Electromagnetics:
Electromagnetic Field Theory

Dispersion Relation

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Lecture Outline

- Dispersion Relation
- Index Ellipsoids

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Dispersion Relation

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Derivation in LHI Media

We started with the wave equation.

$$\nabla^2 \vec{E} + k^2 \vec{E} = 0$$

The solution was found to be plane waves.

$$\vec{E}(\vec{r}) = \vec{P} e^{-j\vec{k} \bullet \vec{r}}$$

If the solution is substituted back into the wave equation, the dispersion relation is derived.

$$k^2 = \left(\frac{\omega n}{c_0} \right)^2 = (k_0 n)^2 = k_x^2 + k_y^2 + k_z^2$$

The dispersion relation relates frequency ω to wave number k. For LHI media, it fixes the magnitude of the wave vector to be a constant.

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Index Ellipsoids

Index Ellipsoids

From the previous slide, the dispersion relation for a LHI material was:

$$k_x^2 + k_y^2 + k_z^2 = k_0^2 n^2$$

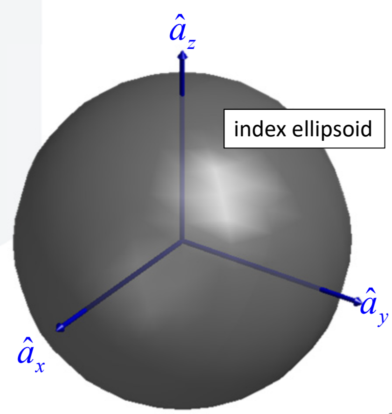
This defines a sphere called an "index ellipsoid."

The vector connecting the origin to a point on the surface of the sphere is the k-vector for that direction. Refractive index is calculated from this.

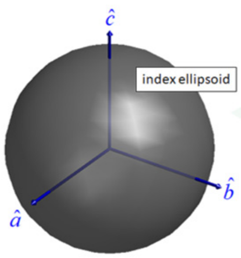
$$\left|\vec{k}\right| = k_0 n$$

For LHI materials, the refractive index is the same in all directions.

Think of this as a map of the refractive index as a function of the wave's direction through the medium.



What About Anisotropic Materials?

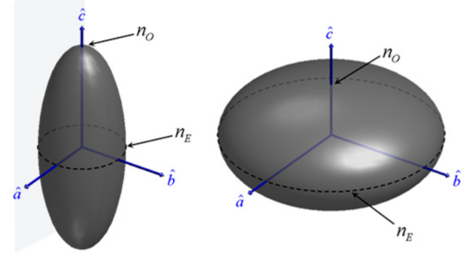


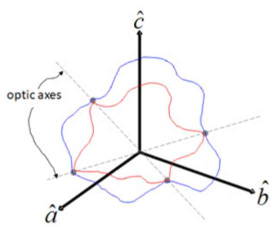
Isotropic Materials

$$k_a^2 + k_b^2 + k_c^2 = k_0^2 n^2$$

Uniaxial Materials

$$\left(\frac{k_a^2 + k_b^2 + k_c^2}{n_O^2} - k_0^2 \right) \left(\frac{k_a^2 + k_b^2}{n_E^2} + \frac{k_c^2}{n_O^2} - k_0^2 \right) = 0$$





Biaxial Materials

$$\frac{k_a^2}{|\vec{k}|^2 - k_0^2 n_a^2} + \frac{k_b^2}{|\vec{k}|^2 - k_0^2 n_b^2} + \frac{k_c^2}{|\vec{k}|^2 - k_0^2 n_c^2} = 1$$

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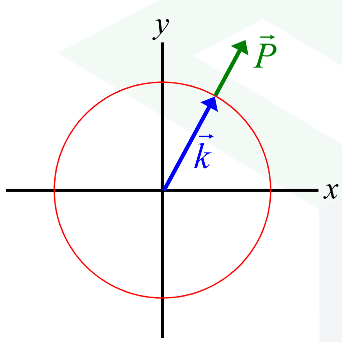
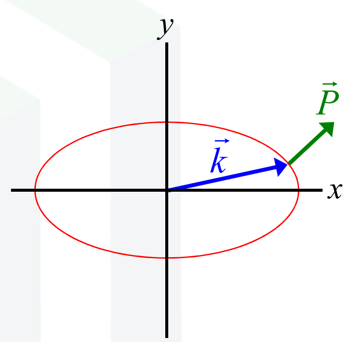
Direction of Power Flow

Isotropic Materials



Anisotropic Materials



Phase propagates in the direction of $\vec{k}$. Therefore, the refractive index derived from $|\vec{k}|$ is best described as the phase refractive index. Velocity is thus the phase velocity.

Power propagates in the direction of $\vec{P}$ which is always normal to the surface of the index ellipsoid. From this, we can define a group velocity and a group refractive index.

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Illustration of $\vec{k}$ versus $\vec{P}$