



Electromagnetics:  
Electromagnetic Field Theory

# Constitutive Relations

1

## Outline

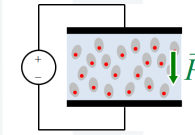
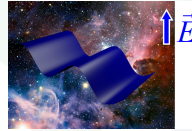
- Electric Response of Materials
- Magnetic Response of Materials
- Classification of Materials

2

## Electric Response of Materials

$$\vec{D} = \underbrace{\epsilon_0 \vec{E}}_{\text{Vacuum response}} + \underbrace{\vec{P}}_{\text{Material response}}$$

Note:  $\epsilon_0$  is the free space permittivity and multiplies  $E$  so that  $\epsilon_0 E$  has the same units as  $P$ .



Electric Susceptibility,  $\chi_e$

The electric susceptibility  $\chi_e$  is a measure of how easily bound charges are displaced due to an applied electric field.

$$\vec{P} = \epsilon_0 \chi_e \vec{E}$$

3

## Electric Polarization

In general, the relation between the applied electric field  $\vec{E}$  and the electric polarization  $\vec{P}$  is nonlinear so it can be expressed as a polynomial.

$$P = \underbrace{\epsilon_0 \chi_e^{(1)} E}_{\text{Linear response}} + \underbrace{\epsilon_0 \chi_e^{(2)} E^2 + \epsilon_0 \chi_e^{(3)} E^3 + \dots}_{\text{Nonlinear response}}$$

These terms are usually ignored. They tend to only become significant when the electric field is very strong.

$\chi_e^{(2)}$  is pronounced "chi two"

$\chi_e^{(3)}$  is pronounced "chi three"

⋮

$\chi_e^{(n)} \equiv$  electric susceptibility

$\chi_e^{(1)}$  no units

$\chi_e^{(2)}$  m/V

$\chi_e^{(3)}$  m<sup>2</sup>/V<sup>2</sup>

⋮

$\chi_e^{(n)}$  m<sup>n-1</sup>/V<sup>n-1</sup>

4

## Relation Between Permittivity & Susceptibility

The permittivity is related to the electric susceptibility through

$$\vec{D} = \epsilon_0 \vec{E} + \vec{P} = \epsilon_0 \vec{E} + \epsilon_0 \chi_e^{(1)} \vec{E} = \epsilon_0 (1 + \chi_e^{(1)}) \vec{E}$$

The constitutive relation can also be written in terms of the relative permittivity  $\epsilon_r$ .

$$\vec{D} = \epsilon_0 \epsilon_r \vec{E}$$

$$\epsilon_r = 1 + \chi_e^{(1)}$$

Vacuum response

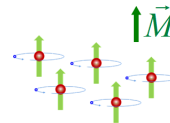
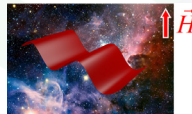
Dielectric response

5

## Magnetic Response of Materials

$$\vec{B} = \underbrace{\mu_0 \vec{H}}_{\text{Vacuum response}} + \underbrace{\vec{M}}_{\text{Material response}}$$

Note:  $\mu_0$  is the free space permeability and multiples  $\vec{H}$  so that  $\mu_0 \vec{H}$  has the same units as  $\vec{M}$ .



Magnetic Susceptibility,  $\chi_m$

The magnetic susceptibility  $\chi_m$  is a measure of how easily magnetic dipoles are aligned due to an applied magnetic field.

$$\vec{M} = \mu_0 \chi_m \vec{H}$$

6

# Magnetic Polarization

In general, the relation between the applied magnetic field  $\vec{H}$  and the magnetic polarization  $\vec{M}$  is nonlinear so it can be expressed as a polynomial.

$$M = \underbrace{\mu_0 \chi_m^{(1)} H}_{\text{Linear response}} + \underbrace{\mu_0 \chi_m^{(2)} H^2 + \mu_0 \chi_m^{(3)} H^3 + \dots}_{\text{Nonlinear response}}$$

Linear  
response

Nonlinear response

These terms are usually ignored. They tend to only become significant when the magnetic field is very strong.

$\chi_m^{(2)}$  is pronounced "chi two"

$\chi_m^{(3)}$  is pronounced "chi three"

⋮

$\chi_m^{(n)} \equiv$  magnetic susceptibility

$\chi_m^{(1)}$  no units

$\chi_m^{(2)}$  m/A

$\chi_m^{(3)}$  m<sup>2</sup>/A<sup>2</sup>

⋮

$\chi_m^{(n)}$  m<sup>n-1</sup>/A<sup>n-1</sup>

7

# Relation Between Permeability & Susceptibility

The permeability is related to the magnetic susceptibility through

$$\vec{B} = \mu_0 \mu_r \vec{H} = \mu_0 \vec{H} + \mu_0 \chi_m^{(1)} \vec{H} = \mu_0 (1 + \chi_m^{(1)}) \vec{H}$$

$$\mu_r = 1 + \chi_m^{(1)}$$

Vacuum response

Magnetic response

8

# Types of Magnetic Materials

- Diamagnetic
  - Negative magnetic susceptibility ( $\chi_m < 0$ )
  - Tends to oppose an applied magnetic field.
  - All materials are diamagnetic, but usually very weak.
  - Copper, silver, gold
- Paramagnetic
  - Small positive susceptibility ( $\chi_m > 0$  but small)
  - Material is magnetizable and is attracted to an applied magnetic field.
  - Does not retain magnetization when the external field is removed.
- Ferromagnetic
  - Large positive susceptibility
  - Like paramagnetic, but they retain their magnetism to some degree when the external field is removed.
  - Iron, nickel, cobalt, and some alloys.

9

# Anisotropy

The dielectric response of a material arises due to the electric field displacing charges.

Due to structural and bonding effects at the atomic scale, charges are often more easily displaced in some directions than others.

This gives rise to anisotropy where the electric field may experience an entirely different permittivity depending on what direction it is oriented.

$$\vec{D} = [\boldsymbol{\varepsilon}] \vec{E}$$

↑  
tensor

$$\begin{bmatrix} D_x \\ D_y \\ D_z \end{bmatrix} = \underbrace{\begin{bmatrix} \varepsilon_{xx} & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{yx} & \varepsilon_{yy} & \varepsilon_{yz} \\ \varepsilon_{zx} & \varepsilon_{zy} & \varepsilon_{zz} \end{bmatrix}}_{[\boldsymbol{\varepsilon}]} \begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix}$$

$\varepsilon_{ij}$  = how much of  $E_j$  contributes to  $D_i$

10

## Types of Anisotropy (1 of 2)

### Isotropic Media

$$\begin{bmatrix} D_a \\ D_b \\ D_c \end{bmatrix} = \begin{bmatrix} \epsilon & 0 & 0 \\ 0 & \epsilon & 0 \\ 0 & 0 & \epsilon \end{bmatrix} \begin{bmatrix} E_a \\ E_b \\ E_c \end{bmatrix}$$

This is the typical approximation made in electromagnetics. The permittivity tensor reduces to a scalar.

$$\vec{D} = \epsilon \vec{E}$$

### Uniaxial Media

$$\begin{bmatrix} D_a \\ D_b \\ D_c \end{bmatrix} = \begin{bmatrix} \epsilon_o & 0 & 0 \\ 0 & \epsilon_o & 0 \\ 0 & 0 & \epsilon_e \end{bmatrix} \begin{bmatrix} E_a \\ E_b \\ E_c \end{bmatrix}$$

Anisotropic materials are said to be birefringent.

$$\Delta\epsilon = \epsilon_e - \epsilon_o$$

Positive birefringence:  $\Delta\epsilon > 0$

Negative birefringence:  $\Delta\epsilon < 0$

$\epsilon_o \equiv$  ordinary permittivity

$\epsilon_e \equiv$  extraordinary permittivity

### Biaxial Media

$$\begin{bmatrix} D_a \\ D_b \\ D_c \end{bmatrix} = \begin{bmatrix} \epsilon_a & 0 & 0 \\ 0 & \epsilon_b & 0 \\ 0 & 0 & \epsilon_c \end{bmatrix} \begin{bmatrix} E_a \\ E_b \\ E_c \end{bmatrix}$$

When orientation is not important, it is a convention to order the tensor elements according to

$$\epsilon_a < \epsilon_b < \epsilon_c$$

## Types of Anisotropy (2 of 2)

### Doubly Anisotropic

$$\vec{D} = [\epsilon] \vec{E} \quad \text{and} \quad \vec{B} = [\mu] \vec{H}$$

### Chiral Materials

$$\begin{bmatrix} D_a \\ D_b \\ D_c \end{bmatrix} = \begin{bmatrix} \epsilon_a & -j\epsilon_b & 0 \\ j\epsilon_b & \epsilon_a & 0 \\ 0 & 0 & \epsilon_c \end{bmatrix} \begin{bmatrix} E_a \\ E_b \\ E_c \end{bmatrix}$$

Gyroelectric

$$\begin{bmatrix} B_a \\ B_b \\ B_c \end{bmatrix} = \begin{bmatrix} \mu_a & -j\mu_b & 0 \\ j\mu_b & \mu_a & 0 \\ 0 & 0 & \mu_c \end{bmatrix} \begin{bmatrix} H_a \\ H_b \\ H_c \end{bmatrix}$$

Gryomagnetic

# Ordinary and Bi- Materials

	Ordinary Materials	Bi- Materials
Isotropic Materials	<p><b>Isotropic Materials</b></p> $\vec{D} = \epsilon \vec{E}$ $\vec{B} = \mu \vec{H}$	<p><b>Bi-Isotropic Materials</b></p> $\vec{D} = \epsilon \vec{E} + \xi \vec{H}$ $\vec{B} = \xi \vec{E} + \mu \vec{H}$ <p><math>\xi \equiv</math> magnetoelectric coupling coefficient</p>
Anisotropic Materials	<p><b>Anisotropic Materials</b></p> $\vec{D} = [\epsilon] \vec{E}$ $\vec{B} = [\mu] \vec{H}$	<p><b>Bi-Anisotropic Materials</b></p> $\vec{D} = [\epsilon] \vec{E} + [\xi] \vec{H}$ $\vec{B} = [\xi]^T \vec{E} + [\mu] \vec{H}$