Advanced Electromagnetics:
21st Century Electromagnetics

Calculation Examples of Periodic Structures

1

Lecture Outline

- The square lattice
- The hexagonal lattice

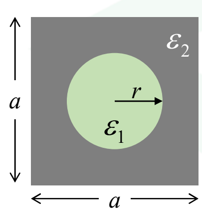
2

Square Lattice

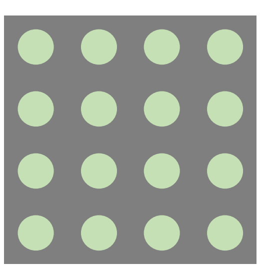
3

The Square Lattice

A unit cell of the lattice…



Extended lattice…



$$r = 0.35a$$

$$\varepsilon_1 = 1.0$$

$$\varepsilon_2 = 9.0$$

What is there to know about this lattice?

1. Direct lattice vectors.
2. Wigner-Seitz primitive unit cell.
3. Reciprocal lattice vectors.
4. Brillouin zone.
5. Irreducible Brillouin zone.
6. Key points of symmetry.
7. Electromagnetic band diagram.
8. Isofrequency contours.
9. Where does the lattice self-collimate?
10. Design a self-collimating lattice.
11. ???

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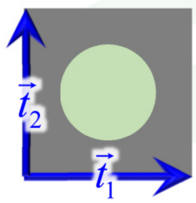
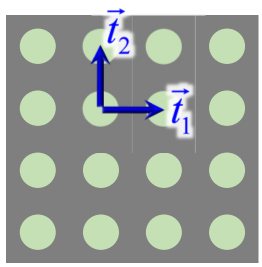
Direct Lattice Vectors





Our direct lattice vectors are…

$$\vec{t}_1 = a\hat{x} = \begin{bmatrix} a \\ 0 \end{bmatrix}$$

$$\vec{t}_2 = a\hat{y} = \begin{bmatrix} 0 \\ a \end{bmatrix}$$

EMPossible 5

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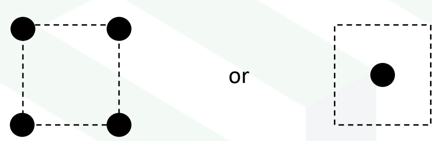
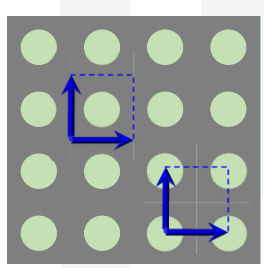
Wigner-Seitz Primitive Unit Cell

We think of the unit cell in terms of just its symmetry, not the pattern.





EMPossible 6

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Reciprocal Lattice Vectors

The direct lattice vectors are…

$$\vec{t}_1 = a\hat{x} = \begin{bmatrix} a \\ 0 \end{bmatrix} \qquad \vec{t}_2 = a\hat{y} = \begin{bmatrix} 0 \\ a \end{bmatrix}$$

The reciprocal lattice vectors in 2D are calculated according to

$$\vec{T}_1 = \frac{2\pi}{\left|\vec{t}_1 \times \vec{t}_2\right|}\begin{bmatrix} t_{2,y} \\ -t_{2,x} \end{bmatrix} \qquad \vec{T}_2 = \frac{2\pi}{\left|\vec{t}_1 \times \vec{t}_2\right|}\begin{bmatrix} -t_{1,y} \\ t_{1,x} \end{bmatrix}$$

The reciprocal lattice vectors are calculated to be

$$\vec{T}_1 = \frac{2\pi}{a}\hat{x} \qquad \vec{T}_2 = \frac{2\pi}{a}\hat{y}$$

EMPossible 7

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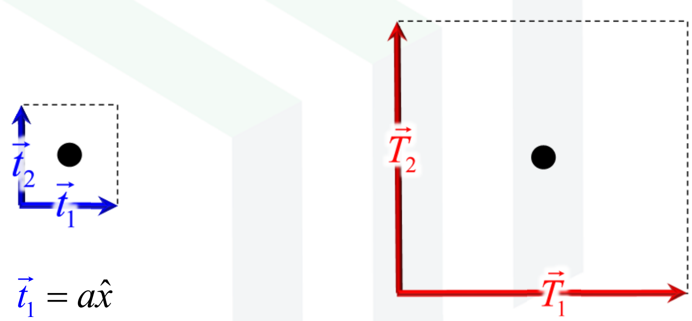
Direct Vs. Reciprocal Lattices

Direct Lattice

Reciprocal Lattice



$$\vec{t}_1 = a\hat{x}$$
$$\vec{t}_2 = a\hat{y}$$

$$\vec{T}_1 = \frac{2\pi}{a}\hat{x} \qquad \vec{T}_2 = \frac{2\pi}{a}\hat{y}$$

EMPossible 8

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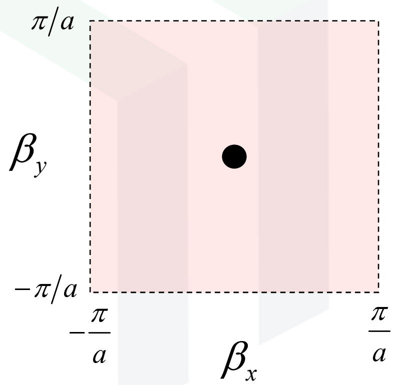
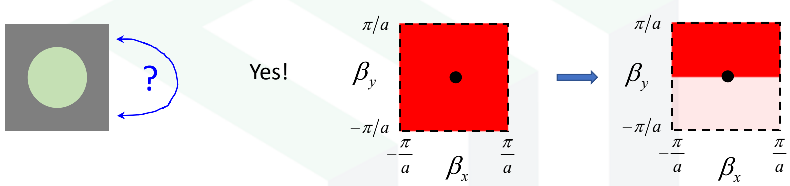
Brillouin Zone

The Brillouin zone is simply the primitive unit cell in reciprocal space. Typically, the unit cell is centered around zero.

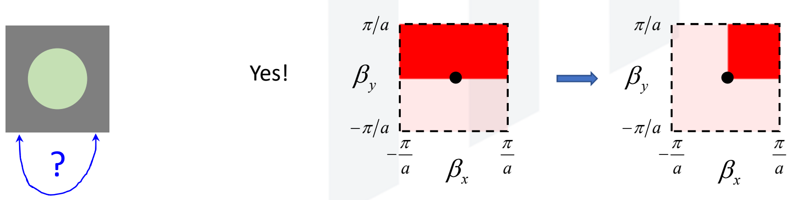


Irreducible Brillouin Zone (1 of 2)

Does the unit cell have up/down reflection symmetry?

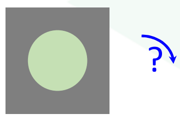


Does the unit cell have left/right reflection symmetry?

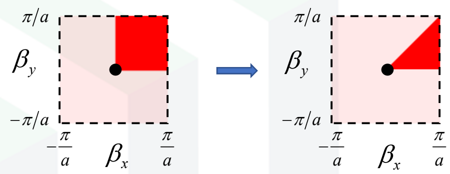


Irreducible Brillouin Zone (2 of 2)

Does the unit cell have 90° rotational symmetry?

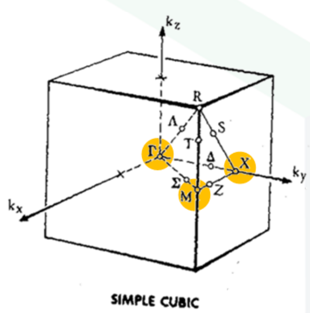


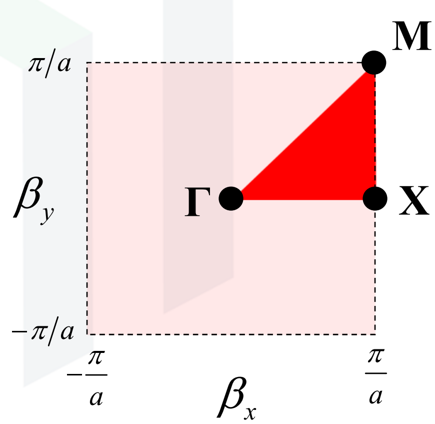
Yes!



Identify the Key Points of Symmetry

Refer to the Brillouin zone table.





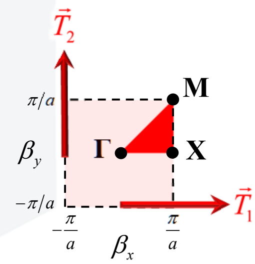
Calculate the Key Points of Symmetry

The key points of symmetry are calculated from the reciprocal lattice vectors as follows:

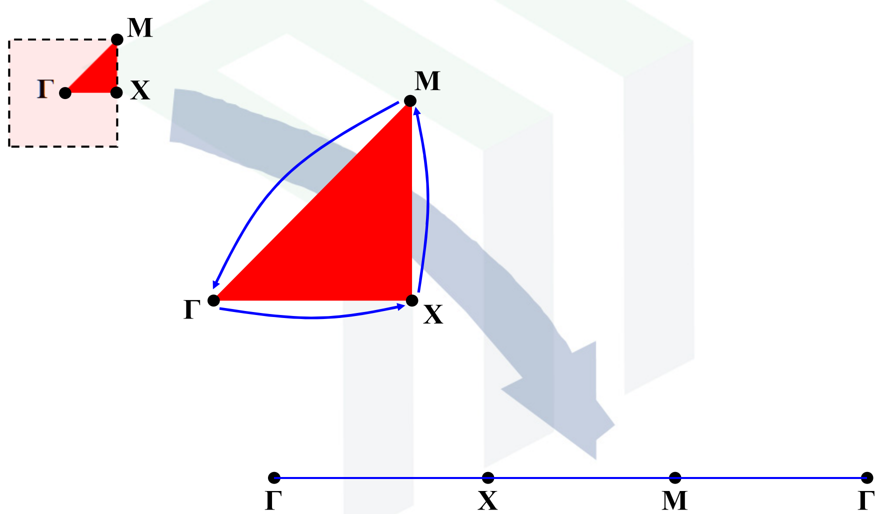
$$\mathbf{\Gamma} = \mathbf{0} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\mathbf{X} = \tfrac{1}{2}\vec{T}_1 = \begin{bmatrix} \pi/a \\ 0 \end{bmatrix}$$

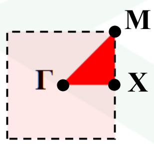
$$\mathbf{M} = \tfrac{1}{2}\vec{T}_1 + \tfrac{1}{2}\vec{T}_2 = \begin{bmatrix} \pi/a \\ \pi/a \end{bmatrix}$$



Choose Path Around IBZ



Calculate List of Bloch Wave Vectors



Notice the scale of the lengths along the side of the IBZ are conveyed in the list.

$$\beta = \begin{bmatrix} 0.00 & 0.52 & 1.05 & 1.57 & 2.09 & 2.62 & 3.14 & 3.14 & 3.14 & 3.14 & 3.14 & 3.14 & 3.14 & 2.79 & 2.44 & 2.09 & 1.75 & 1.40 & 1.05 & 0.70 & 0.35 & 0.00 \\ 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.52 & 1.05 & 1.57 & 2.09 & 2.62 & 3.14 & 2.79 & 2.44 & 2.09 & 1.75 & 1.40 & 1.05 & 0.70 & 0.35 & 0.00 \end{bmatrix}$$

EMPossible 15

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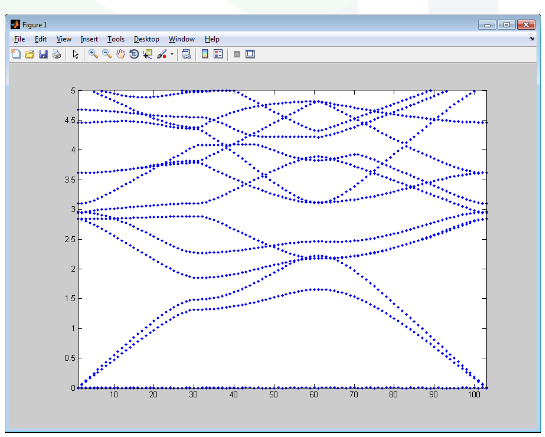
Constructing the Electromagnetic Band Diagram

The electromagnetic band diagram is constructed by plotting the eigen-values along the vertical axis as the Bloch wave vector is varied across the horizontal axis.



EMPossible 16

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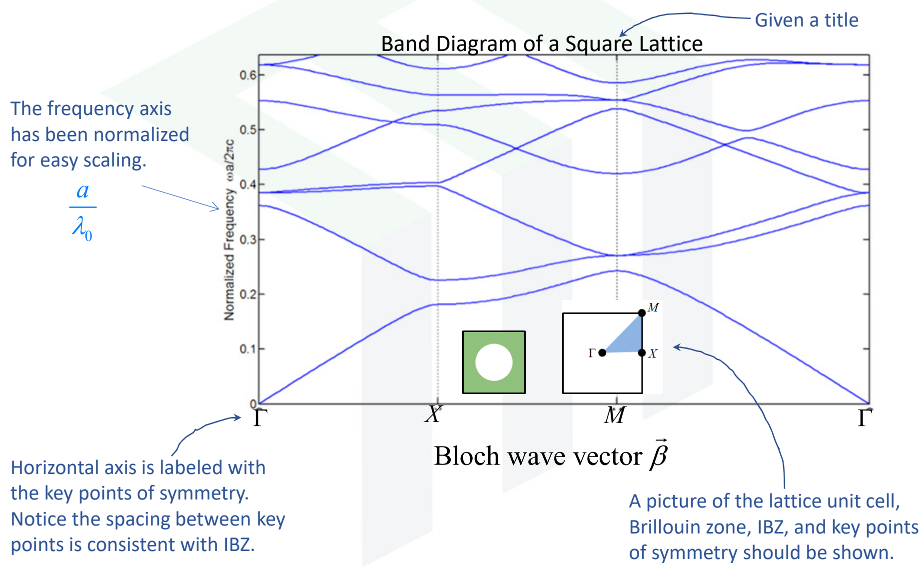
Generate a Professional Looking Band Diagram



EMPossible

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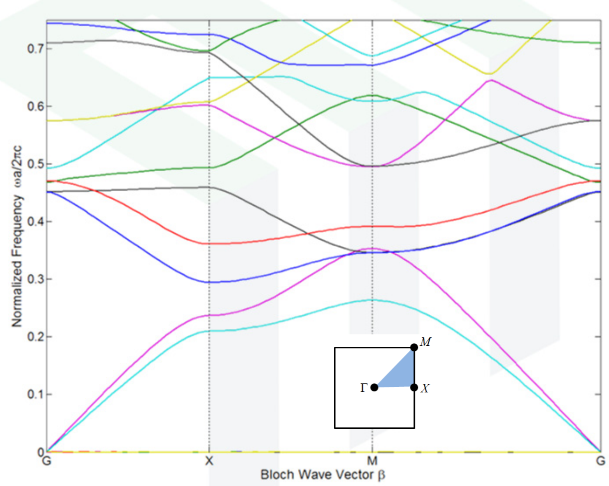
The Final Band Diagram



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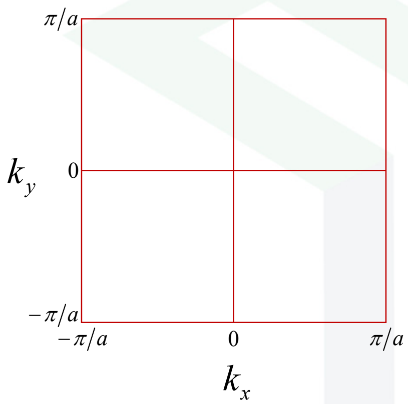
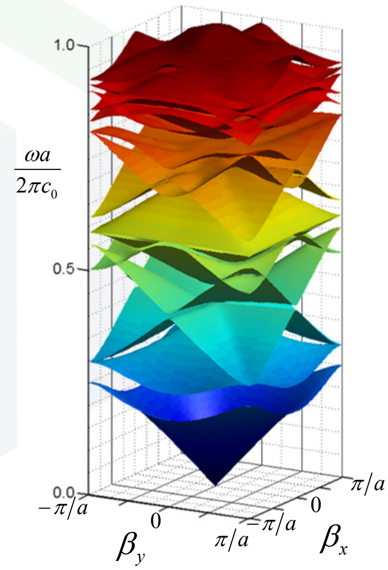
The Complete Band Diagram

The Full Brillouin Zone





There is an infinite set of eigen-frequencies associated with each point in the Brillouin zone. These form "sheets" as shown at right.

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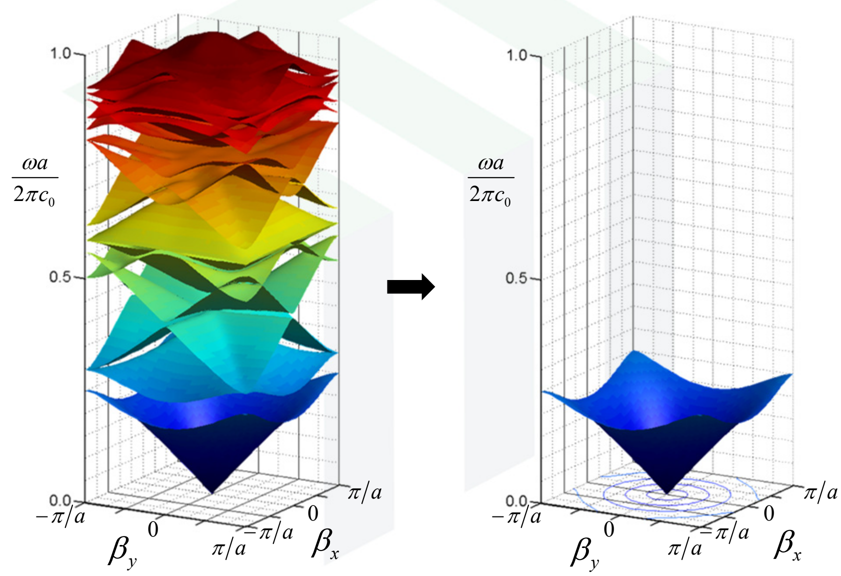
Index Ellipsoids From First-Order Band



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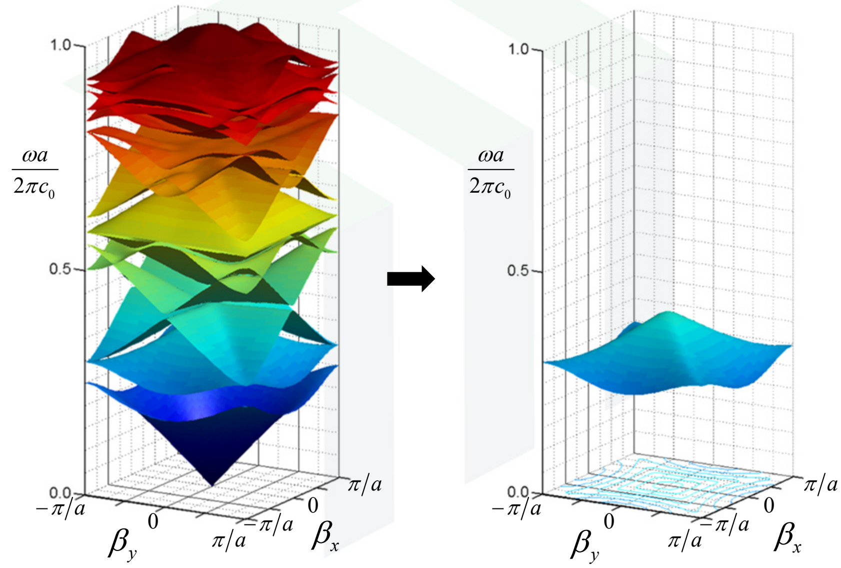
Index Ellipsoids From Second-Order Band



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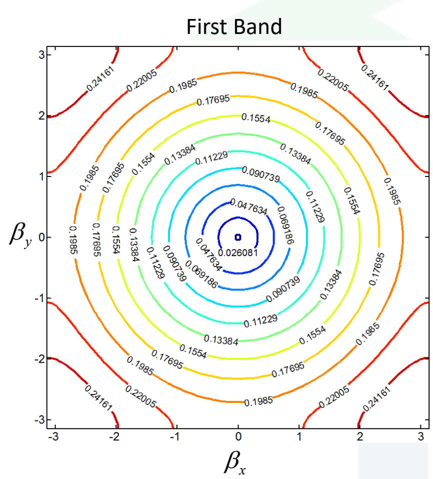
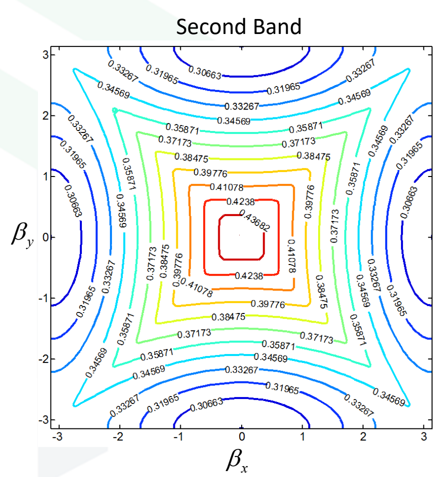
Isofrequency Contours





Self-collimation occurs at

$$\frac{\omega a}{2\pi c_0} \approx 0.4238$$

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Design Self-Collimating Lattice

What is the desired frequency of operation?

Let's say 10 GHz

The free space wavelength is then…

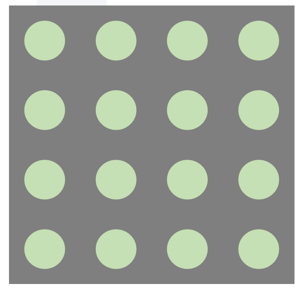
$$\lambda_0 = \frac{c_0}{f} = \frac{3\times10^8\ \frac{m}{s}}{10\times10^9\ \text{Hz}} = 3.0\ \text{cm}$$

The lattice constant should be…

$$\frac{\omega a}{2\pi c_0} = \frac{a}{\lambda_0} = 0.4238$$

$$a = 0.4238\lambda_0 = 0.4238(3.0\ \text{cm}) = 1.27\ \text{cm}$$

$$a = 1.27\ \text{cm}$$
$$r = 0.35a$$
$$\varepsilon_1 = 1.0$$
$$\varepsilon_2 = 9.0$$



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Hexagonal Lattice

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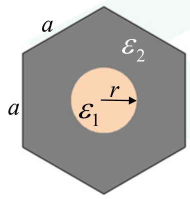
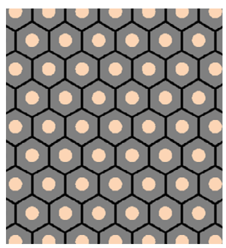
The Hexagonal Lattice

A unit cell of the lattice…



Extended lattice…



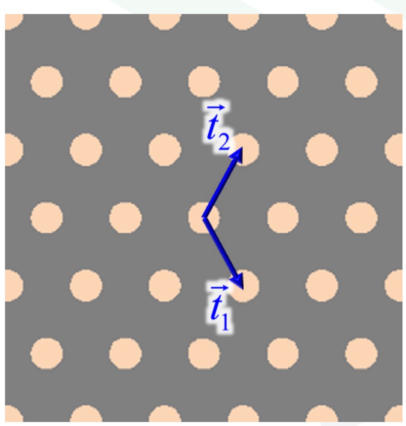
$$r = 0.35a$$
$$\varepsilon_1 = 1.0$$
$$\varepsilon_2 = 9.0$$

What is there to know about this lattice?

1. Direct lattice vectors.
2. Wigner-Seitz primitive unit cell.
3. Reciprocal lattice vectors.
4. Brillouin zone.
5. Irreducible Brillouin zone.
6. Key points of symmetry.
7. Electromagnetic band diagram.
8. Isofrequency contours.
9. ???

Direct Lattice Vectors

The primitive translation vectors (direct lattice vectors) are vectors that connect adjacent sites in the lattice. All sites in the lattice should be “reachable” through an integer combination of the lattice vectors.

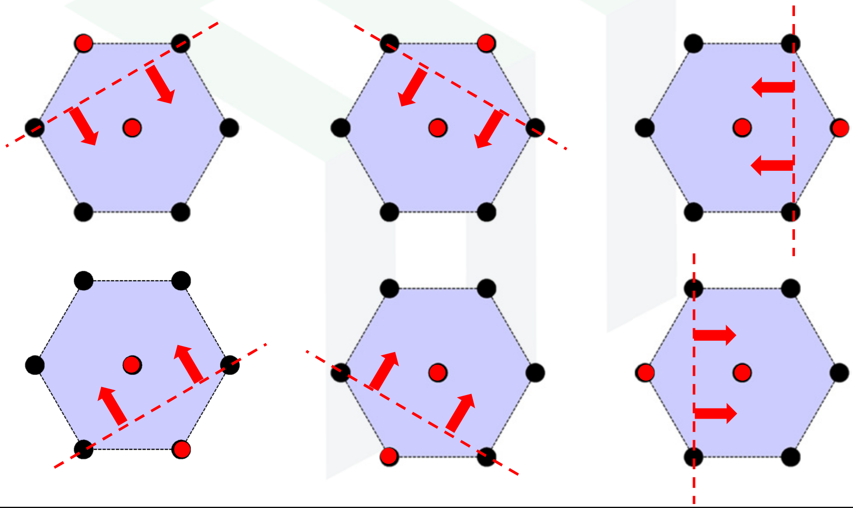


$$\vec{t}_1 = \frac{a}{2}\hat{x} - \frac{a\sqrt{3}}{2}\hat{y} = \begin{bmatrix} a/2 \\ -a\sqrt{3}/2 \end{bmatrix}$$

$$\vec{t}_2 = \frac{a}{2}\hat{x} + \frac{a\sqrt{3}}{2}\hat{y} = \begin{bmatrix} a/2 \\ a\sqrt{3}/2 \end{bmatrix}$$

Wigner-Seitz Primitive Unit Cell (1 of 2)

The Wigner-Seitz unit cell is that volume of space surrounding a point in the lattice that is closer to that point than any other point.



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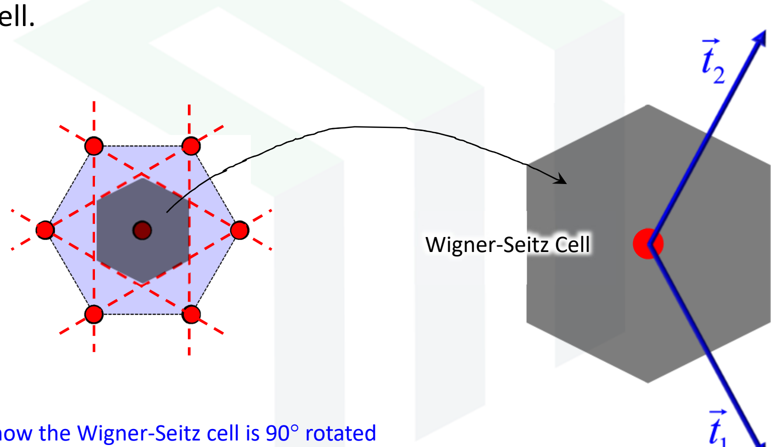
Wigner-Seitz Primitive Unit Cell (2 of 2)

Putting all of these together, we construct the Wigner-Seitz primitive unit cell.



Note how the Wigner-Seitz cell is 90° rotated from the "conventional" unit cell.

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Reciprocal Lattice Vectors

The direct lattice vectors were…

$$\vec{t}_1 = \frac{a}{2}\hat{x} - \frac{a\sqrt{3}}{2}\hat{y} \qquad \vec{t}_2 = \frac{a}{2}\hat{x} + \frac{a\sqrt{3}}{2}\hat{y}$$

The reciprocal lattice vectors in 2D are calculated according to

$$\vec{T}_1 = \frac{2\pi}{\left|\vec{t}_1 \times \vec{t}_2\right|}\begin{bmatrix} t_{2,y} \\ -t_{2,x} \end{bmatrix} \qquad \vec{T}_2 = \frac{2\pi}{\left|\vec{t}_1 \times \vec{t}_2\right|}\begin{bmatrix} -t_{1,y} \\ t_{1,x} \end{bmatrix}$$

The reciprocal lattice vectors are calculated to be

$$\vec{T}_1 = \frac{2\pi}{a}\hat{x} - \frac{2\pi}{a\sqrt{3}}\hat{y} \qquad \vec{T}_2 = \frac{2\pi}{a}\hat{x} + \frac{2\pi}{a\sqrt{3}}\hat{y}$$

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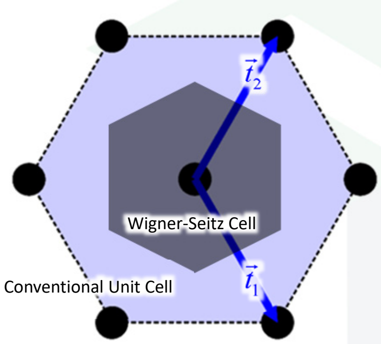
Direct Vs. Reciprocal Lattices

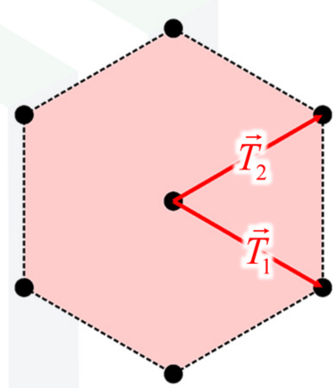
Direct Lattice



$$\vec{t}_1 = \frac{a}{2}\hat{x} - \frac{a\sqrt{3}}{2}\hat{y}$$

$$\vec{t}_2 = \frac{a}{2}\hat{x} + \frac{a\sqrt{3}}{2}\hat{y}$$

Reciprocal Lattice



$$\vec{T}_1 = \frac{2\pi}{a}\hat{x} - \frac{2\pi}{a\sqrt{3}}\hat{y} \qquad \vec{T}_2 = \frac{2\pi}{a}\hat{x} + \frac{2\pi}{a\sqrt{3}}\hat{y}$$

EMPossible 30

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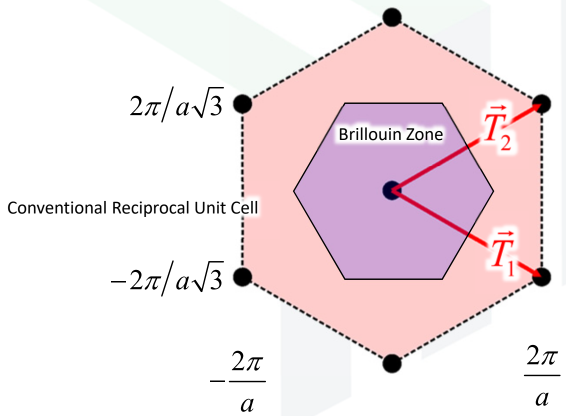
Brillouin Zone

The Brillouin zone is simply the “Wigner-Seitz” cell constructed in reciprocal space. Typically, the unit cell is centered around zero.



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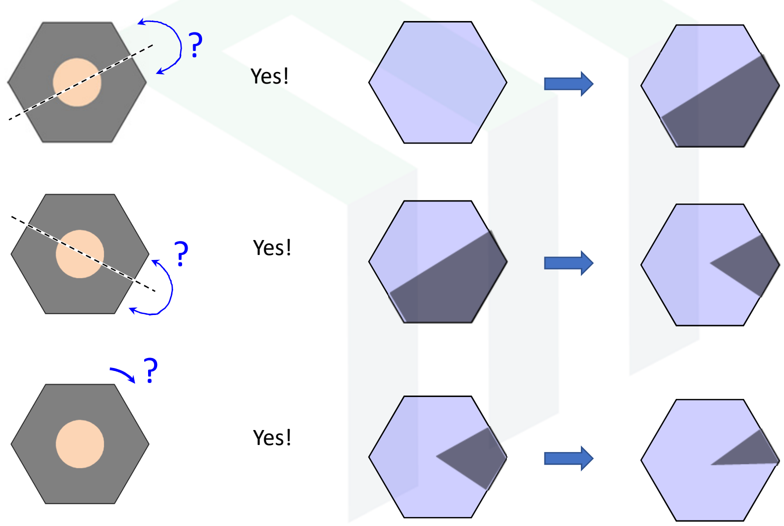
Irreducible Brillouin Zone

What additional symmetry does this lattice have?



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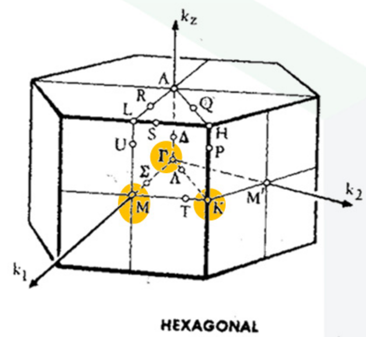
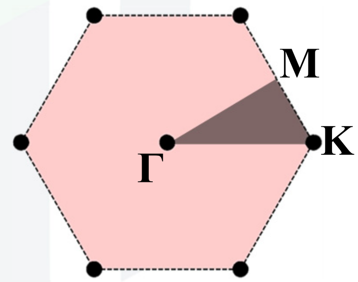
Identify the Key Points of Symmetry

Refer to the Brillouin zone table.





EMPossible

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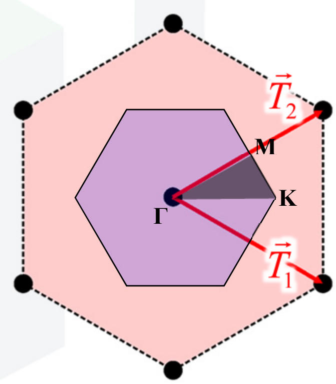
Calculate the Key Points of Symmetry

The key points of symmetry are calculated from the reciprocal lattice vectors as follows:

$$\mathbf{\Gamma} = \mathbf{0}$$

$$\mathbf{M} = \tfrac{1}{2}\vec{T}_2$$

$$\mathbf{K} = \tfrac{1}{3}\vec{T}_1 + \tfrac{1}{3}\vec{T}_2$$



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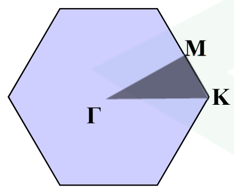
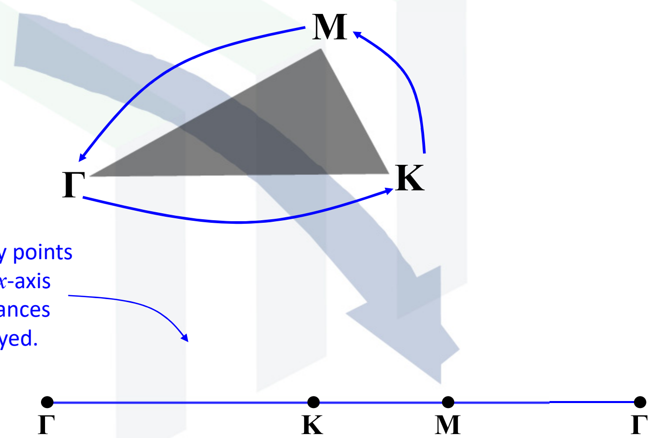
Choose Path Around IBZ





Be sure to space the key points of symmetry along the x-axis so that the relative distances between them is conveyed.

EMPossible

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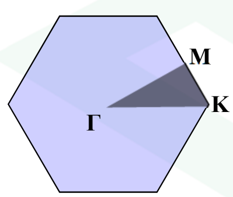
Calculate List of Bloch Wave Vectors



Notice the scale of the lengths along the side of the IBZ are conveyed in the list.

$$\beta = \begin{bmatrix} 0 & 0.60 & 1.20 & 1.80 & 2.39 & 2.99 & 3.59 & 4.19 & 3.93 & 3.67 & 3.40 & 3.14 & 2.62 & 2.09 & 1.57 & 1.05 & 0.52 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.45 & 0.91 & 1.36 & 1.81 & 1.51 & 1.21 & 0.91 & 0.60 & 0.30 & 0 \end{bmatrix}$$

EMPossible

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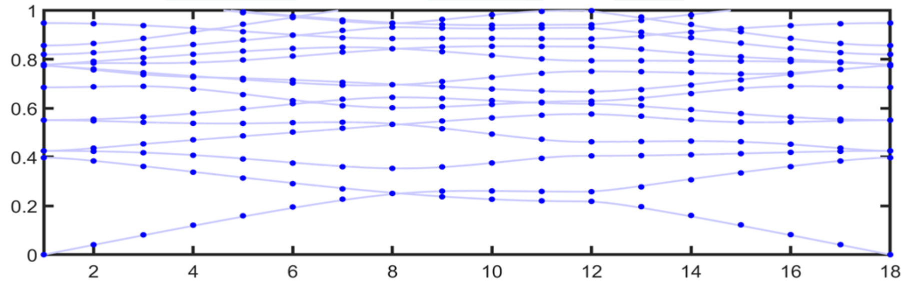
Constructing the Electromagnetic Band Diagram

The electromagnetic band diagram is constructed by plotting the eigen-values along the vertical axis as the Bloch wave vector is varied across the horizontal axis.



$$\beta = \begin{bmatrix} 0 & 0.60 & 1.20 & 1.80 & 2.39 & 2.99 & 3.59 & 4.19 & 3.93 & 3.67 & 3.40 & 3.14 & 2.62 & 2.09 & 1.57 & 1.05 & 0.52 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.45 & 0.91 & 1.36 & 1.81 & 1.51 & 1.21 & 0.91 & 0.60 & 0.30 & 0 \end{bmatrix}$$

EMPossible 37

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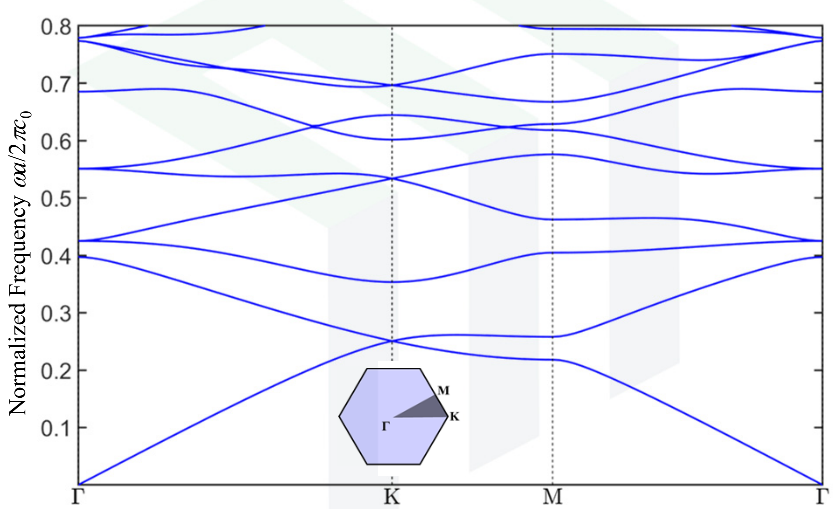
The Final Band Diagram



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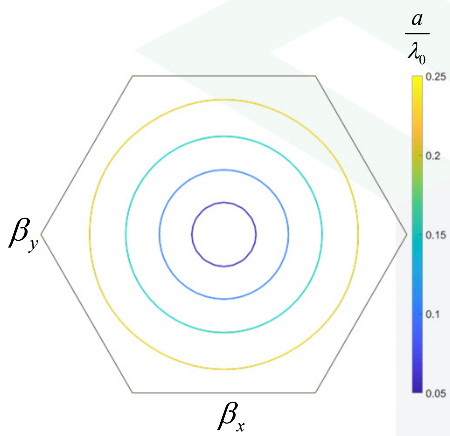
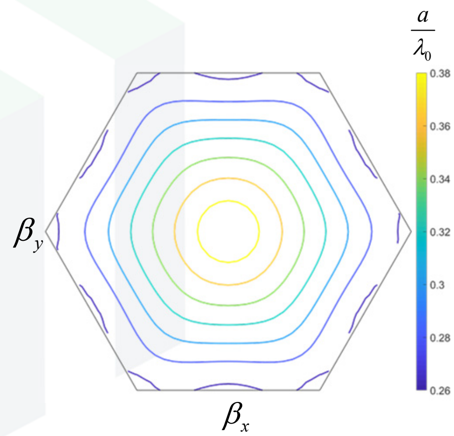
Isofrequency Contours





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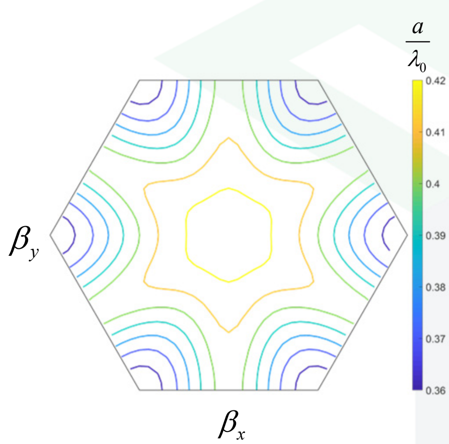
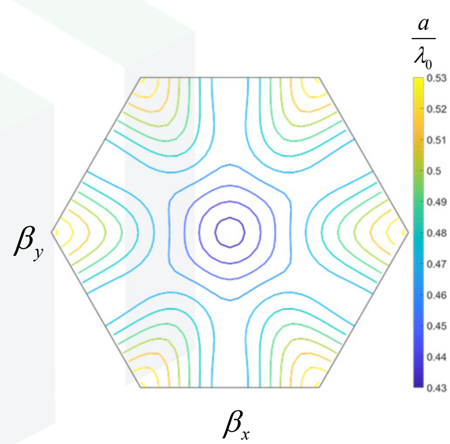
Isofrequency Contours





EMPossible

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