finite Element Method
- Method of Moments
- Other Worthy Methods
 - Boundary element method
 - Spectral domain method

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Overview

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Classification of Variational Methods

Finite Element Method

- Utilizes a volume mesh
- Matrices are sparse
- Requires boundary conditions

Boundary Element Method

- Utilizes a conformal surface mesh
- Matrices are full
- Good for devices described efficiently by their surfaces
- Does not require boundary conditions

Method of Moments

- Typically makes a PEC approximation
- Utilizes a conformal surface mesh
- Can be 1D for thin wire structures
- Matrices are full
- Good for devices described efficiently by their surfaces
- Does not require boundary conditions

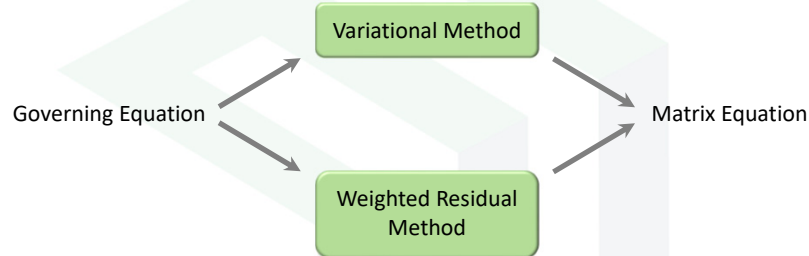
Spectral Domain Method

- BEM/MoM in Fourier-Space
- Matrices are full
- Excellent for periodic structures
- Does not require boundary conditions

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Discretization



Both the variational method and the method of weighted residuals can be used to write a governing equation in matrix form.

Both approaches yield exactly the same matrices.

The Galerkin method is the most popular special case of weighted residual methods.

Galerkin Method



Boris Grigoryevich Galerkin
1871 - 1945

Linear Equations

Consider the following linear homogeneous equation.

$$L[f(x)] = g(x)$$

$L[\] \equiv$ Linear operation
 $f(x) \equiv$ unknown solution
 $g(x) \equiv$ known driving function

The linear operator $L[\]$ has the following properties

$$L[f_1(x) + f_2(x)] = L[f_1(x)] + L[f_2(x)]$$

$$L[af(x)] = aL[f(x)]$$

$$L_1[L_2[f(x)]] = L_2[L_1[f(x)]]$$

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Inner Product

An inner product is a scalar quantity that provides a measure of similarity between two functions.

$\langle f(x), g(x) \rangle \equiv$ inner product between $f(x)$ and $g(x)$

$\langle f, g \rangle = 0$	f and g are orthogonal	It is common to scale functions such that
$\langle f, g \rangle = \text{small number}$	f and g are very different	
$\langle f, g \rangle = \text{big number}$	f and g are very similar	

$$\langle f, f^* \rangle = 1$$

An appropriate inner product must satisfy

$$\langle f, g \rangle = \langle g, f \rangle$$

$$\langle \alpha f + \beta g, h \rangle = \alpha \langle f, h \rangle + \beta \langle g, h \rangle$$

$$\langle f, f^* \rangle = \begin{cases} > 0 & f \neq 0 \\ = 0 & f = 0 \end{cases}$$

We will use the following inner product: $\langle f(z), g(z) \rangle = \int_z f(z) g(z) dz$

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Expansion Into a Set of Basis Functions

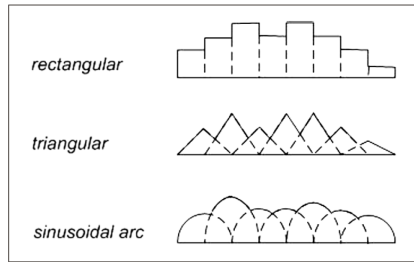
Let $f(x)$ be expanded into a set of basis functions.

$$f(x) = \sum_n a_n v_n(x)$$

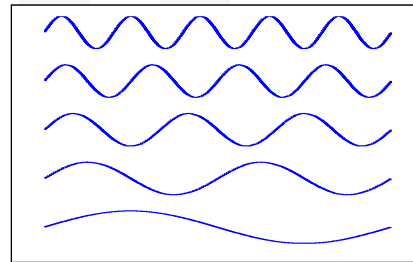
$f(x) \equiv$ unknown function

$a_n \equiv$ coefficient of n^{th} basis function

$v_n(x) \equiv n^{\text{th}}$ basis function



Subdomain Basis Functions



Entire Domain Basis Functions

We choose the basis functions with two considerations: (1) ease of calculations, and (2) minimize how many are needed in the expansion to accurately portray the field.

Linear Equation in Terms of Basis Functions

First we substitute the expansion into the original linear equation.

$$L[f(x)] = g(x)$$

$$f(x) = \sum_n a_n v_n(x)$$

Using the properties of linear operations, we get

$$L[f(x)] = g(x)$$

$$L\left[\sum_n a_n v_n(x)\right] = g(x)$$

$$\sum_n L[a_n v_n(x)] = g(x) \longrightarrow \sum_n a_n L[v_n(x)] = g(x)$$

We try to choose $v_n(x)$ so that $L[v_n(x)]$ is easy to calculate and efficiently represents $f(x)$.

Method of Weighted Residuals (1 of 2)

Similar to how we choose a set of basis functions, we choose another set of weighting functions.

$$w_n(x)$$

We start with our linear inhomogeneous equation and calculate the inner product with $w_m(x)$ of both sides. Here we "test" both sides with $w_m(x)$.

$$\sum_n a_n L[v_n(x)] = g(x)$$

$$\left\langle w_m(x), \sum_n a_n L[v_n(x)] \right\rangle = \left\langle w_m(x), g(x) \right\rangle$$

$$\sum_n a_n \left\langle w_m(x), L[v_n(x)] \right\rangle = \left\langle w_m(x), g(x) \right\rangle$$

Method of Weighted Residuals (2 of 2)

This equation can be written in matrix form as

$$\sum_n a_n \left\langle w_m(x), L[v_n(x)] \right\rangle = \left\langle w_m(x), g(x) \right\rangle \rightarrow [z_{mn}][a_n] = [g_m]$$

$$[z_{mn}] = \begin{bmatrix} \langle w_1, L[v_1] \rangle & \langle w_1, L[v_2] \rangle & & \\ \langle w_2, L[v_1] \rangle & \langle w_2, L[v_2] \rangle & & \\ & & \ddots & \\ & & & \langle w_M, L[v_N] \rangle \end{bmatrix}$$

$$[a_n] = \begin{bmatrix} a_1 \\ a_1 \\ \vdots \\ a_N \end{bmatrix}$$

$$[g_m] = \begin{bmatrix} \langle w_1, g \rangle \\ \langle w_2, g \rangle \\ \vdots \\ \langle w_M, g \rangle \end{bmatrix}$$

Galerkin Method

The Galerkin method is the method of weighted residuals, but the weighting functions are made to be the same as the basis functions.

$$w_m(x) = v_m(x)$$

The matrix equation becomes

$$\sum_n a_n \langle v_m(x), L[v_n(x)] \rangle = \langle v_m(x), g(x) \rangle \rightarrow [z_{mn}][a_n] = [g_m]$$

$$[z_{mn}] = \begin{bmatrix} \langle v_1, L[v_1] \rangle & \langle v_1, L[v_2] \rangle & & \\ \langle v_2, L[v_1] \rangle & \langle v_2, L[v_2] \rangle & & \\ & & \ddots & \\ & & & \langle v_M, L[v_N] \rangle \end{bmatrix} \quad [a_n] = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_N \end{bmatrix} \quad [g_m] = \begin{bmatrix} \langle v_1, g \rangle \\ \langle v_2, g \rangle \\ \vdots \\ \langle v_M, g \rangle \end{bmatrix}$$

Summary of the Galerkin Method

The Galerkin method can be used to find the solution to any linear inhomogeneous equation.

$$L[f] = g$$

Step 1 – Expand the unknown function into a set of basis functions.

$$f = \sum_n a_n v_n \quad L\left[\sum_n a_n v_n\right] = g \rightarrow \sum_n a_n L[v_n] = g$$

Step 2 – Test both sides of the equation with the basis functions using an inner product

$$\left\langle v_m, \sum_n a_n L[v_n] \right\rangle = \langle v_m, g \rangle \rightarrow \sum_n a_n \langle v_m, L[v_n] \rangle = \langle v_m, g \rangle$$

Step 3 – Form a matrix equation

$$[z_{mn}][a_n] = [g_m] \quad [z_{mn}] = \begin{bmatrix} \langle v_1, L[v_1] \rangle & \langle v_1, L[v_2] \rangle & & \\ \langle v_2, L[v_1] \rangle & \langle v_2, L[v_2] \rangle & & \\ & & \ddots & \\ & & & \langle v_M, L[v_N] \rangle \end{bmatrix} \quad [a_n] = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_N \end{bmatrix} \quad [g_m] = \begin{bmatrix} \langle v_1, g \rangle \\ \langle v_2, g \rangle \\ \vdots \\ \langle v_M, g \rangle \end{bmatrix}$$

Example

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Governing Equation and Its Solution

Governing Equation

We will apply the Galerkin method to solve the following differential equation.

$$-\frac{d^2 f}{dx^2} = 1 + 4x^2 \quad f(0) = f(1) = 0 \quad 0 \leq x \leq 1$$

Simple Analytical Solution

This is a simple boundary value problem with the following solution.

$$f(x) = \frac{5x}{6} - \frac{x^2}{2} - \frac{x^4}{3}$$

We wish to solve this using the Galerkin method.

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Choose Basis Functions

Basis Functions

For this problem, it will be convenient to choose as basis functions:

$$v_n = x - x^{n+1}$$

Expansion of the Function into the Basis

We expand our function into this set of basis functions as

$$f(x) = \sum_{n=1}^N a_n (x - x^{n+1})$$

Choose Testing Functions

Testing Functions

The Galerkin method uses the same function for basis and testing.

$$w_n = v_n = x - x^{n+1}$$

Form of Final Solution

Recall that we are converting a linear equation into a matrix equation according to

$$L[f] = g \quad \Longrightarrow \quad [z_{mn}][a_n] = [g_m]$$

$$[z_{mn}] = \begin{bmatrix} \langle v_1, L[v_1] \rangle & \langle v_1, L[v_2] \rangle & \dots \\ \langle v_2, L[v_1] \rangle & \langle v_2, L[v_2] \rangle & \dots \\ \vdots & \vdots & \ddots \end{bmatrix}$$

$$[a_n] = \begin{bmatrix} a_1 \\ a_1 \\ \vdots \\ a_N \end{bmatrix} \quad [g_m] = \begin{bmatrix} \langle v_1, g \rangle \\ \langle v_2, g \rangle \\ \vdots \\ \langle v_M, g \rangle \end{bmatrix}$$

To do this, we need to evaluate $\langle v_m, L[v_n] \rangle$ and $\langle v_m, g \rangle$.

First Inner Product

$$\begin{aligned} \langle v_m, L[v_n] \rangle &= \int_0^1 v_m L[v_n] dx \\ &= \int_0^1 (x - x^{m+1}) \left\{ -\frac{d^2}{dx^2} [x - x^{n+1}] \right\} dx \\ &= -\int_0^1 (x - x^{m+1}) [-n(n+1)x^{n-1}] dx \\ &= n(n+1) \int_0^1 (x^n - x^{m+n}) dx \\ &= n(n+1) \cdot \left[\frac{x^{n+1}}{n+1} - \frac{x^{m+n+1}}{m+n+1} \right]_0^1 \\ &= n(n+1) \cdot \left[\frac{1}{n+1} - \frac{1}{m+n+1} \right] \\ &= n(n+1) \cdot \left[\frac{m+n+1}{(n+1)(m+n+1)} - \frac{n+1}{(n+1)(m+n+1)} \right] \\ &= n(n+1) \cdot \left[\frac{m}{(n+1)(m+n+1)} \right] \\ &= \frac{mn}{(m+n+1)} \end{aligned}$$

$$z_{mn} = \langle v_m, L[v_n] \rangle = \frac{mn}{(m+n+1)}$$

Second Inner Product

$$\begin{aligned}
 \langle v_m, g \rangle &= \int_x v_m g dx \\
 &= \int_0^1 (x - x^{m+1})(1 + 4x^2) dx \\
 &= \int_0^1 (x + 4x^3 - x^{m+1} - 4x^{m+3}) dx \\
 &= \left[\frac{x^2}{2} + x^4 - \frac{x^{m+2}}{m+2} - \frac{4x^{m+4}}{m+4} \right]_0^1 \\
 &= \frac{1}{2} + 1 - \frac{1}{m+2} - \frac{4}{m+4} \\
 &= \frac{3}{2} - \frac{1}{m+2} - \frac{4}{m+4} \\
 &= \frac{3(m+2)(m+4) - 2(m+4) - 8(m+2)}{2(m+2)(m+4)} \\
 &= \frac{3(m^2 + 6m + 8) - 10m - 24}{2(m+2)(m+4)} \\
 &= \frac{3m^2 + 8m}{2(m+2)(m+4)} \\
 &= \frac{m(3m+8)}{2(m+2)(m+4)}
 \end{aligned}$$

$$g_m = \langle v_m, g \rangle = \frac{m(3m+8)}{2(m+2)(m+4)}$$

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Try $N=1$

For $N=1$, our matrix equation isn't even a matrix equation.

$$[z_{11}][a_1] = [g_1] \rightarrow z_{11}a_1 = g_1$$

$$z_{mn} = \frac{mn}{(m+n+1)}$$

Applying our equations for z_{mn} and g_m , we get

$$g_m = \frac{m(3m+8)}{2(m+2)(m+4)}$$

$$z_{11} = \frac{1 \cdot 1}{(1+1+1)} = \frac{1}{3}$$

$$g_1 = \frac{1(3 \cdot 1 + 8)}{2(1+2)(1+4)} = \frac{11}{30}$$

The coefficient is

$$a_1 = \frac{g_1}{z_{11}} = \frac{11/30}{1/3} = \frac{11}{10}$$

Finally, the solution for $N=1$ is

$$f(x) = a_1(x - x^2) = \frac{11x}{10} - \frac{11x^2}{10} \quad \leftarrow \text{Not correct. Need larger } N.$$

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Try $N=2$

For $N=2$, our matrix equation is

$$\begin{bmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} g_1 \\ g_2 \end{bmatrix}$$

$$z_{mn} = \frac{mn}{(m+n+1)}$$

$$g_m = \frac{m(3m+8)}{2(m+2)(m+4)}$$

Applying our equations for z_{mn} and g_m , we get

$$z_{11} = \frac{1 \cdot 1}{(1+1+1)} = \frac{1}{3} \quad z_{12} = \frac{1 \cdot 2}{(1+2+1)} = \frac{1}{2} \quad z_{21} = \frac{2 \cdot 1}{(2+1+1)} = \frac{1}{2} \quad z_{22} = \frac{2 \cdot 2}{(2+2+1)} = \frac{4}{5}$$

$$g_1 = \frac{1(3 \cdot 1 + 8)}{2(1+2)(1+4)} = \frac{11}{30} \quad g_2 = \frac{2(3 \cdot 2 + 8)}{2(2+2)(2+4)} = \frac{7}{12}$$

The coefficients are

$$\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{bmatrix}^{-1} \begin{bmatrix} g_1 \\ g_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & \frac{1}{2} \\ \frac{1}{2} & \frac{4}{5} \end{bmatrix}^{-1} \begin{bmatrix} \frac{11}{30} \\ \frac{7}{12} \end{bmatrix} = \begin{bmatrix} \frac{1}{10} \\ \frac{2}{3} \end{bmatrix}$$

Finally, the solution for $N=2$ is

$$f(x) = a_1(x - x^2) + a_2(x - x^3) = \frac{23x}{30} - \frac{x^2}{10} - \frac{2x^3}{3} \quad \leftarrow \text{Still not correct. Need larger } N.$$

Try $N=3$

For $N=3$, are matrix equation is

$$\begin{bmatrix} z_{11} & z_{12} & z_{13} \\ z_{21} & z_{22} & z_{23} \\ z_{31} & z_{32} & z_{33} \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} g_1 \\ g_2 \\ g_3 \end{bmatrix}$$

$$z_{mn} = \frac{mn}{(m+n+1)}$$

$$g_m = \frac{m(3m+8)}{2(m+2)(m+4)}$$

Applying our equations for z_{mn} and g_m , we get

$$\begin{bmatrix} \frac{1}{3} & \frac{1}{2} & \frac{3}{5} \\ \frac{1}{2} & \frac{4}{5} & 1 \\ \frac{3}{5} & 1 & \frac{9}{7} \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} \frac{11}{30} \\ \frac{7}{12} \\ \frac{51}{70} \end{bmatrix} \quad \Rightarrow \quad \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & \frac{1}{2} & \frac{3}{5} \\ \frac{1}{2} & \frac{4}{5} & 1 \\ \frac{3}{5} & 1 & \frac{9}{7} \end{bmatrix}^{-1} \begin{bmatrix} \frac{11}{30} \\ \frac{7}{12} \\ \frac{51}{70} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ 0 \\ \frac{1}{3} \end{bmatrix}$$

Finally, the solution for $N=3$ is

$$f(x) = a_1(x - x^2) + a_2(x - x^3) + a_3(x - x^4) = \frac{5x}{6} - \frac{x^2}{2} - \frac{x^4}{3} \quad \leftarrow \text{Exact solution!}$$

Try $N=4$

For $N=4$, we get a larger matrix equation, but it also converges to the exact solution.

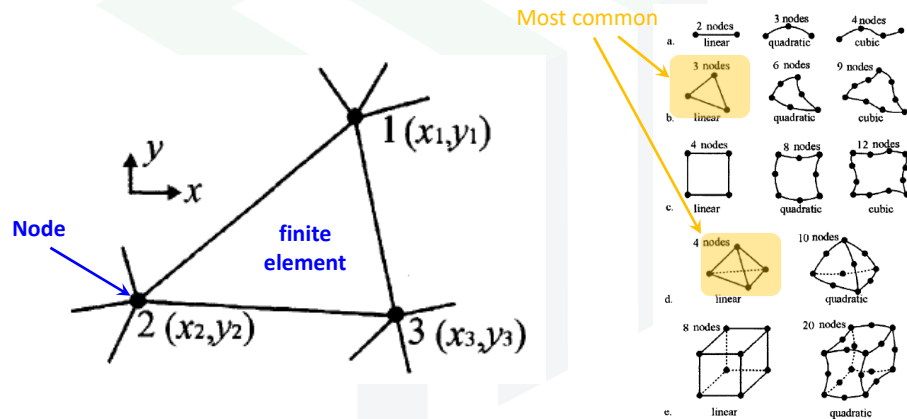
In fact, the method converges to the exact solution for all $N \geq 3$.

$$f(x) = \sum_{n=1}^N a_n (x - x^{n+1}) = \frac{5x}{6} - \frac{x^2}{2} - \frac{x^4}{3} \leftarrow \text{Exact solution}$$

Finite Element Method

What is a Finite Element?

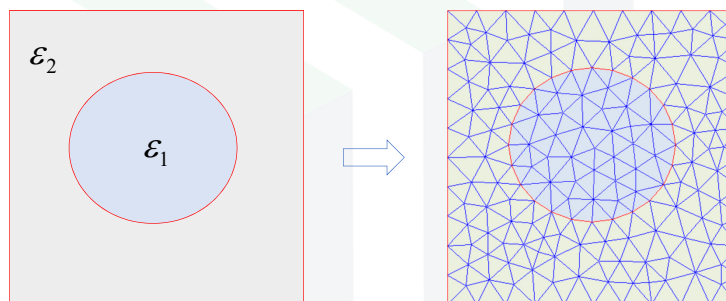
Like the finite-difference method, the finite element method (FEM) discretizes the problem space. In FEM, it is divided into small regions called finite elements.



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Meshing

Meshing describes the "intelligent" process of subdividing space into non-overlapping finite elements. Usually this is done to conform perfectly to the shapes of devices.

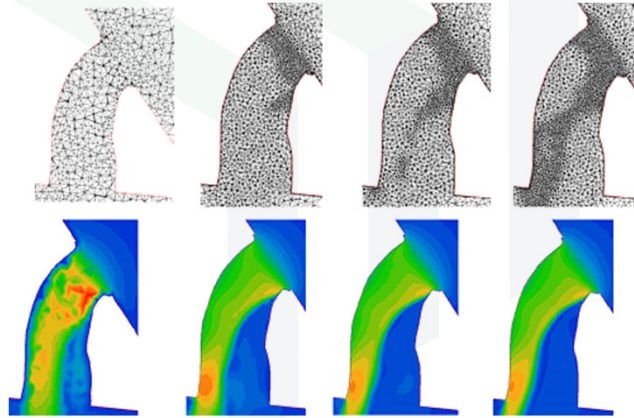


- FEM mesh conforms very well to curved geometries.
- Mesh can be locally refined to resolve small features or abruptly varying fields.
- Adaptive meshing – Refines the mesh based on a prior solution.

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Adaptive Meshing

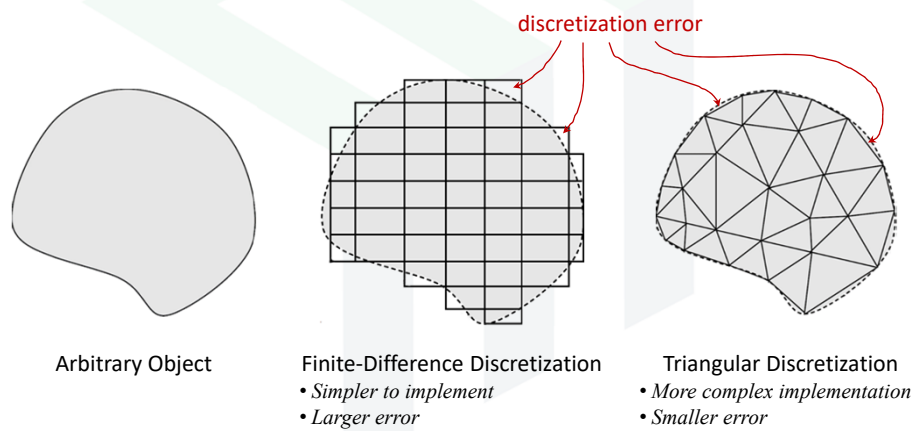
A model is performed using a coarse mesh. The mesh is then refined where the field is varying rapidly and the model is run again. This continues until convergence.



http://www.cradle-cfd.com/product/tetnav7/tetra_pre.html

FEM Vs. Finite-Difference Grids

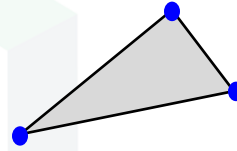
The finite element mesh offers minimal discretization error.



Node Elements Vs. Edge Elements

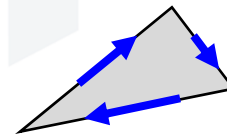
Node Elements

- Field is known and stored at the element nodes.
- Elsewhere, field is interpolated.
- Suffers from spurious solutions due to lack of enforcing divergence conditions.
- Matrices better conditioned.



Edge Elements

- Vector field is known at the edges of the elements.
- Elsewhere, field is interpolated.
- Solves the spurious solution problem because the divergence conditions are enforced through continuity of the tangential field components.
- Matrices have poorer conditioning.



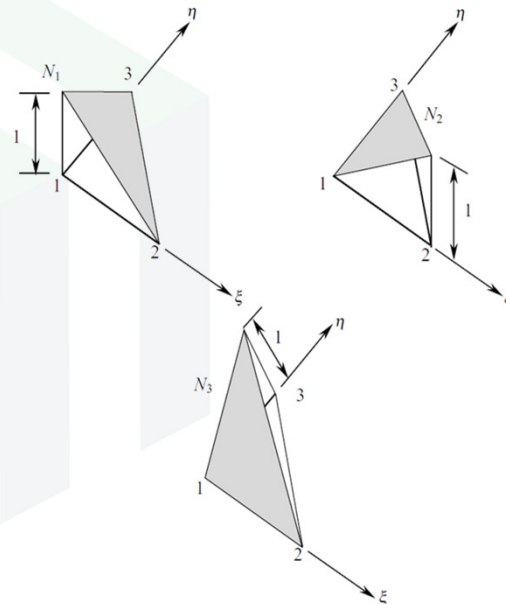
Shape Functions

The field within the elements is expanded into a set of basis functions that are called "shape" functions.

$$E(x, y) = \sum_{i=1}^3 u_i N_i(x, y)$$

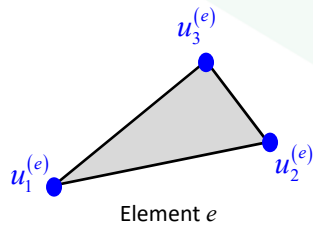
For linear triangular elements, the shape functions are

$$N_i(x, y) = a_i x + b_i y + c_i$$



Element Matrix \mathbf{K}

The element matrix is $N \times N$ where N is the number of nodes in an element. It is derived to enforce Maxwell's equations within an element.



Nodes are normally labeled going counter-clockwise.

$$\nabla \times \left(\frac{1}{\mu} \nabla \times \vec{E} \right) - \omega^2 \epsilon \vec{E} = 0$$

Galerkin Method

Variational Method

$$\underbrace{\begin{bmatrix} K_{11}^{(e)} & K_{12}^{(e)} & K_{13}^{(e)} \\ K_{21}^{(e)} & K_{22}^{(e)} & K_{23}^{(e)} \\ K_{31}^{(e)} & K_{32}^{(e)} & K_{33}^{(e)} \end{bmatrix}}_{\text{Element Matrix } \mathbf{K}^{(e)}} \begin{bmatrix} u_1^{(e)} \\ u_2^{(e)} \\ u_3^{(e)} \end{bmatrix} = \begin{bmatrix} b_1^{(e)} \\ b_2^{(e)} \\ b_3^{(e)} \end{bmatrix}$$