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Electromagnetics:
Electromagnetic Field Theory

Electric Potential

1

Outline

- Concept of electric potential V
- Potential difference V_{AB}
- Electric potential due to charge
- Relationship between $\vec{E}$ and V
- Electric potential example

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2

Concept of Electric Potential

Vector Calculus Derivation of Electric Potential

Recall from vector calculus that the gradient of a scalar function cannot have any curl.

$$\nabla \times \left(\nabla V \right) = 0$$

Recall for electrostatics that the electric field has zero curl.

$$\nabla \times \vec{E} = 0$$

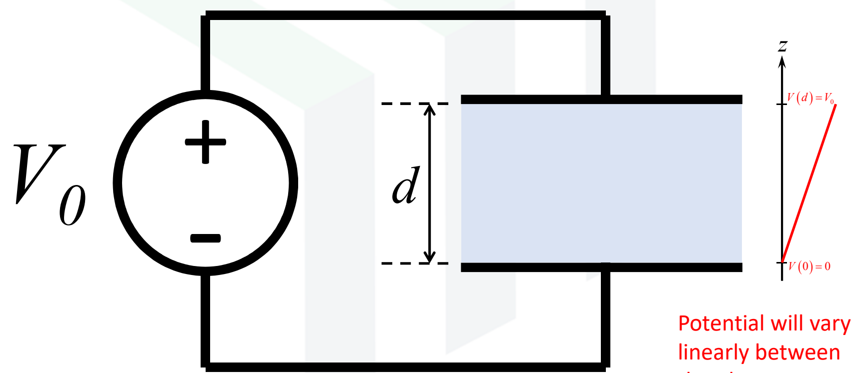
This means that the electric field intensity can be written as the gradient of a scalar function V.

$$\boxed{\vec{E} = -\nabla V}$$

A negative sign is incorporated to enforce the sign convention that electric fields are directed from high to low potential.

Intuitive Derivation of Electric Potential

Suppose a voltage V_0 is applied across two plates bounding some medium.



Potential will vary linearly between the plates.

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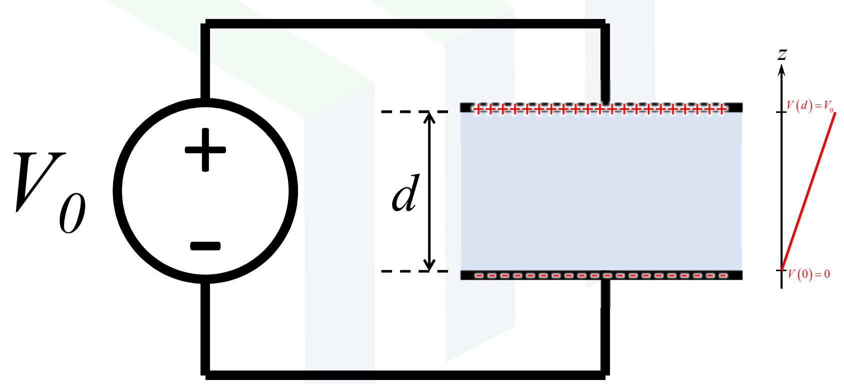
Intuitive Derivation of Electric Potential

This induces charge in the plates at either side of the medium.



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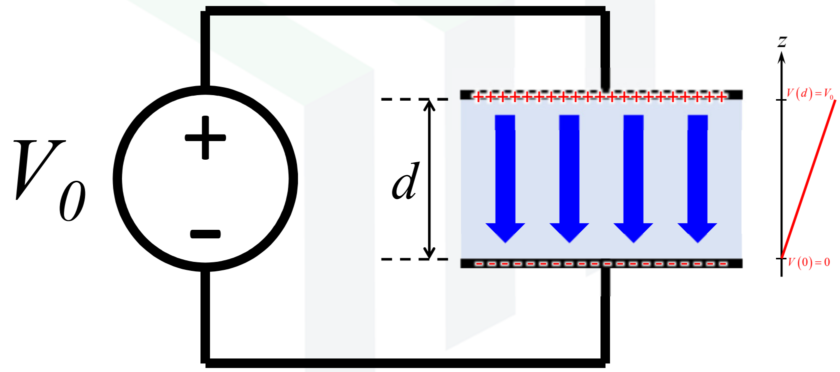
Intuitive Derivation of Electric Potential

The charge creates an electric field between the plates that is known to be uniform.



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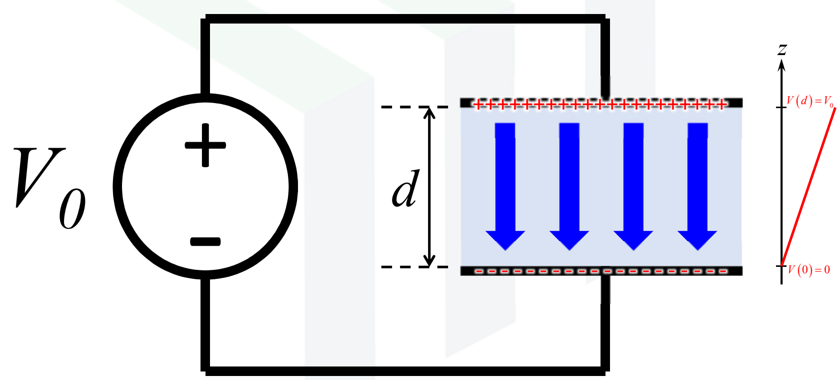
Slide 7

7

Intuitive Derivation of Electric Potential

Conclusion – the electric field is the slope of the voltage.

$$\vec{E} = -\frac{\partial V}{\partial z}\hat{a}_z$$



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Slide 8

8

Intuitive Derivation of Electric Potential

This can be generalized to 3D using the gradient operation.

$$\vec{E} = -\frac{\partial V}{\partial z}\hat{a}_z \quad \Longrightarrow \quad \boxed{\vec{E} = -\nabla V}$$

This is the equation used to calculate the electric field from the electric potential.

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Slide 9

9

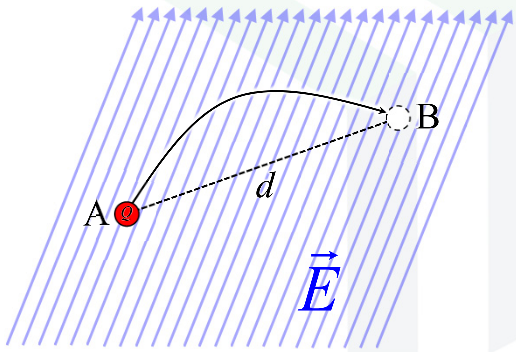
Potential Difference

Slide 10

10

Work to Move a Charge

Suppose a point charge Q is moved from point A to point B, a distance of d, in the presence of an electric field $\vec{E}$.



Force on the charge

$$\vec{F} = Q\vec{E}$$

Work done to move charge

$$W = -Fd = -Qd\left|\vec{E}\right|$$

The negative sign indicates the force is external.

This can be generalized to a differential distance

$$dW = -\vec{F} \bullet d\vec{\ell} = -Q\vec{E} \bullet d\vec{\ell}$$

Integrate to get total work.

$$W = \int_{A}^{B} dW = -Q\int_{A}^{B} \vec{E} \bullet d\vec{\ell}$$

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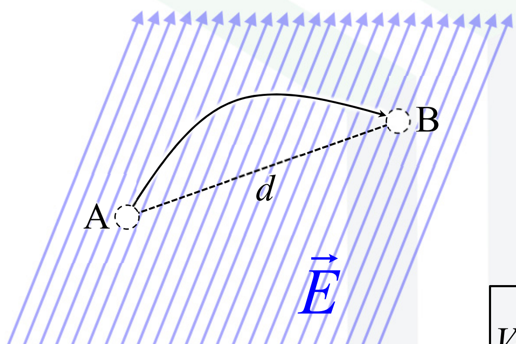
11

Potential Difference

The potential difference between points A and B is the potential energy per unit charge. Potential energy is the same as the work it would take to move the charge.



Divide total work by Q

$$W = -Q\int_{A}^{B} \vec{E} \bullet d\vec{\ell}$$

$$\frac{W}{Q} = -\int_{A}^{B} \vec{E} \bullet d\vec{\ell}$$

This is the potential difference between points A and B.

$$V_{AB} = V_B - V_A = \frac{W}{Q} = -\int_{A}^{B} \vec{E} \bullet d\vec{\ell}$$

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12

One Application of Potential Difference

The equation below is used to calculate the electric potential from the electric field.

$$V_{AB} = V_B - V_A = -\int_A^B \vec{E} \bullet d\vec{\ell}$$

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Slide 13

13

Notes About Potential Difference

- A is the initial point and B is the final point. This is important for the sign convention.
- $V_{AB} < 0$ indicates a loss in potential energy because work is being done by the field.
- $V_{AB} > 0$ indicates a gain in potential energy because an external agent must be doing the work.
- V_{AB} is independent of the path taken from A to B.
- V_{AB} is measured in joules per Coulomb (J/C), or volts (V).

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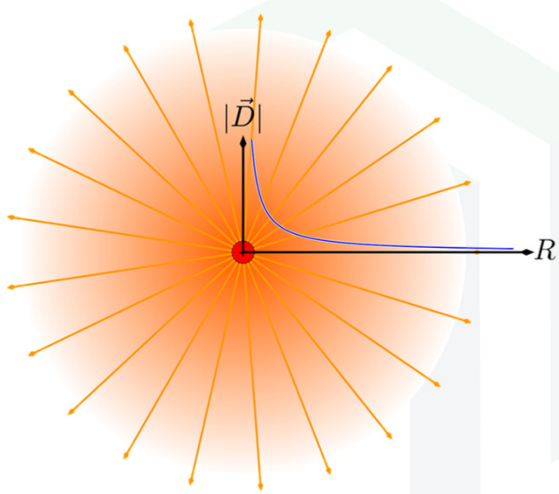
Slide 14

14

Electric Potential Due to Charge

Recall Electric Field Around a Charge



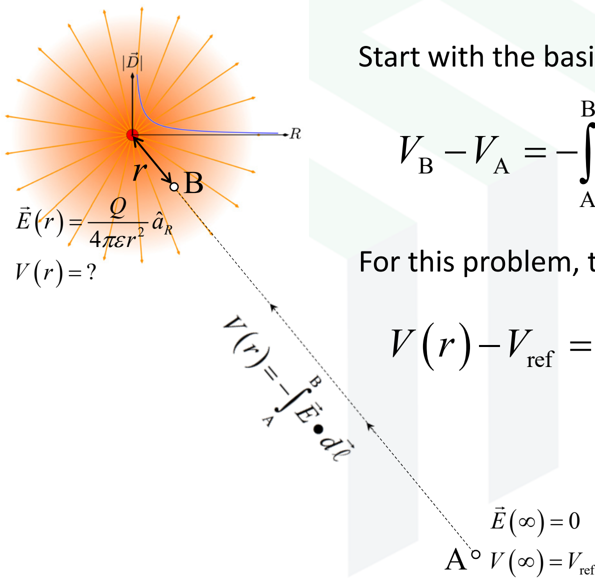
Electric flux density

$$\vec{D} = \frac{Q}{4\pi R^2}\hat{a}_R$$

Electric Field Intensity

$$\vec{E} = \frac{\vec{D}}{\varepsilon} = \frac{Q}{4\pi\varepsilon R^2}\hat{a}_R$$

Derivation Setup



Start with the basic equation to calculate V from $\vec{E}$.

$$V_B - V_A = -\int_A^B \vec{E} \bullet d\vec{\ell}$$

For this problem, the equation becomes

$$V(r) - V_{\text{ref}} = -\int_\infty^r \frac{Q}{4\pi\varepsilon r'^2}\hat{a}_R \bullet dr'\hat{a}_R$$

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Slide 17

17

The Derivation

$$V(r) - V_{\text{ref}} = -\int_\infty^r \frac{Q}{4\pi\varepsilon r'^2}\hat{a}_R \bullet dr'\hat{a}_R$$ Equation from last slide.

$$= -\frac{Q}{4\pi\varepsilon}\int_\infty^r \frac{1}{r'^2}dr'$$ Bring constants to outside of integral. Perform dot product $\hat{a}_R \bullet dr'\hat{a}_R = dr'$

$$= -\frac{Q}{4\pi\varepsilon}\left(-\frac{1}{r'}\right)\Bigg|_\infty^R$$ Calculate anti-derivative.

$$= \frac{Q}{4\pi\varepsilon}\left(\frac{1}{r'}\right)\Bigg|_\infty^r$$ Cancel negative signs.

$$= \frac{Q}{4\pi\varepsilon}\left[\frac{1}{r} - \frac{1}{\infty}\right]$$ Evaluate anti-derivative at limits.

$$V(r) = \frac{Q}{4\pi\varepsilon r} + V_{\text{ref}}$$ Simplify and bring V_{ref} to right side.

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Slide 18

18

Potential Due V to Point Charge Q

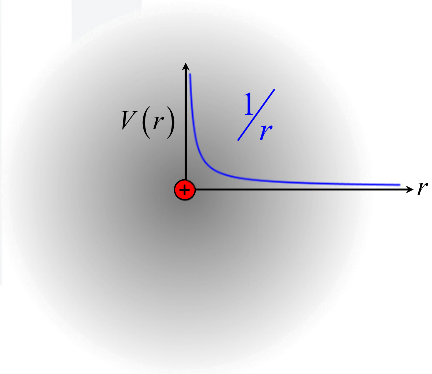
$$V(r)=\frac{Q}{4\pi\varepsilon r}+V_{\text{ref}} \quad \text{or} \quad V(\vec{r})=\frac{Q}{4\pi\varepsilon\left|\vec{r}-\vec{r}_Q\right|}+V_{\text{ref}}$$

A single potential has almost no meaning.

Only the potential difference between two points is ever of interest.

Any background potential, or reference potential V_{ref}, can be chosen.

Usually, the reference potential V_{ref} is not written explicitly.



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Slide 19

19

Potential Due to Charge Distribution

Point Charge

Q

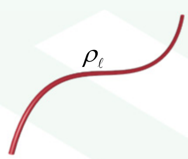
Charge

Q (C)

Electric Potential

$$V=\frac{Q}{4\pi\varepsilon\left|\vec{r}-\vec{r}_Q\right|}$$

Line Charge



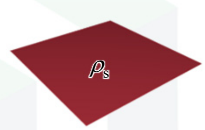
Line Charge Density

ρ_ℓ (C/m)

Electric Potential

$$V=\frac{1}{4\pi\varepsilon}\int_\ell\frac{\rho_\ell}{\left|\vec{r}-\vec{r}'\right|}d\ell$$

Sheet Charge



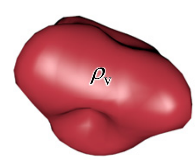
Surface Charge Density

ρ_s $\left(\text{C/m}^2\right)$

Electric Potential

$$V=\frac{1}{4\pi\varepsilon}\iint_S\frac{\rho_s}{\left|\vec{r}-\vec{r}'\right|}ds$$

Volume Charge



Volume Charge Density

ρ_v $\left(\text{C/m}^3\right)$

Electric Potential

$$V=\frac{1}{4\pi\varepsilon}\iiint_v\frac{\rho_v}{\left|\vec{r}-\vec{r}'\right|}dv$$

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Slide 20

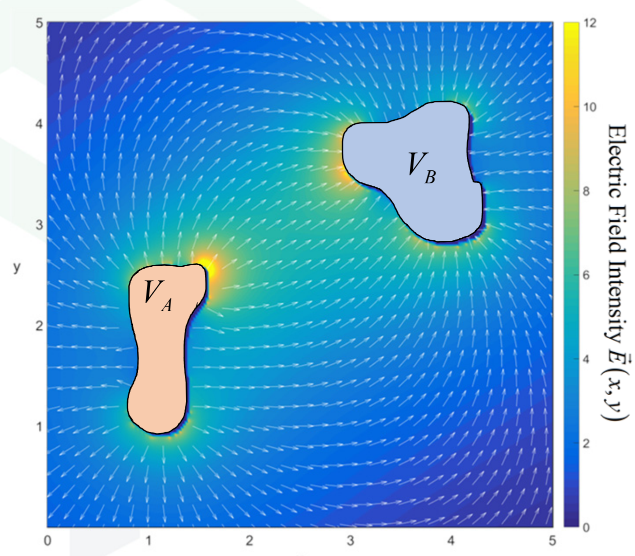
20

Electric Potential Example

Example Setup

What is the potential difference between these objects?

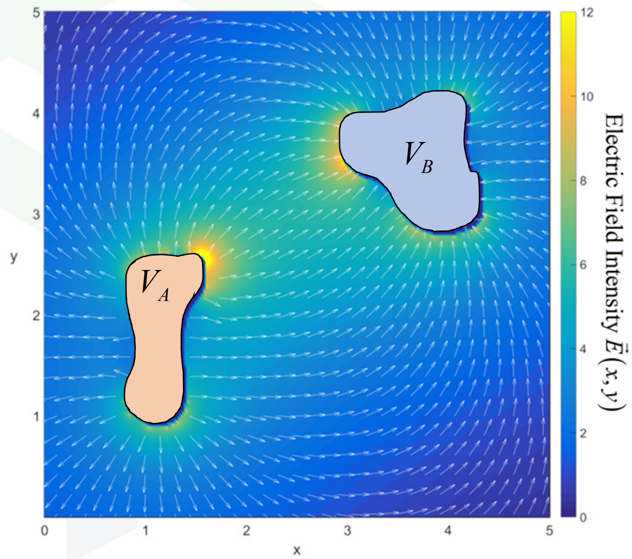
$$V = V_B - V_A$$



Solution (1 of 8)

Calculate V from $\vec{E}$ using a line integration.

$$V = V_B - V_A = -\int_A^B \vec{E} \bullet d\vec{\ell}$$



Solution (2 of 8)

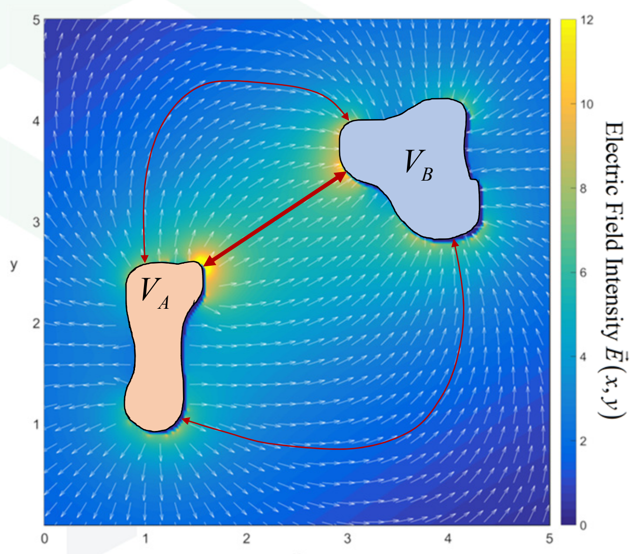
Calculate V from $\vec{E}$ using a line integration.

$$V = V_B - V_A = -\int_A^B \vec{E} \bullet d\vec{\ell}$$

For easy integration, a path is chosen where the electric field is always parallel to that path.

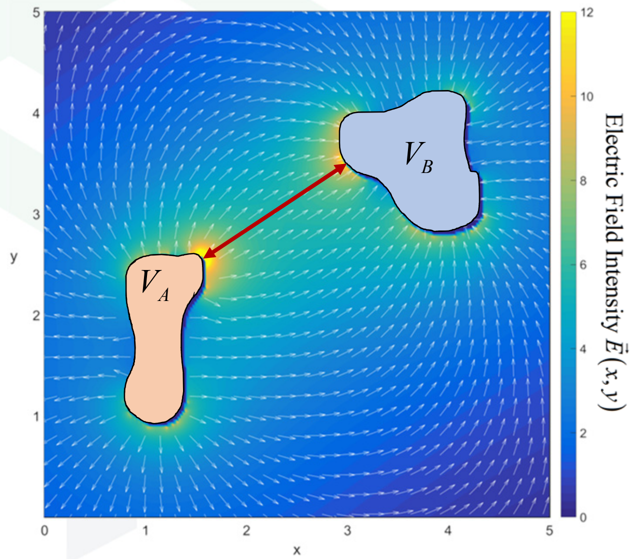
This will simplify the dot product to

$$\vec{E} \bullet d\vec{\ell} \approx \left|\vec{E}\right| d\ell$$



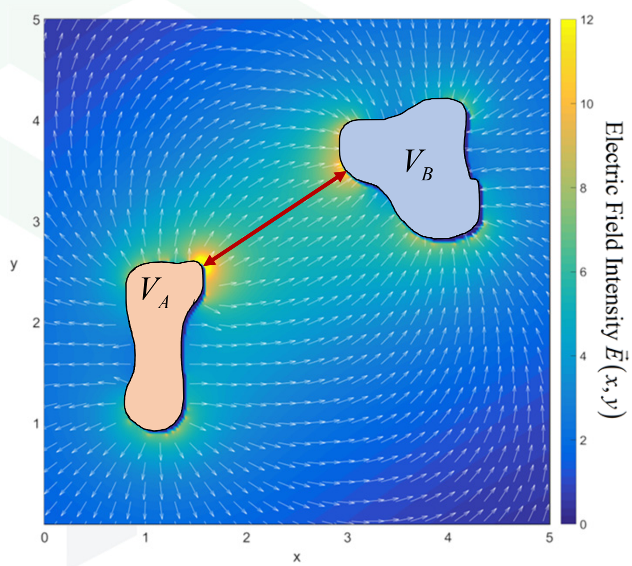
Solution (3 of 8)

How about this path?



Solution (4 of 8)

From the figure, the electric field is almost constant along this path.



Solution (5 of 8)

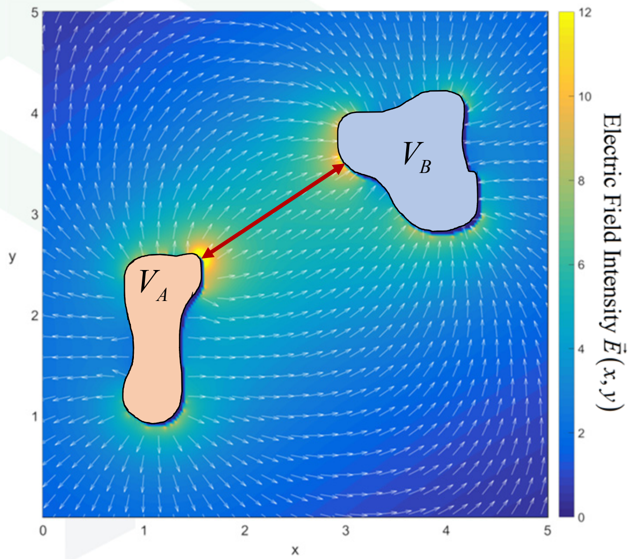
From the figure, the electric field is almost constant along this path.

This let's the integral be approximated as

$$V = -\int_A^B \vec{E} \bullet d\vec{\ell}$$

$$\approx -\int_A^B \left|\vec{E}\right| d\ell$$

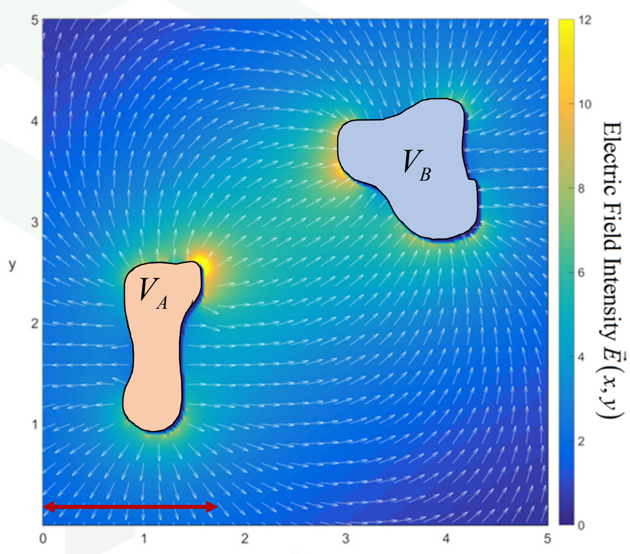
$$\approx -\left|\vec{E}\right| L$$



Solution (6 of 8)

To estimate L, the line is moved down to the x-axis.

$$L \approx 1.7$$



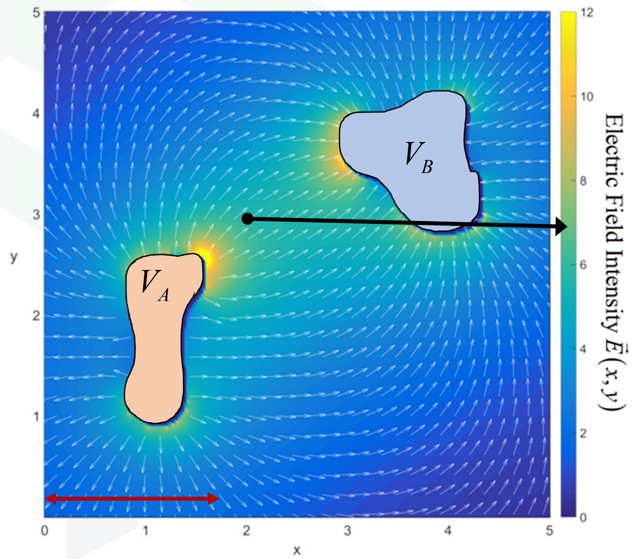
Solution (7 of 8)

To estimate L, the line is moved down to the x-axis.

$$L \approx 1.7$$

To estimate $|E|$, the color is read from the color bar.

$$\left|\vec{E}\right| \approx 7.0$$



Solution (8 of 8)

To estimate L, the line is moved down to the x-axis.

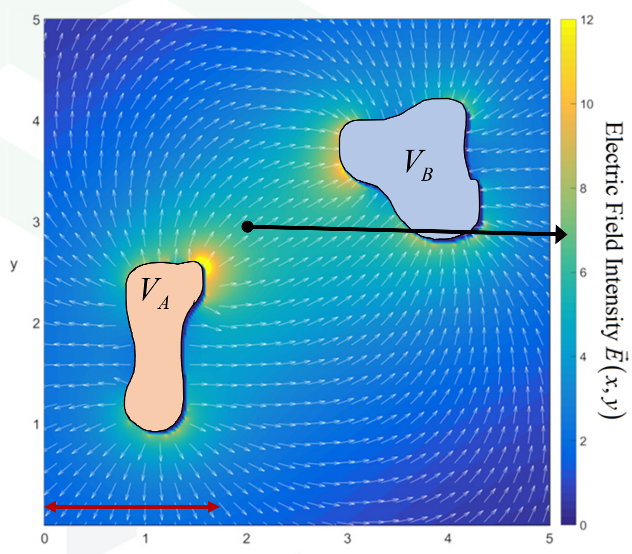
$$L \approx 1.7$$

To estimate $|E|$, the color is read from the color bar.

$$\left|\vec{E}\right| \approx 7.0$$

Finally, the potential difference V is approximately

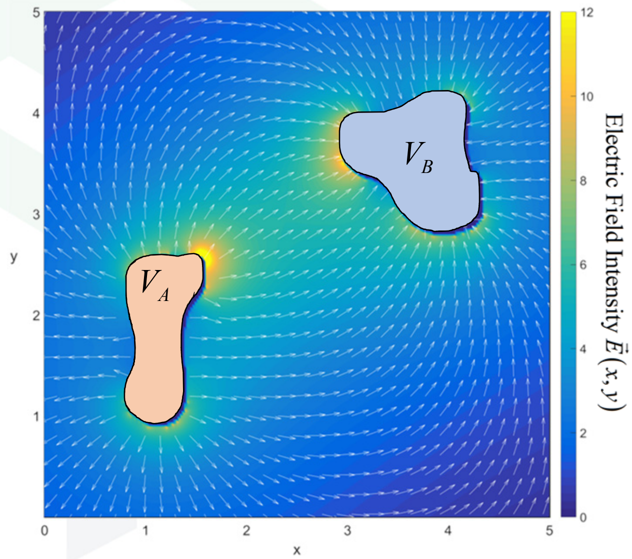
$$\begin{aligned} V &= -\left|\vec{E}\right| L \\ &= -(7.0)(1.7) \\ &= \boxed{-11.9\ \text{V}} \end{aligned}$$



Interpreting the Sign (1 of 2)

Why is the voltage negative?

$$V = -11.9 \text{ V}$$



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Slide 31

31

Interpreting the Sign (2 of 2)

Why is the voltage negative?

$$V = -11.9 \text{ V}$$

Recall the definition of V.

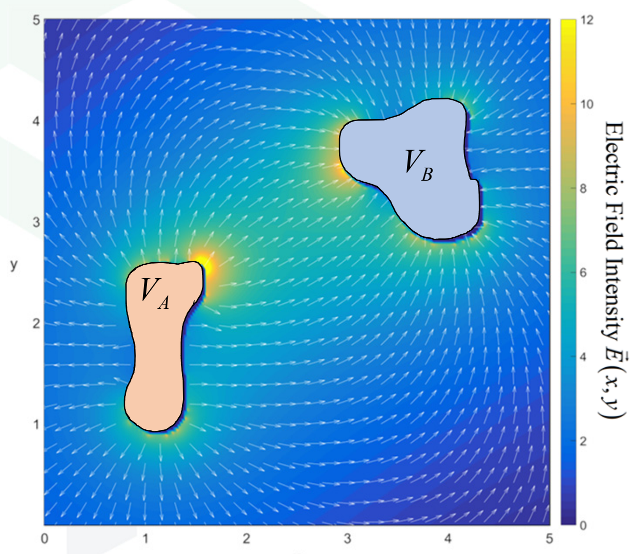
$$V = V_B - V_A = -\int_A^B \vec{E} \bullet d\vec{\ell}$$

From the figure, it is concluded that

$V_A > V_B$

Therefore

$V = V_B - V_A$ is negative.



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Slide 32

32