



Electromagnetics:  
Electromagnetic Field Theory

# The Del Operator & Field Operations



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## Outline

- Scalar vs. Vector Fields
- The Del Operator  $\nabla$
- Gradient of a Scalar Field  $\nabla V$
- Divergence of a Vector Field  $\nabla \cdot \vec{A}$
- Curl of a Vector Field  $\nabla \times \vec{A}$
- Laplacian Operation  $\nabla^2$

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# Scalar vs. Vector Fields

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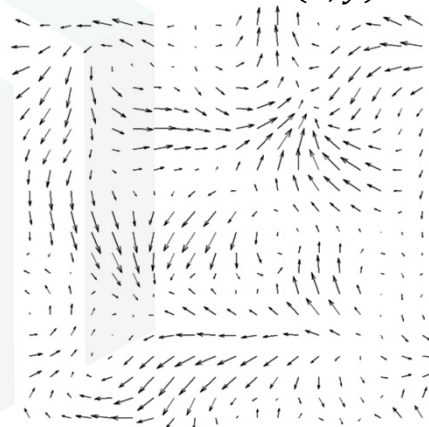
## Scalar Field Vs. Vector Field (2D)

Scalar Field  $f(x, y)$



- Magnitude only

Vector Field  $\vec{F}(x, y)$



- Magnitude
- Direction

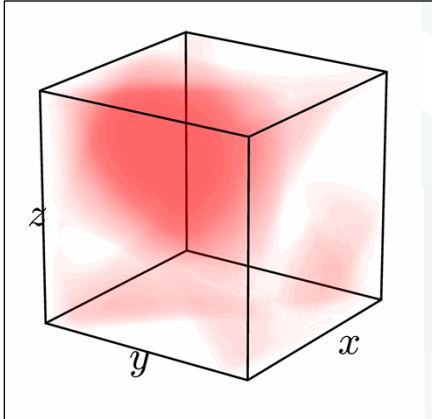
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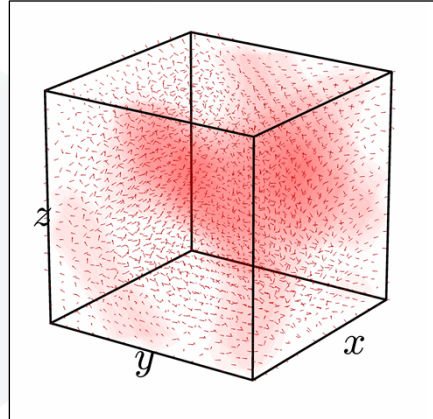
## Scalar Field Vs. Vector Field (3D)

Scalar Field  $f(x, y, z)$



- Magnitude only

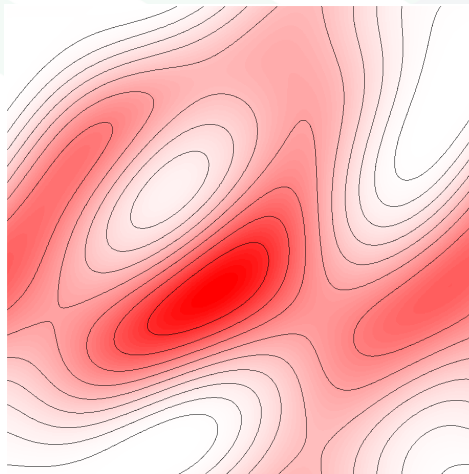
Vector Field  $\vec{F}(x, y, z)$



- Magnitude
- Direction

## Isocontour Lines

Isocontour lines trace the paths of equal value. Closely spaced isocontours convey that the function is varying rapidly.



# The Del Operator $\nabla$

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## The Del Operator $\nabla$

The del operator  $\nabla$  is the vector differential operator. It is sort of a 3D derivative.

Coordinate System	Del Operator
Cartesian	$\nabla = \frac{\partial}{\partial x} \hat{a}_x + \frac{\partial}{\partial y} \hat{a}_y + \frac{\partial}{\partial z} \hat{a}_z$
Cylindrical	$\nabla = \frac{\partial}{\partial \rho} \hat{a}_\rho + \frac{1}{\rho} \frac{\partial}{\partial \phi} \hat{a}_\phi + \frac{\partial}{\partial z} \hat{a}_z$
Spherical	$\nabla = \frac{\partial}{\partial r} \hat{a}_r + \frac{1}{r} \frac{\partial}{\partial \theta} \hat{a}_\theta + \frac{1}{r \sin \theta} \frac{\partial}{\partial \phi} \hat{a}_\phi$

Even though  $\nabla$  is a vector, it is never written as  $\vec{\nabla}$ .

Derived from the Cartesian equation using coordinate transformation.

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## Summary of Vector Operations

Operation	Input	Output
Vector Addition & Subtraction $\vec{U} + \vec{V}$	Vectors	Vector
Dot Product $\vec{U} \cdot \vec{V}$	Vectors	Scalar
Cross Product $\vec{U} \times \vec{V}$	Vectors	Vector
Gradient $\nabla f$	Scalar Function	Vector Function
Divergence $\nabla \cdot \vec{U}$	Vector Function	Scalar Function
Curl $\nabla \times \vec{U}$	Vector Function	Vector Function
Scalar Laplacian $\nabla^2 V$	Scalar Function	Scalar Function
Vector Laplacian $\nabla^2 \vec{U}$	Vector Function	Vector Function

## Gradient of a Scalar Field

## Gradient of a Scalar Field (1 of 3)

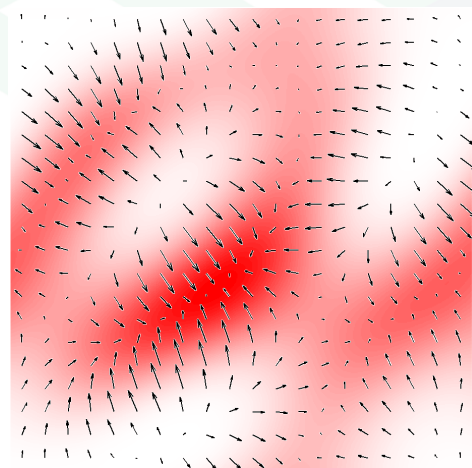
Start with a scalar field  $f(x, y)$ .



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## Gradient of a Scalar Field (2 of 3)

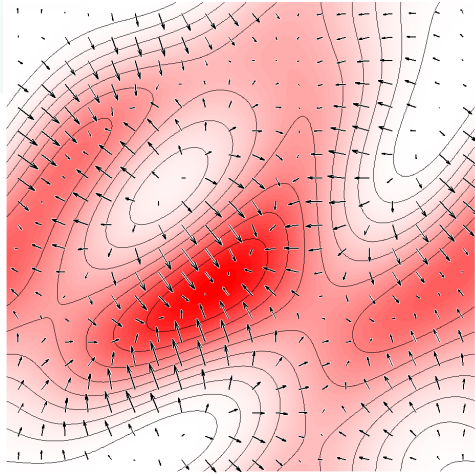
Plot the gradient  $\nabla f(x, y)$  on top of  $f(x, y)$ . The background color is the original scalar field  $f(x, y)$ .



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## Gradient of a Scalar Field (3 of 3)

The gradient will always be perpendicular to the isocontour lines.



## The Gradient $\nabla V$

The gradient calculates how rapidly, and in what direction, a scalar function is increasing.

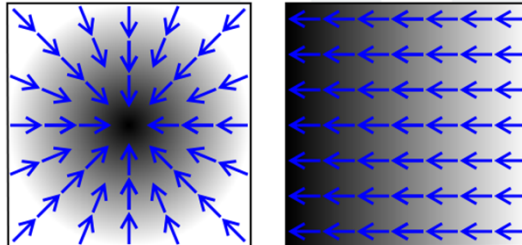
Coordinate System	Gradient Operator
Cartesian	$\nabla V = \frac{\partial V}{\partial x} \hat{a}_x + \frac{\partial V}{\partial y} \hat{a}_y + \frac{\partial V}{\partial z} \hat{a}_z$
Cylindrical	$\nabla V = \frac{\partial V}{\partial \rho} \hat{a}_\rho + \frac{1}{\rho} \frac{\partial V}{\partial \phi} \hat{a}_\phi + \frac{\partial V}{\partial z} \hat{a}_z$
Spherical	$\nabla V = \frac{\partial V}{\partial r} \hat{a}_r + \frac{1}{r} \frac{\partial V}{\partial \theta} \hat{a}_\theta + \frac{1}{r \sin \theta} \frac{\partial V}{\partial \phi} \hat{a}_\phi$

## Algebra Rules for the Gradient

$\nabla(U \pm V) = \nabla U \pm \nabla V$	Sum/Difference Rule
$\nabla(UV) = U\nabla V + V\nabla U$	Product Rule
$\nabla\left(\frac{U}{V}\right) = \frac{V\nabla U - U\nabla V}{V^2}$	Quotient Rule
$\nabla V^n = nV^{n-1}\nabla V$	Power Rule

## Properties of the Gradient

1. The gradient of a scalar function is a vector function.
2. The magnitude of  $\nabla V$  is the local maximum rate of change in  $V$ .
3.  $\nabla V$  points in the direction of maximum rate of change in  $V$ .
4.  $\nabla V$  at any point is perpendicular to the constant  $V$  surface that passes through that point.
5.  $\nabla V$  points toward increasing numbers in  $V$ .



# Divergence of a Vector Field

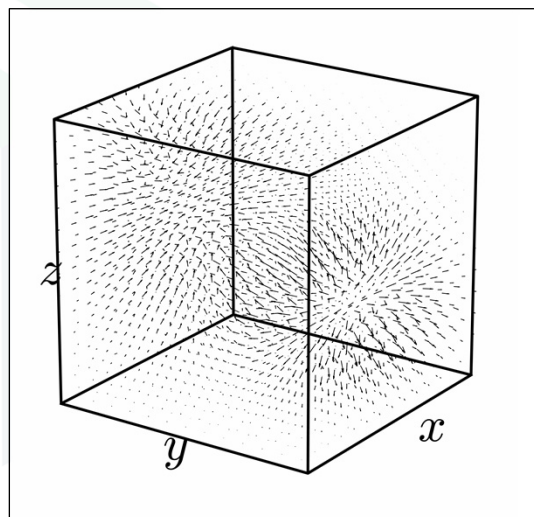
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## Divergence of a Vector Field (1 of 2)

Start with the following vector field  $\vec{F}(x, y, z)$ .

Observe how the field seems to be converging to point in the upper left and diverging from a point in the low right.



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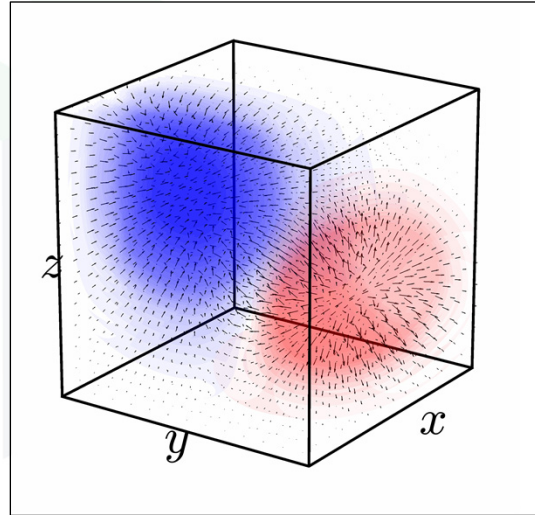
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## Divergence of a Vector Field (2 of 2)

The divergence is  $\nabla \cdot \vec{F}(x, y, z)$ .

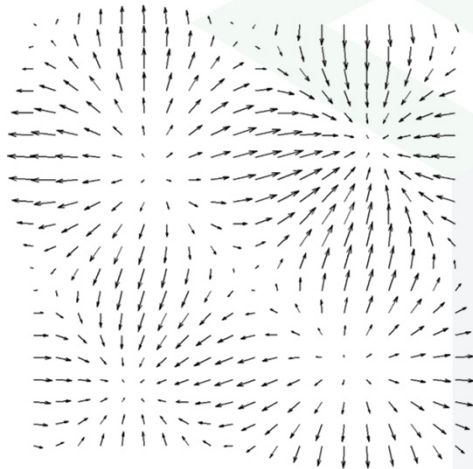
*Divergence* measures the tendency of a vector field to diverge from a point or converge to a point.

Divergence is a scalar quantity and is shown as the colored regions. **Blue indicates negative divergence (convergence)** and **red indicates positive divergence**.

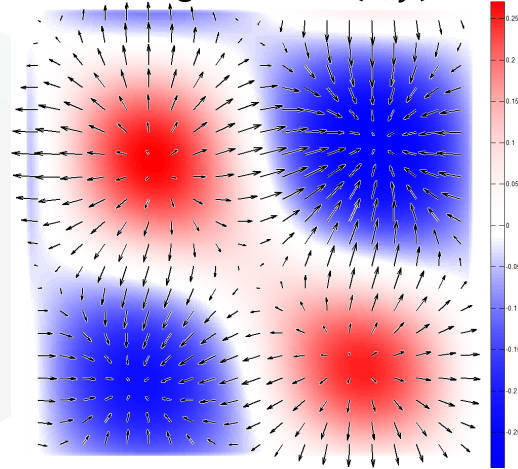


## Divergence of a Vector Field (2D)

Vector Field  $\vec{A}(x, y)$



Divergence  $\nabla \cdot \vec{A}(x, y)$



## Divergence $\nabla \cdot \vec{A}$

The divergence of a vector field is a scalar field that measures the tendency of a vector field to diverge from a point or converge to a point.

Coordinate System	Gradient Operation
Cartesian	$\nabla \cdot \vec{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$
Cylindrical	$\nabla \cdot \vec{A} = \frac{1}{\rho} \frac{\partial(\rho A_\rho)}{\partial \rho} + \frac{1}{\rho} \frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z}$
Spherical	$\nabla \cdot \vec{A} = \frac{1}{r^2} \frac{\partial(r^2 A_r)}{\partial r} + \frac{1}{r \sin \theta} \frac{\partial(A_\theta \sin \theta)}{\partial \theta} + \frac{1}{r \sin \theta} \frac{\partial A_\phi}{\partial \phi}$

## Algebra Rules for Divergence

$$\nabla \cdot (\vec{A} + \vec{B}) = \nabla \cdot \vec{A} + \nabla \cdot \vec{B} \quad \text{Distributive Rule}$$

## Properties of Divergence $\nabla \cdot \vec{A}$

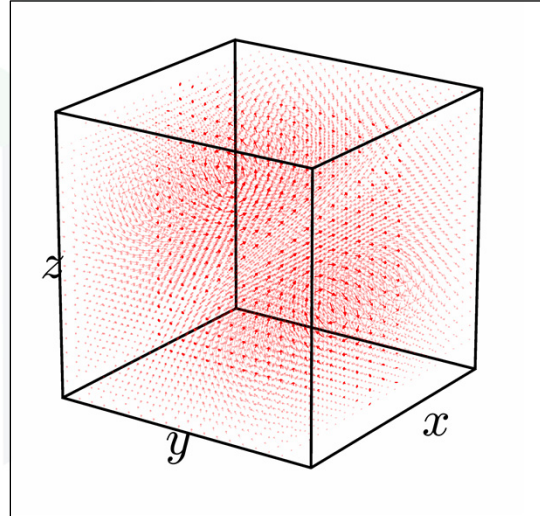
1. The divergence of a vector function is a scalar function.
2. The divergence of a scalar field does not make sense.
3. The original vector field will point from larger to smaller numbers in the scalar field calculated from the divergence.

## Curl of a Vector Field

## Curl of a Vector Field (1 of 2)

Start with the following vector field  $\vec{F}(x, y, z)$ .

Observe how the field seems to be rotating around two different axes.

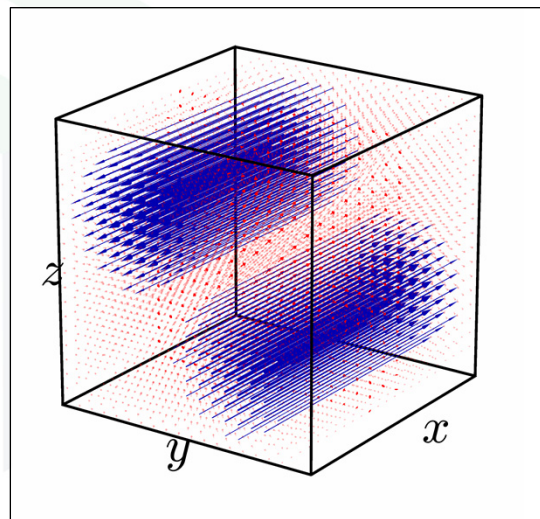


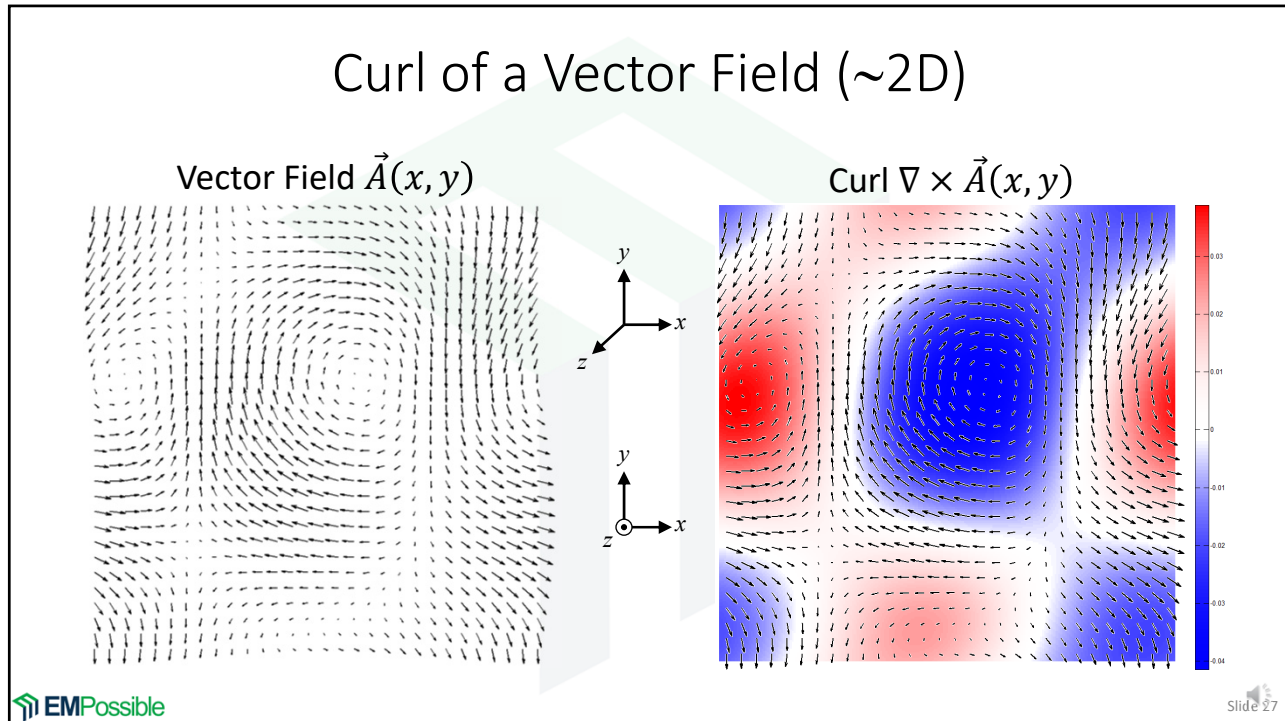
## Curl of a Vector Field (1 of 2)

The curl is  $\nabla \times \vec{F}(x, y, z)$ .

*Curl* measures the tendency of the vector field to circulate around an axis.

Curl is a vector quantity and is shown as the blue arrows. The magnitude of the curl conveys the strength of the circulation. The direction of the curl is the axis of the circulation.





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## Curl

The curl of a vector function is a vector function that quantifies the tendency of the vector field to circulate around an axis. The magnitude of the curl conveys the strength of the circulation. The direction of the curl is the axis of the circulation.

Coordinate System	Curl Operation
<b>Cartesian</b>	$\nabla \times \vec{A} = \left( \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} \right) \hat{a}_x + \left( \frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} \right) \hat{a}_y + \left( \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right) \hat{a}_z$
<b>Cylindrical</b>	$\nabla \times \vec{A} = \left( \frac{1}{\rho} \frac{\partial A_z}{\partial \phi} - \frac{\partial A_\phi}{\partial z} \right) \hat{a}_\rho + \left( \frac{\partial A_\rho}{\partial z} - \frac{\partial A_z}{\partial \rho} \right) \hat{a}_\phi + \frac{1}{\rho} \left( \frac{\partial(\rho A_\phi)}{\partial \rho} - \frac{\partial A_\rho}{\partial \phi} \right) \hat{a}_z$
<b>Spherical</b>	$\nabla \times \vec{A} = \frac{1}{r \sin \theta} \left[ \frac{\partial(A_\phi \sin \theta)}{\partial \theta} - \frac{\partial A_\theta}{\partial \phi} \right] \hat{a}_r + \frac{1}{r} \left[ \frac{1}{\sin \theta} \frac{\partial A_r}{\partial \phi} - \frac{\partial(r A_\phi)}{\partial r} \right] \hat{a}_\theta + \frac{1}{r} \left[ \frac{\partial(r A_\theta)}{\partial r} - \frac{\partial A_r}{\partial \theta} \right] \hat{a}_\phi$

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## Algebra Rules for Curl

$$\nabla \times (\vec{A} + \vec{B}) = \nabla \times \vec{A} + \nabla \times \vec{B} \quad \text{Distributive Rule}$$

$$\nabla \times (\vec{A} \times \vec{B}) = \vec{A}(\nabla \cdot \vec{B}) - \vec{B}(\nabla \cdot \vec{A}) + (\vec{B} \cdot \nabla)\vec{A} - (\vec{A} \cdot \nabla)\vec{B}$$

Triple Product

## Properties of Curl

1. The curl of a vector function is a vector function.
2. The curl of a scalar field does not make sense.
3. Curl follows the right-hand rule.
4. The divergence of the curl of a vector field is always zero.  $\nabla \cdot (\nabla \times \vec{A}) = 0$
5. The curl of the gradient of a scalar field is always zero.  $\nabla \times (\nabla V) = 0$
6. Note that  $\nabla \cdot \vec{A} \neq \vec{A} \cdot \nabla$ .  $\nabla \cdot \vec{A}$  calculates the derivative of  $\vec{A}$ , whereas  $\vec{A} \cdot \nabla$  sets up a derivative operation that is scaled by  $\vec{A}$ .



# Laplacian Operation

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## Scalar Laplacian $\nabla^2 V$

The scalar Laplacian is defined as the divergence of the gradient. It is sort of a measure of the tendency of the scalar function to form bowls.

$$\nabla^2 V = \nabla \cdot (\nabla V)$$

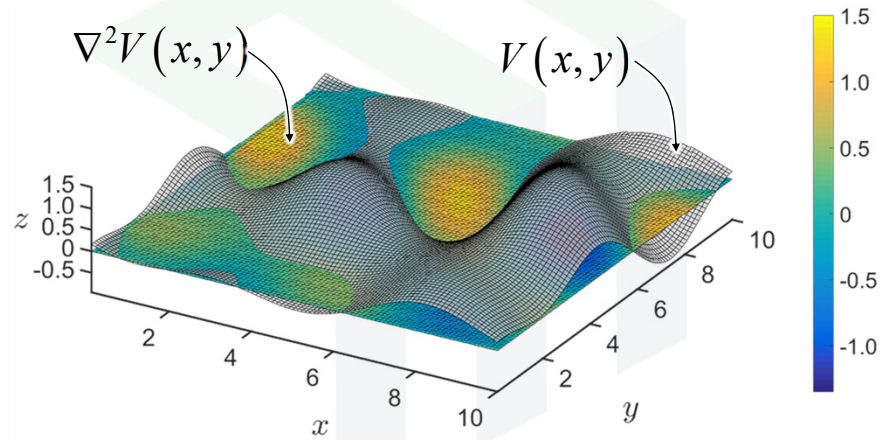
Coordinate System	Scalar Laplacian Operation
<b>Cartesian</b>	$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2}$
<b>Cylindrical</b>	$\nabla^2 V = \frac{1}{\rho} \frac{\partial}{\partial \rho} \left( \rho \frac{\partial V}{\partial \rho} \right) + \frac{1}{\rho^2} \frac{\partial^2 V}{\partial \phi^2} + \frac{\partial^2 V}{\partial z^2}$
<b>Spherical</b>	$\nabla^2 V = \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial V}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial V}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 V}{\partial \phi^2}$

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## Visualization of Scalar Laplacian $\nabla^2 V$



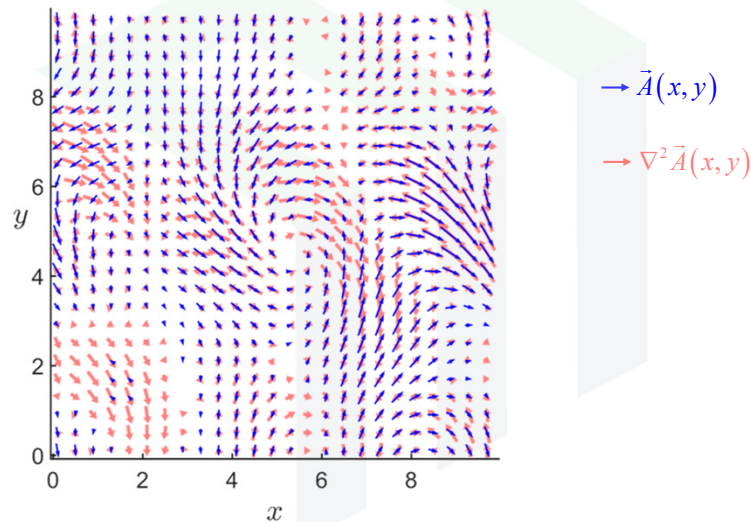
## Vector Laplacian $\nabla^2 \vec{A}$

The vector Laplacian is defined as the gradient of the divergence minus the curl.

$$\nabla^2 \vec{A} = \nabla(\nabla \cdot \vec{A}) - \nabla \times \nabla \times \vec{A}$$

Coordinate System	Scalar Laplacian Operation
<b>Cartesian</b>	$\nabla^2 \vec{A} = \nabla^2 A_x \hat{a}_x + \nabla^2 A_y \hat{a}_y + \nabla^2 A_z \hat{a}_z$
<b>Cylindrical</b>	$\nabla^2 \vec{A} = \nabla^2 A_\rho \hat{a}_\rho + \nabla^2 A_\phi \hat{a}_\phi + \nabla^2 A_z \hat{a}_z$
<b>Spherical</b>	$\nabla^2 \vec{A} = \nabla^2 A_r \hat{a}_r + \nabla^2 A_\theta \hat{a}_\theta + \nabla^2 A_\phi \hat{a}_\phi$

## Visualization of Vector Laplacian $\nabla^2 \vec{A}$



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## Properties of the Laplacian

1. The Laplacian of a scalar function is a scalar function.
2. The scalar Laplacian arises when dealing with electric potential.
3. The Laplacian of a vector function is a vector function.
4. The vector Laplacian arises in the wave equation.

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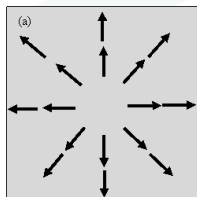
# Classification of Vector Fields

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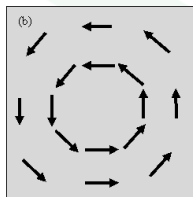
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## Irrotational

A vector field is irrotational if  $\nabla \times \vec{A} = 0$



Irrotational



Rotational

[Examples of Irrotational Fields](#)

Gravity

Electric potential

### Consequences:

1. Irrotational fields must flow in essentially straight lines.
2. If  $\nabla \times \vec{A} = 0$ , then  $\oint_L \vec{A} \cdot d\vec{\ell} = 0$  and the field is said to be *conservative*.

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# Solenoidal (Divergenceless)

A vector field is solenoidal if  $\nabla \cdot \vec{A} = 0$

Examples of Solenoidal Fields  
 Fluid flows  
 Magnetic fields

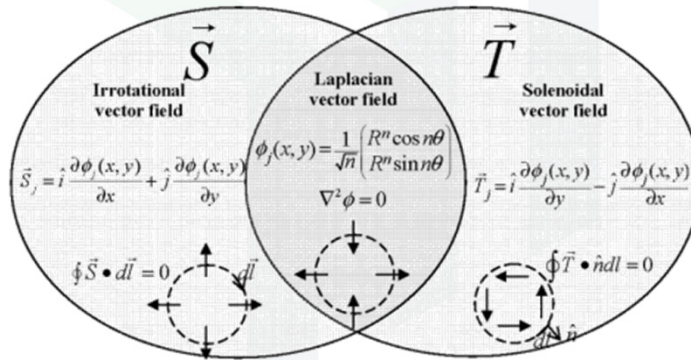
Consequences:

1. If the vector function does not diverge from any points, it must form loops.
2. If  $\nabla \cdot \vec{A} = 0$ , then  $\oiint_S \vec{A} \cdot d\vec{s} = 0$  and the field will have zero net flux.

# Laplacian Vector Fields

A vector field is Laplacian if it is both irrotational and solenoidal

$$\nabla \times \vec{A} = 0 \quad \nabla \cdot \vec{A} = 0$$





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