Electromagnetics:

Electromagnetic Field Theory

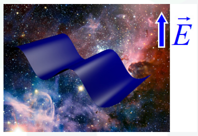
Constitutive Relations

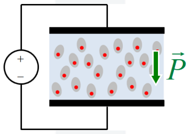
Outline

- Electric Response of Materials
- Magnetic Response of Materials
- Classification of Materials

Electric Response of Materials

$$\vec{D} = \underbrace{\varepsilon_0 \vec{E}}_{\text{Vacuum response}} + \underbrace{\vec{P}}_{\text{Material response}}$$





Note: ε_0 is the free space permittivity and multiples $\vec{E}$ so that $\varepsilon_0\vec{E}$ has the same units as $\vec{P}$.

Electric Susceptibility, χ_e

The electric susceptibility χ_e is a measure of how easily bound charges are displaced due to an applied electric field.

$$\vec{P} = \varepsilon_0 \chi_e \vec{E}$$

Electric Polarization $\vec{P}$

In general, the relation between the applied electric field $\vec{E}$ and the electric polarization $\vec{P}$ is nonlinear so it can be expressed as a polynomial.

$$P = \underbrace{\varepsilon_0 \chi_e^{(1)} E}_{\text{Linear response}} + \underbrace{\varepsilon_0 \chi_e^{(2)} E^2 + \varepsilon_0 \chi_e^{(3)} E^3 + \cdots}_{\text{Nonlinear response}}$$

These terms are usually ignored. They tend to only become significant when the electric field is very strong.

$\chi_e^{(2)}$ is pronounced "chi two"

$\chi_e^{(3)}$ is pronounced "chi three"

$\vdots$

$$\chi_e^{(n)} \equiv \text{electric susceptibility}$$

$\chi_e^{(1)}$ no units

$\chi_e^{(2)}$ $\mathrm{m/V}$

$\chi_e^{(3)}$ $\mathrm{m^2/V^2}$

$\vdots$

$\chi_e^{(n)}$ $\mathrm{m^{n-1}/V^{n-1}}$

Relation Between Permittivity & Susceptibility

The permittivity ε is related to the electric susceptibility χ_e through

$$\vec{D} = \varepsilon_0 \vec{E} + \vec{P} = \varepsilon_0 \vec{E} + \varepsilon_0 \chi_\text{e}^{(1)} \vec{E} = \varepsilon_0 \left(1 + \chi_\text{e}^{(1)} \right) \vec{E}$$

The constitutive relation can also be written in terms of the relative permittivity ε_r.

$$\vec{D} = \varepsilon_0 \varepsilon_\text{r} \vec{E}$$

$$\varepsilon_\text{r} = 1 + \chi_\text{e}^{(1)}$$

Vacuum response

Dielectric response

Magnetic Response of Materials

$$\vec{B} = \underbrace{\mu_0 \vec{H}}_{\text{Vacuum response}} + \underbrace{\vec{M}}_{\text{Material response}}$$

Note: μ_0 is the free space permeability and multiples $\vec{H}$ so that $\mu_0 \vec{H}$ has the same units as $\vec{M}$.

Magnetic Susceptibility, χ_m

The magnetic susceptibility χ_m is a measure of how easily magnetic dipoles are aligned due to an applied magnetic field.

$$\vec{M} = \mu_0 \chi_\text{m} \vec{H}$$

Magnetic Polarization $\vec{M}$

In general, the relation between the applied magnetic field $\vec{H}$ and the magnetic polarization $\vec{M}$ is nonlinear so it can be expressed as a polynomial.

$$M = \underbrace{\mu_0 \chi_{\mathrm{m}}^{(1)} H}_{\text{Linear response}} + \underbrace{\mu_0 \chi_{\mathrm{m}}^{(2)} H^2 + \mu_0 \chi_{\mathrm{m}}^{(3)} H^3 + \cdots}_{\text{Nonlinear response}}$$

These terms are usually ignored. They tend to only become significant when the magnetic field is very strong.

$\chi_{\mathrm{m}}^{(2)}$ is pronounced "chi two"

$\chi_{\mathrm{m}}^{(3)}$ is pronounced "chi three"

$\vdots$

$\chi_{\mathrm{m}}^{(n)} \equiv$ magnetic susceptibility

$\chi_{\mathrm{m}}^{(1)}$ no units

$\chi_{\mathrm{m}}^{(2)}$ $\mathrm{m/A}$

$\chi_{\mathrm{m}}^{(3)}$ $\mathrm{m^2/A^2}$

$\vdots$

$\chi_{\mathrm{m}}^{(n)}$ $\mathrm{m}^{n-1}/\mathrm{A}^{n-1}$

Relation Between Permeability & Susceptibility

The permeability μ is related to the magnetic susceptibility χ_{m} through

$$\vec{B} = \mu_0 \mu_{\mathrm{r}} \vec{H} = \mu_0 \vec{H} + \mu_0 \chi_{\mathrm{m}}^{(1)} \vec{H} = \mu_0 \left(1 + \chi_{\mathrm{m}}^{(1)}\right) \vec{H}$$

$$\mu_{\mathrm{r}} = \underset{\text{Vacuum response}}{1} + \underset{\text{Magnetic response}}{\chi_{\mathrm{m}}^{(1)}}$$

Types of Magnetic Materials

- Diamagnetic
  - Negative magnetic susceptibility ($\chi_\mathrm{m} < 0$)
  - Tends to oppose an applied magnetic field.
  - All materials are diamagnetic, but usually very week.
  - Copper, silver, gold
- Paramagnetic
  - Small positive susceptibility ($\chi_\mathrm{m} > 0$ but small)
  - Material is magnetizable and is attracted to an applied magnetic field.
  - Does not retain magnetization when the external field is removed.
- Ferromagnetic
  - Large positive susceptibility
  - Like paramagnetic, but they retain their magnetism to some degree when the external field is removed.
  - Iron, nickel, cobalt, and some aloys.

Anisotropy

The dielectric response of a material arises due to the electric field displacing charges.

Due to structural and bonding effects at the atomic scale, charges are often more easily displaced in some directions than others.

This gives rise to anisotropy where the electric field may experience an entirely different permittivity depending on what direction it is oriented.

$$\vec{D} = \underset{\uparrow\ \text{tensor}}{[\varepsilon]}\vec{E}$$

$$\begin{bmatrix} D_x \\ D_y \\ D_z \end{bmatrix} = \underbrace{\begin{bmatrix} \varepsilon_{xx} & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{yx} & \varepsilon_{yy} & \varepsilon_{yz} \\ \varepsilon_{zx} & \varepsilon_{zy} & \varepsilon_{zz} \end{bmatrix}}_{[\varepsilon]} \begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix}$$

ε_{ij} = how much of E_j contributes to D_i

Types of Anisotropy (1 of 2)

Isotropic Media

$$\begin{bmatrix} D_a \\ D_b \\ D_c \end{bmatrix} = \begin{bmatrix} \varepsilon & 0 & 0 \\ 0 & \varepsilon & 0 \\ 0 & 0 & \varepsilon \end{bmatrix} \begin{bmatrix} E_a \\ E_b \\ E_c \end{bmatrix}$$

This is the typical approximation made in electromagnetics. The permittivity tensor reduces to a scalar.

$$\vec{D} = \varepsilon \vec{E}$$

Uniaxial Media

$$\begin{bmatrix} D_a \\ D_b \\ D_c \end{bmatrix} = \begin{bmatrix} \varepsilon_{\text{o}} & 0 & 0 \\ 0 & \varepsilon_{\text{o}} & 0 \\ 0 & 0 & \varepsilon_{\text{e}} \end{bmatrix} \begin{bmatrix} E_a \\ E_b \\ E_c \end{bmatrix}$$

Anisotropic materials are said to be birefringent.

$$\Delta\varepsilon = \varepsilon_{\text{e}} - \varepsilon_{\text{o}}$$

Positive birefringence: $\Delta\varepsilon > 0$

Negative birefringence: $\Delta\varepsilon < 0$

$\varepsilon_{\text{o}} \equiv$ ordinary permittivity

$\varepsilon_{\text{e}} \equiv$ extraordinary permittivity

Biaxial Media

$$\begin{bmatrix} D_a \\ D_b \\ D_c \end{bmatrix} = \begin{bmatrix} \varepsilon_a & 0 & 0 \\ 0 & \varepsilon_b & 0 \\ 0 & 0 & \varepsilon_c \end{bmatrix} \begin{bmatrix} E_a \\ E_b \\ E_c \end{bmatrix}$$

When orientation is not important, it is convention to order the tensor elements according to

$$\varepsilon_a < \varepsilon_b < \varepsilon_c$$

EMPossible

Types of Anisotropy (2 of 2)

Doubly Anisotropic

$$\vec{D} = [\varepsilon]\vec{E} \quad \text{and} \quad \vec{B} = [\mu]\vec{H}$$

Chiral Materials

$$\begin{bmatrix} D_a \\ D_b \\ D_c \end{bmatrix} = \begin{bmatrix} \varepsilon_a & -j\varepsilon_b & 0 \\ j\varepsilon_b & \varepsilon_a & 0 \\ 0 & 0 & \varepsilon_c \end{bmatrix} \begin{bmatrix} E_a \\ E_b \\ E_c \end{bmatrix}$$

Gyroelectric

$$\begin{bmatrix} B_a \\ B_b \\ B_c \end{bmatrix} = \begin{bmatrix} \mu_a & -j\mu_b & 0 \\ j\mu_b & \mu_a & 0 \\ 0 & 0 & \mu_c \end{bmatrix} \begin{bmatrix} H_a \\ H_b \\ H_c \end{bmatrix}$$

Gryomagnetic

EMPossible

Ordinary and Bi- Materials

	Ordinary Materials	Bi- Materials
Isotropic Materials	**Isotropic Materials** $\vec{D} = \varepsilon \vec{E}$ $\vec{B} = \mu \vec{H}$	**Bi-Isotropic Materials** $\vec{D} = \varepsilon \vec{E} + \xi \vec{H}$ $\vec{B} = \xi \vec{E} + \mu \vec{H}$ $\xi \equiv$ magnetoelectric coupling coefficient
Anisotropic Materials	**Anisotropic Materials** $\vec{D} = [\varepsilon]\vec{E}$ $\vec{B} = [\mu]\vec{H}$	**Bi-Anisotropic Materials** $\vec{D} = [\varepsilon]\vec{E} + [\xi]\vec{H}$ $\vec{B} = [\xi]^{\mathrm{T}}\vec{E} + [\mu]\vec{H}$