Electromagnetics:
Electromagnetic Field Theory

Electric Potential

Outline

• Concept of electric potential $V$
• Potential difference $V_{AB}$
• Electric potential due to charge
• Relationship between $\vec{E}$ and $V$
• Electric potential example
Recall for electrostatics that the electric field has zero curl.

\[ \nabla \times \vec{E} = 0 \]

Recall from vector calculus that the gradient of a scalar function cannot have any curl.

\[ \nabla \times (\nabla V) = 0 \]

This means that the electric field intensity can be written as the gradient of a scalar function \( V \).

\[ \vec{E} = -\nabla V \]

A negative sign is incorporated to enforce the sign convention that electric fields are directed from high to low potential.
Intuitive Derivation of Electric Potential

Suppose a voltage $V_0$ is applied across two plates bounding some medium.

This induces charge in the plates at either side of the medium.
Intuitive Derivation of Electric Potential

The charge creates an electric field between the plates that is known to be uniform.

\[ V_0 \]

\[ \begin{align*}
V(z) &= V_0 \\
\frac{\partial V}{\partial z} &= \frac{\epsilon}{\mu} \end{align*} \]

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Conclusion – the electric field is the slope of the voltage.

\[ \vec{E} = -\frac{\partial V}{\partial z} \hat{a}_z \]
Intuitive Derivation of Electric Potential

This can be generalized to 3D using the gradient operation.

\[
\vec{E} = -\frac{\partial V}{\partial z} \hat{a}_z \quad \Rightarrow \quad \vec{E} = -\nabla V
\]

This is the equation used to calculate the electric field from the electric potential.

Potential Difference
Work to Move a Charge

Suppose a point charge $Q$ is moved from point A to point B, a distance of $d$, in the presence of an electric field $\vec{E}$.

[Diagram showing a point charge moving from A to B in an electric field $\vec{E}$]

Force on the charge:
$$\vec{F} = Q\vec{E}$$

Work done to move charge:
$$W = -Fd = -Qd\vec{E}$$

The negative sign indicates the force is external.

This can be generalized to a differential distance:
$$dW = -\vec{F} \cdot d\vec{l} = -Q\vec{E} \cdot d\vec{l}$$

Integrate to get total work:
$$W = \int_{A}^{B} dW = \int_{A}^{B} -Q\vec{E} \cdot d\vec{l}$$

Potential Difference

The potential difference between points A and B is the potential energy per unit charge. Potential energy is the same as the work it would take to move the charge.

Divide total work by $Q$:
$$W = -Q\int_{A}^{B} \vec{E} \cdot d\vec{l}$$

$$\frac{W}{Q} = \int_{A}^{B} \vec{E} \cdot d\vec{l}$$

This is the potential difference between points A and B.

$$V_{AB} = V_{B} - V_{A} = \frac{W}{Q} = -\int_{A}^{B} \vec{E} \cdot d\vec{l}$$
One Application of Potential Difference

The equation below is used to calculate the electric potential from the electric field.

\[ V_{AB} = V_B - V_A = -\int_{A}^{B} \vec{E} \cdot d\vec{l} \]

Notes About Potential Difference

• A is the initial point and B is the final point. This is important for the sign convention.
• \( V_{AB} < 0 \) indicates a loss in potential energy because work is being done by the field.
• \( V_{AB} > 0 \) indicates a gain in potential energy because an external agent must be doing the work.
• \( V_{AB} \) is independent of the path taken from A to B.
• \( V_{AB} \) is measured in joules per Coulomb (J/C), or volts (V).
Electric Potential Due to Charge

Recall Electric Field Around a Charge

Electric flux density
\[ \vec{D} = \frac{Q}{4\pi R^2} \hat{a}_R \]

Electric Field Intensity
\[ \vec{E} = \frac{\vec{D}}{\varepsilon} = \frac{Q}{4\pi \varepsilon R^2} \hat{a}_R \]
Derivation Setup

Start with the basic equation to calculate $V$ from $\vec{E}$.

$$V_B - V_A = -\int_A^B \vec{E} \cdot d\vec{\ell}$$

For this problem, the equation becomes

$$V(r) - V_{\text{ref}} = -\int_{\infty}^r \frac{Q}{4\pi \varepsilon_0 r'^2} \hat{a}_R \cdot dr' \hat{a}_R$$

The Derivation

$$V(r) - V_{\text{ref}} = -\int_{\infty}^r \frac{Q}{4\pi \varepsilon_0 r'^2} \hat{a}_R \cdot dr' \hat{a}_R$$

Equation from last slide.

Bring constants to outside of integral.

Perform dot product $\hat{a}_R \cdot dr' \hat{a}_R = dr'$

Calculate antiderivative.

Cancel negative signs.

Evaluate antiderivative at limits.

Simplify and bring $V_{\text{ref}}$ to right side.
Potential Due \( V \) to Point Charge \( Q \)

\[
V(r) = \frac{Q}{4\pi \varepsilon r} + V_{\text{ref}} \quad \text{or} \quad V(\vec{r}) = \frac{Q}{4\pi \varepsilon |\vec{r} - \vec{r}_Q|} + V_{\text{ref}}
\]

A single potential has almost no meaning. Only the potential difference between two points is ever of interest. Any background potential, or reference potential \( V_{\text{ref}} \), can be chosen. Usually, the reference potential \( V_{\text{ref}} \) is not written explicitly.

Potential Due to Charge Distribution

- **Point Charge**
  - Charge \( Q \) (C)
  - Electric Potential \( V = \frac{Q}{4\pi \varepsilon |\vec{r} - \vec{r}_Q|} \)

- **Line Charge**
  - Line Charge Density \( \rho_1 \) (C/m)
  - Electric Potential \( V = \frac{1}{4\pi \varepsilon} \int \rho_1 \, d\ell \)

- **Sheet Charge**
  - Surface Charge Density \( \rho_s \) (C/m\(^2\))
  - Electric Potential \( V = \frac{1}{4\pi \varepsilon} \int \int \rho_s \, ds \)

- **Volume Charge**
  - Volume Charge Density \( \rho_v \) (C/m\(^3\))
  - Electric Potential \( V = \frac{1}{4\pi \varepsilon} \int \int \int \rho_v \, dv \)
What is the potential difference between these objects?

\[ V = V_B - V_A \]
Solution (1 of 8)

Calculate $V$ from $\vec{E}$ using a line integration.

$$V = V_B - V_A = -\oint_A \vec{E} \cdot d\vec{\ell}$$

Solution (2 of 8)

Calculate $V$ from $\vec{E}$ using a line integration.

$$V = V_B - V_A = -\oint_A \vec{E} \cdot d\vec{\ell}$$

For easy integration, a path is chosen where the electric field is always parallel to that path.

This will simplify the dot product to

$$\vec{E} \cdot d\vec{\ell} \approx |\vec{E}| d\ell$$
How about this path?

From the figure, the electric field is almost constant along this path.
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This lets the integral be approximated as

\[
V = \vec{E} \cdot d\vec{\ell} = \int_{A}^{B} |\vec{E}| d\ell \\
\approx -|\vec{E}|L
\]

To estimate \( L \), the line is moved down to the \( x \)-axis.

\( L \approx 1.7 \)
Solution (7 of 8)

To estimate $L$, the line is moved down to the $x$-axis.

$$L \approx 1.7$$

To estimate $|E|$, the color is read from the color bar.

$$|E| \approx 7.0$$

Finally, the potential difference $V$ is approximately

$$V = -|E|L$$

$$= -(7.0)(1.7)$$

$$= -11.9 \text{ V}$$

Solution (8 of 8)

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Interpreting the Sign (1 of 2)

Why is the voltage negative?

\[ V = -11.9 \text{ V} \]

Interpreting the Sign (2 of 2)

Recall the definition of \( V \).

\[ V = V_B - V_A = - \int_{A}^{B} \mathbf{E} \cdot d\mathbf{l} \]

From the figure, it is concluded that \( V_A > V_B \)

Therefore

\[ V = V_B - V_A \text{ is negative.} \]