Electromagnetics:
Electromagnetic Field Theory

Electromagnetic Waves

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Lecture Outline

- Maxwell's Equations Predict Waves
- Derivation of the Wave Equation
- Solution to the Wave Equation

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Maxwell's Equations Predict Waves

Recall Maxwell's Equations in Source Free Media

In source-free media, $\vec{J} = 0$ and $\rho_{\mathrm{v}} = 0$.
Maxwell's equations in the frequency-domain become

Curl Equations

$$\nabla \times \vec{E} = -j\omega\vec{B}$$

$$\nabla \times \vec{H} = j\omega\vec{D}$$

Divergence Equations

$$\nabla \bullet \vec{D} = 0$$

$$\nabla \bullet \vec{B} = 0$$

Constitutive Relations

$$\vec{D} = \varepsilon\vec{E}$$

$$\vec{B} = \mu\vec{H}$$

The Curl Equations Predict Waves

Substituting the constitutive relations into the curl equations gives

$$\nabla \times \vec{E} = -j\omega\mu\vec{H}$$

⇩

A time-harmonic magnetic field $\vec{H}$ will induce a time-harmonic electric field $\vec{E}$ circulating about the magnetic field $\vec{H}$.

A time-harmonic circulating electric field $\vec{E}$ will induce a time-harmonic magnetic field $\vec{H}$ through the axis of circulation.

$$\nabla \times \vec{H} = j\omega\varepsilon\vec{E}$$

⇩

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An $\vec{H}$ induces an $\vec{E}$. That $\vec{E}$ induces another $\vec{H}$. That new $\vec{H}$ induces another $\vec{E}$. That $\vec{E}$ induces yet another $\vec{H}$. And so on.

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How Waves Propagate

Start with an oscillating electric field.

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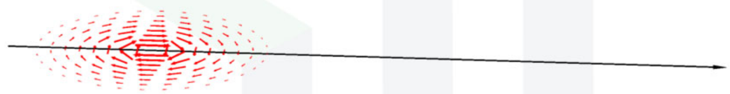
How Waves Propagate



This induces a circulating magnetic field.

$$\nabla \times \vec{H} = j\omega\varepsilon\vec{E}$$

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How Waves Propagate

Now examine the magnetic field on axis.

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How Waves Propagate

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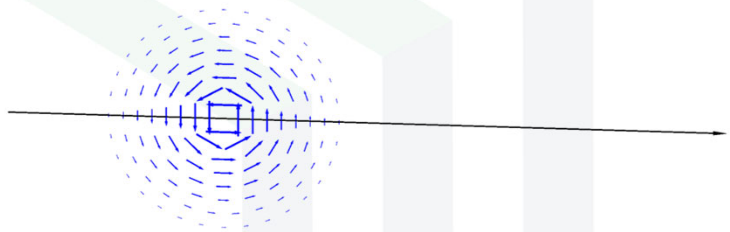
circulating electric field.

$$\nabla \times \vec{E} = -j\omega\mu\vec{H}$$



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How Waves Propagate

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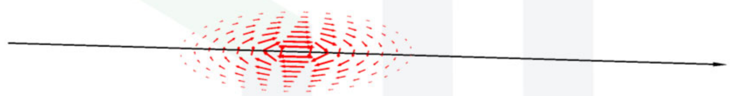
How Waves Propagate



This induces a circulating magnetic field.

$$\nabla \times \vec{H} = j\omega\varepsilon\vec{E}$$

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How Waves Propagate

…and so on…

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Derivation of the Wave Equation

Wave Equation in Linear Media (1 of 2)

Since the curl equations predict propagation, it makes sense that the wave equation is derived by combining the curl equations.

$$\nabla \times \vec{E} = -j\omega[\mu]\vec{H} \qquad \nabla \times \vec{H} = j\omega[\varepsilon]\vec{E}$$

Solve for $\vec{H}$

$$\vec{H} = -\frac{1}{j\omega}[\mu]^{-1}\nabla \times \vec{E}$$

Substitute H

$$\nabla \times \left(-\frac{1}{j\omega}[\mu]^{-1}\nabla \times \vec{E} \right) = j\omega[\varepsilon]\vec{E}$$

Wave Equation in Linear Media (2 of 2)

The last equation is simplified to arrive at the final equation for waves in linear media.

$$\nabla \times [\mu]^{-1} \nabla \times \vec{E} = \omega^2 [\varepsilon] \vec{E}$$

This equation is not very useful for performing derivations. It is typically used in numerical computations.

Note: This cannot be simplified further because the permeability is a function of position and cannot be brought outside of the curl operation.

$$\nabla \times [\mu]^{-1} \nabla \times \vec{E} = \omega^2 [\varepsilon] \vec{E}$$

Wave Equation in LHI Media (1 of 2)

In *linear, homogeneous, and isotropic* (LHI) media, two important simplifications can be made.

First, in isotropic media the permeability μ and permittivity ε reduce to scalar quantities.

$$\nabla \times \mu^{-1} \nabla \times \vec{E} = \omega^2 \varepsilon \vec{E}$$

Second, in homogeneous media, μ is a constant and can be brought to the outside of the curl operation and then brought to the right-hand side of the equation.

$$\nabla \times \nabla \times \vec{E} = \omega^2 \mu \varepsilon \vec{E}$$

Wave Equation in LHI Media (2 of 2)

Now apply the vector identity $\nabla \times \nabla \times \vec{A} = \nabla\left(\nabla \cdot \vec{A}\right) - \nabla^2 \vec{A}$.

$$\nabla \times \nabla \times \vec{E} = \omega^2 \mu\varepsilon \vec{E}$$

$$\nabla\left(\nabla \bullet \vec{E}\right) - \nabla^2 \vec{E} = \omega^2 \mu\varepsilon \vec{E}$$

$$\nabla^2 \vec{E} + \omega^2 \mu\varepsilon \vec{E} = \nabla\left(\nabla \bullet \vec{E}\right)$$

$$\boxed{\nabla^2 \vec{E} + \omega^2 \mu\varepsilon \vec{E} = 0}$$

In LHI media, the divergence equation can be written in terms of $\vec{E}$.

$$\nabla \bullet \vec{D} = 0$$

$$\nabla \bullet \left(\varepsilon \vec{E}\right) = 0$$

$$\varepsilon\left(\nabla \bullet \vec{E}\right) = 0$$

$$\nabla \bullet \vec{E} = 0$$

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Wave Number k and Propagation Constant γ

The term $\omega\sqrt{\mu\varepsilon}$ can be expressed two ways.

$$k^2 = \omega^2 \mu\varepsilon \quad \text{or} \quad \gamma^2 = -\omega^2 \mu\varepsilon \qquad \gamma = jk$$

This provides a way to write the wave equation more simply as

$$\nabla^2 \vec{E} + k^2 \vec{E} = 0 \quad \text{or} \quad \nabla^2 \vec{E} - \gamma^2 \vec{E} = 0$$

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Solution to the Wave Equation

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Components Decouple in LHI Media

The wave equation can be expanded in Cartesian coordinates as

$$\nabla^2 \vec{E} + k^2 \vec{E} = 0$$

$$\nabla^2 \left(E_x \hat{a}_x + E_y \hat{a}_y + E_z \hat{a}_z \right) + k^2 \left(E_x \hat{a}_x + E_y \hat{a}_y + E_z \hat{a}_z \right) = 0$$

$$\nabla^2 E_x \hat{a}_x + \nabla^2 E_y \hat{a}_y + \nabla^2 E_z \hat{a}_z + k^2 E_x \hat{a}_x + k^2 E_y \hat{a}_y + k^2 E_z \hat{a}_z = 0$$

$$\left(\nabla^2 E_x + k^2 E_x \right) \hat{a}_x + \left(\nabla^2 E_y + k^2 E_y \right) \hat{a}_y + \left(\nabla^2 E_z + k^2 E_z \right) \hat{a}_z = 0$$

$$\nabla^2 E_x + k^2 E_x = 0$$

$$\nabla^2 E_y + k^2 E_y = 0$$

$$\nabla^2 E_z + k^2 E_z = 0$$

Observe that the different field components have decoupled from each other.

All three equations have the same numerical form, so they all have the same general solution.

Therefore, only one solution needs to be obtained.

$$\boxed{\nabla^2 E + k^2 E = 0}$$

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General Solution to Scalar Wave Equation

The final wave equation for LHI media is

$$\nabla^2 E + k^2 E = 0$$

This could be handed off to a mathematician to obtain the following general solution.

$$E(\vec{r}) = E_0^+ e^{-j\vec{k}\bullet\vec{r}} + E_0^- e^{+j\vec{k}\bullet\vec{r}}$$

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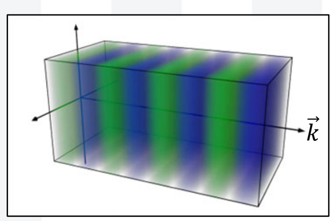
$$E(\vec{r}) = \underbrace{E_0^+ e^{-j\vec{k}\bullet\vec{r}}}_{\text{forward wave}} + E_0^- e^{+j\vec{k}\bullet\vec{r}}$$



forward wave

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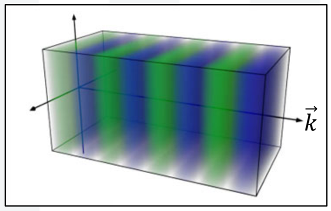
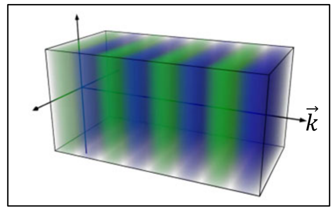


forward wave



backward wave

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General Solution to Vector Wave Equation

Given the solution to the scalar wave equation, the solutions for all three field components can be immediately written.

$$E_x(\vec{r}) = E_x^+ e^{-j\vec{k}\bullet\vec{r}} + E_x^- e^{+j\vec{k}\bullet\vec{r}}$$

$$E_y(\vec{r}) = E_y^+ e^{-j\vec{k}\bullet\vec{r}} + E_y^- e^{+j\vec{k}\bullet\vec{r}}$$

$$E_z(\vec{r}) = E_z^+ e^{-j\vec{k}\bullet\vec{r}} + E_z^- e^{+j\vec{k}\bullet\vec{r}}$$

These three equations are assembled into a single vector equation.

$$\begin{aligned}\vec{E}(\vec{r}) &= E_x(\vec{r})\hat{a}_x + E_y(\vec{r})\hat{a}_y + E_z(\vec{r})\hat{a}_z \\ &= \underbrace{\vec{E}_0^+ e^{-j\vec{k}\bullet\vec{r}}}_{\text{forward wave}} + \underbrace{\vec{E}_0^- e^{+j\vec{k}\bullet\vec{r}}}_{\text{backward wave}}\end{aligned}$$

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General Expression for a Plane Wave

The solution to the wave equation gave two plane waves. From the forward wave, the general expression for plane waves can be extracted.

$$\vec{E}(\vec{r}) = \vec{P}e^{-j\vec{k}\bullet\vec{r}}$$ Frequency-domain

$$\vec{E}(\vec{r},t) = \vec{P}\cos\left(\omega t - \vec{k}\bullet\vec{r}\right)$$ Time-domain

The various parameters are defined as

$\vec{r} = x\hat{a}_x + y\hat{a}_y + z\hat{a}_z \equiv \text{position}$

$\vec{E} \equiv \text{total electric field intensity}$

$\vec{P} \equiv \text{polarization vector}$

$\vec{k} \equiv \text{wave vector}$

$\omega = 2\pi f \equiv \text{angular frequency}$

$t \equiv \text{time}$

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Magnetic Field Component $\vec{H}$

Given that the electric field component of a plane wave is written as

$$\vec{E}(\vec{r}) = \vec{P}e^{-j\vec{k}\bullet\vec{r}}$$

The magnetic field component is derived by substituting this solution into Faraday's law.

$$\nabla\times\vec{E} = -j\omega\mu\vec{H}$$

$$\downarrow$$

$$\nabla\times\left(\vec{P}e^{-j\vec{k}\bullet\vec{r}}\right) = -j\omega\mu\vec{H} \quad \rightarrow \quad \boxed{\vec{H} = \frac{1}{\omega\mu}\left(\vec{k}\times\vec{P}\right)e^{-j\vec{k}\bullet\vec{r}}}$$

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Solution in Terms of the Propagation Constant γ

The wave equation and it's solution in terms of γ is

$$\nabla^2 E - \gamma^2 E = 0 \quad \rightarrow \quad E(\vec{r}) = \underbrace{E_0^+ e^{-\vec{\gamma}\bullet\vec{r}}}_{\text{forward wave}} + \underbrace{E_0^- e^{+\vec{\gamma}\bullet\vec{r}}}_{\text{backward wave}}$$

The general expression for a plane wave is

$$\vec{E}(\vec{r}) = \vec{P} e^{-\vec{\gamma}\bullet\vec{r}}$$

Frequency-domain

The magnetic field component is

$$\vec{H} = \frac{1}{j\omega\mu}\left(\vec{\gamma}\times\vec{P}\right)e^{-\vec{\gamma}\bullet\vec{r}}$$

The wave vector and propagation constant are related through

$$\vec{\gamma} = j\vec{k}$$

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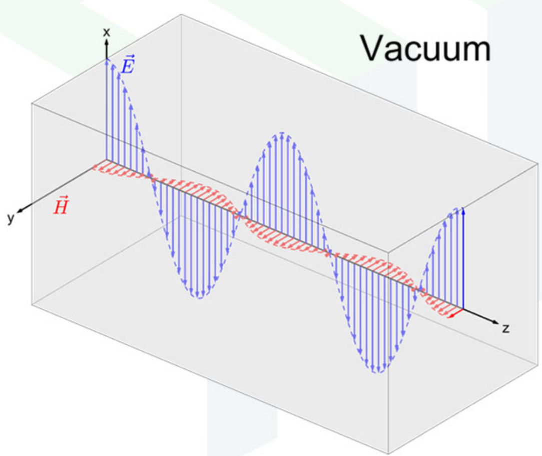
Visualization of an EM Wave (1 of 2)

People tend to draw and think of electromagnetic waves this way…



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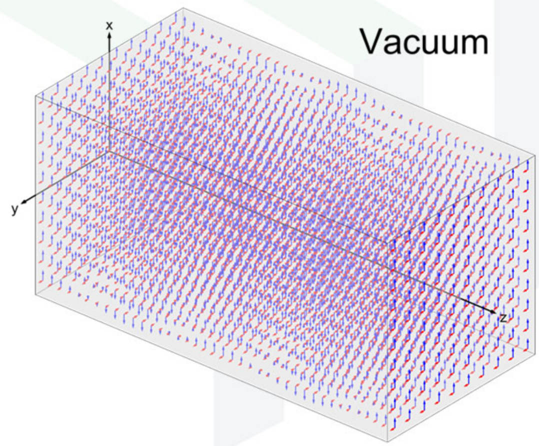
Visualization of an EM Wave (2 of 2)

However, this is a more realistic visualization. It is important to remember that plane waves are volumetric and of infinite extent in all directions.



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