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Electromagnetics:
Electromagnetic Field Theory

Electrostatic Boundary Conditions

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Outline

- General classes of electromagnetic materials
- Boundary conditions for dielectric-dielectric interface
- Refraction of static fields at a dielectric-dielectric interface
- Boundary conditions for dielectric-conductor interface
- Examples

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General Classes of Electromagnetic Materials

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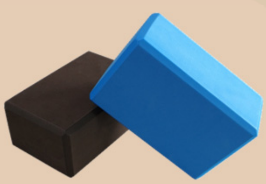
Classification by Conductivity

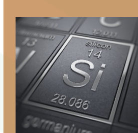
Insulator

$\sigma \ll 1$



- No free charges
- Opposes current
- Most dielectrics are insulating

Semiconductor



- Often switchable and tunable conductivity
- Silicon, gallium arsenide, etc.

Conductor

$\sigma \gg 1$

- Many free charges
- Easily conducts current
- Most metals are conducting

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Boundary Conditions for Dielectric-Dielectric Interface

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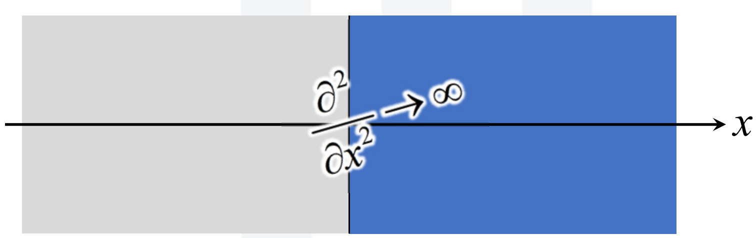
What Are Boundary Conditions?

We often solve electromagnetic problems using differential equations.

$$\frac{d^2E}{dz^2} + k^2E = 0 \qquad \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0$$

The problem is that derivatives are infinite at discontinuities.



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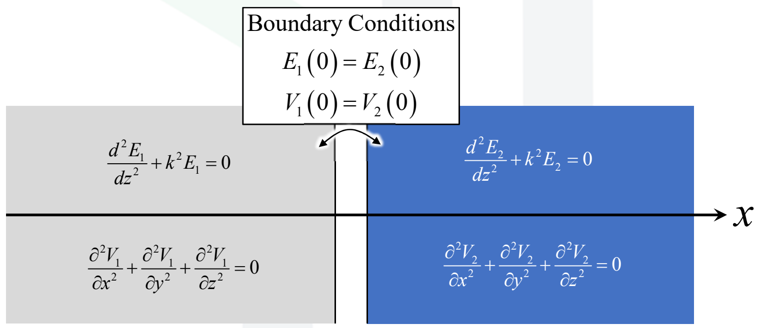
What Are Boundary Conditions?

We are forced to solve our differential equations in each homogeneous region separately.

…and then connect our solutions via boundary conditions.



Deriving Boundary Conditions

Integral equations do not require boundary conditions as long as they do not contain derivatives.

For this reason, we will derive our boundary conditions using Maxwell's equations in integral form.

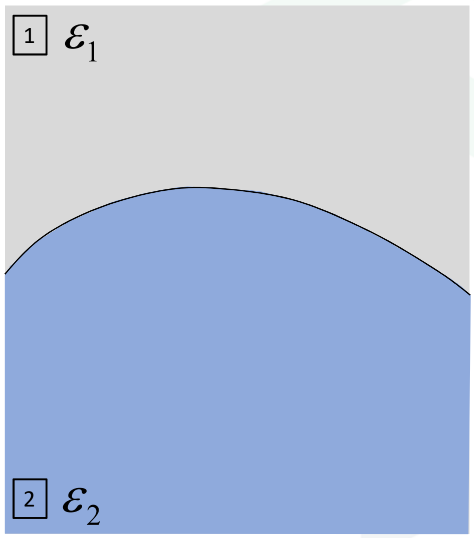
$$0 = \oint_L \vec{E} \bullet d\vec{\ell} \quad \Longrightarrow$$

Boundary conditions for tangential electric fields.

$$Q = \oiint_S \vec{D} \bullet d\vec{s} \quad \Longrightarrow$$

Boundary conditions for normal electric fields.

Analysis Setup



Let's examine the interface between two different dielectrics.

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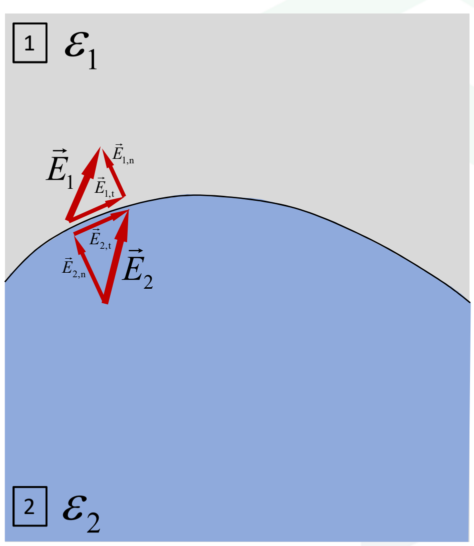
Analysis Setup



Let's examine the interface between two different dielectrics.

We wish to examine the relation between electric fields on either side of the interface, so that if one is known the other can be calculated.

It will be useful to separate the field on either side of the interface into tangential and normal components.

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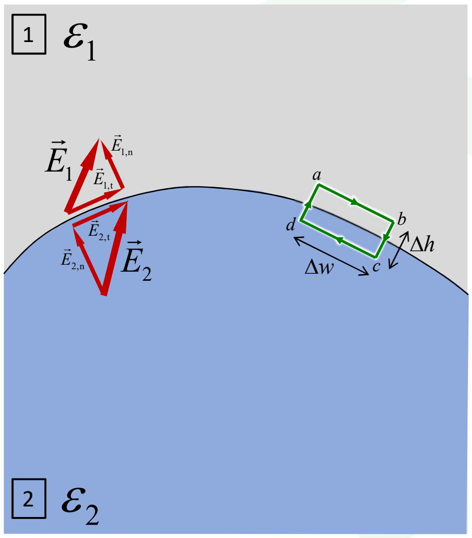
Derivation of Tangential BCs



Apply the following integral to a closed path spanning some section of the interface.

$$0 = \oint_L \vec{E} \bullet d\vec{\ell}$$

$$= \int_a^b \vec{E} \bullet d\vec{\ell} + \int_b^0 \vec{E} \bullet d\vec{\ell} + \int_0^c \vec{E} \bullet d\vec{\ell}$$

$$+ \int_c^d \vec{E} \bullet d\vec{\ell} + \int_d^0 \vec{E} \bullet d\vec{\ell} + \int_0^a \vec{E} \bullet d\vec{\ell}$$

$$= E_{1,t}\Delta w - E_{1,n}\frac{\Delta h}{2} - E_{2,n}\frac{\Delta h}{2}$$

$$- E_{2,t}\Delta w + E_{2,n}\frac{\Delta h}{2} + E_{1,n}\frac{\Delta h}{2}$$

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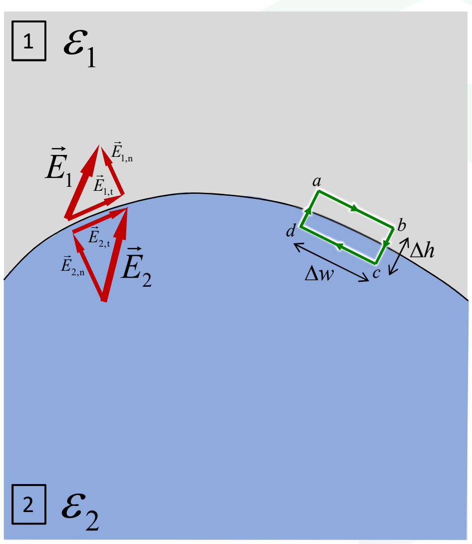
Derivation of Tangential BCs



Cancel like terms with opposite sign.

$$0 = E_{1,t}\Delta w - E_{1,n}\frac{\Delta h}{2} - E_{2,n}\frac{\Delta h}{2}$$

$$- E_{2,t}\Delta w + E_{2,n}\frac{\Delta h}{2} + E_{1,n}\frac{\Delta h}{2}$$

$$= E_{1,t}\Delta w - E_{2,t}\Delta w$$

From this, it is concluded that the tangential component of $\vec{E}$ is continuous across the interface.

$$\boxed{\vec{E}_{1,t} = \vec{E}_{2,t}}$$

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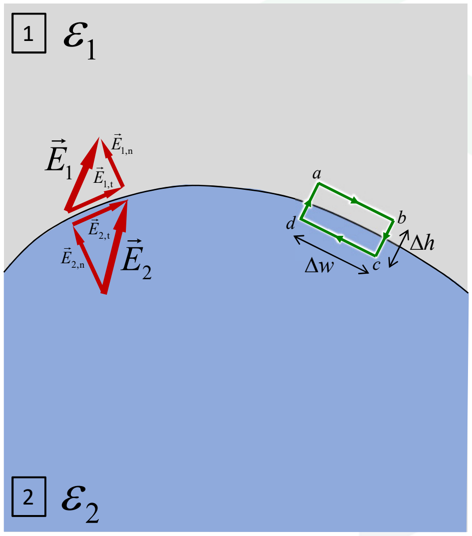
Derivation of Tangential BCs



Apply the constitutive relation to get the boundary condition for $\vec{D}$.

$$\vec{E}_{1,t} = \vec{E}_{2,t}$$

$$\boxed{\frac{\vec{D}_{1,t}}{\varepsilon_1} = \frac{\vec{D}_{2,t}}{\varepsilon_2}}$$

The tangential component of $\vec{D}$ is NOT continuous across the interface, but the ratio of $\vec{D}_t/\varepsilon$ is.

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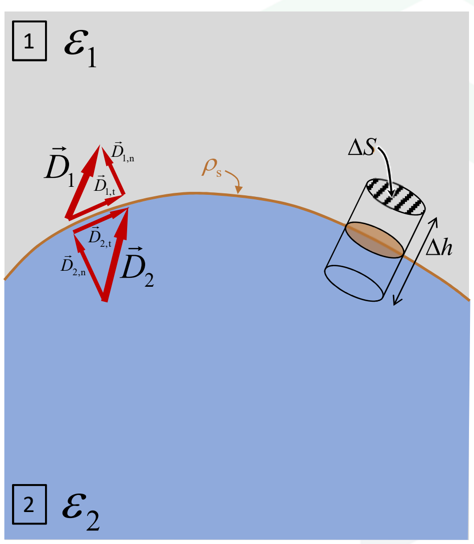
Derivation of Normal BCs



Place some charge density ρ_s on the surface.

Apply the following surface integral to a pillbox spanning the interface.

$$Q = \oiint_S \vec{D} \bullet d\vec{s}$$

Separate the closed-surface integral into three separate surface integrals.

$$Q = \iint_{\text{top}} \vec{D} \bullet d\vec{s} + \iint_{\text{bottom}} \vec{D} \bullet d\vec{s} + \iint_{\text{sides}} \vec{D} \bullet d\vec{s}$$

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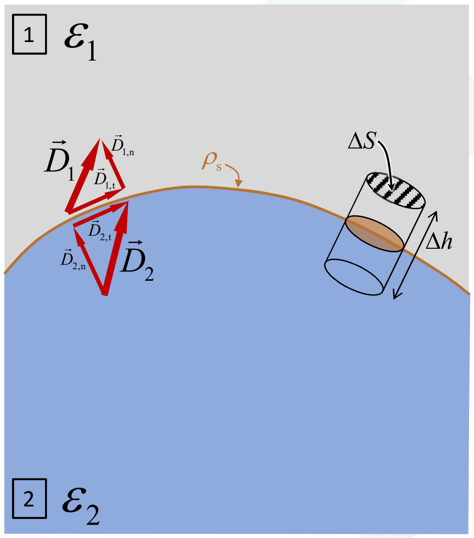
Derivation of Normal BCs



In the limit as $\Delta h \rightarrow 0$

$$Q = \iint_{\text{top}} \vec{D} \bullet d\vec{s} + \iint_{\text{bottom}} \vec{D} \bullet d\vec{s} + \cancel{\iint_{\text{sides}} \vec{D} \bullet d\vec{s}}$$

$$= D_{1,n}\Delta S - D_{2,n}\Delta S$$

The total charge encompassed within the pillbox is

$$Q = \rho_s \Delta S$$

Putting these together gives

$$\rho_s \Delta S = D_{1,n}\Delta S - D_{2,n}\Delta S$$

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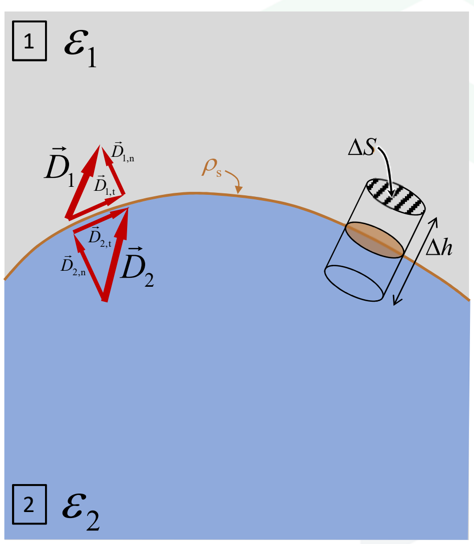
Derivation of Normal BCs



The final boundary condition is then

$$\rho_s \Delta S = D_{1,n}\Delta S - D_{2,n}\Delta S$$

$$\boxed{\vec{D}_{1,n} - \vec{D}_{2,n} = \rho_s}$$

In the absence of charge (i.e. ρ_s = 0)

$$\boxed{\vec{D}_{1,n} = \vec{D}_{2,n} \qquad (\rho_s = 0)}$$

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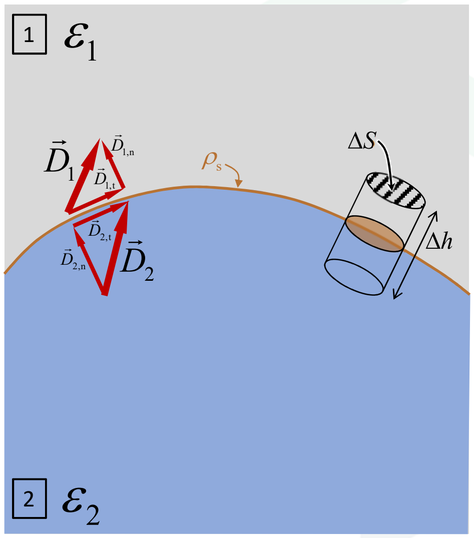
Derivation of Normal BCs



Apply the constitutive relation to get the boundary condition for $\vec{E}$.

$$\vec{D}_{1,n} - \vec{D}_{2,n} = \rho_s$$

$$\boxed{\varepsilon_1 \vec{E}_{1,n} - \varepsilon_2 \vec{E}_{2,n} = \rho_s}$$

In the absence of charge (i.e. $\rho_s = 0$)

$$\boxed{\varepsilon_1 \vec{E}_{1,n} = \varepsilon_2 \vec{E}_{2,n} \qquad (\rho_s = 0)}$$

The normal component of $\vec{E}$ is NOT continuous across the interface, but the product of $\varepsilon \vec{E}_n$ is.

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Refraction of Static Fields at a Dielectric-Dielectric Interface

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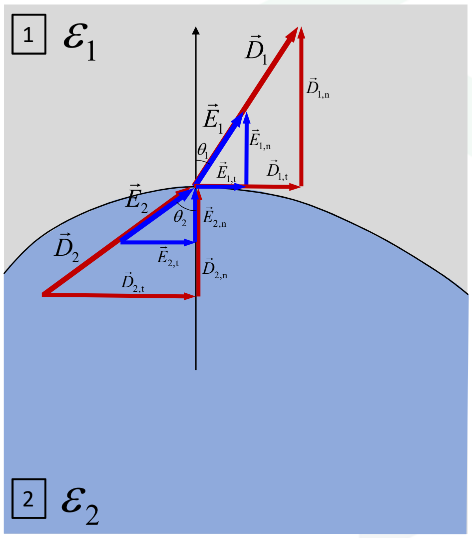
Analysis Setup



We want a single equation that relates θ_1, θ_2, ε_1, and ε_2 without any field quantities in the equation.

Given the angles θ_1 and θ_2, the field components can be written as

$$\vec{E}_1 = E_{1,t}\hat{a}_t + E_{1,n}\hat{a}_n = (E_1 \sin\theta_1)\hat{a}_t + (E_1 \cos\theta_1)\hat{a}_n$$

$$\vec{E}_2 = E_{2,t}\hat{a}_t + E_{2,n}\hat{a}_n = (E_2 \sin\theta_2)\hat{a}_t + (E_2 \cos\theta_2)\hat{a}_n$$

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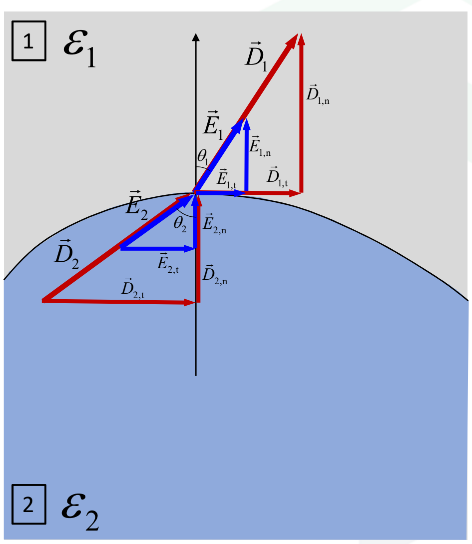
Derivation of Refraction Law



Apply the boundary conditions for tangential components.

$$E_{1,t} = E_{2,t}$$

$$E_1 \sin\theta_1 = E_2 \sin\theta_2$$

Apply the boundary conditions for normal components.

$$\varepsilon_1 E_{1,n} = \varepsilon_2 E_{2,n}$$

$$\varepsilon_1 E_1 \cos\theta_1 = \varepsilon_2 E_2 \cos\theta_2$$

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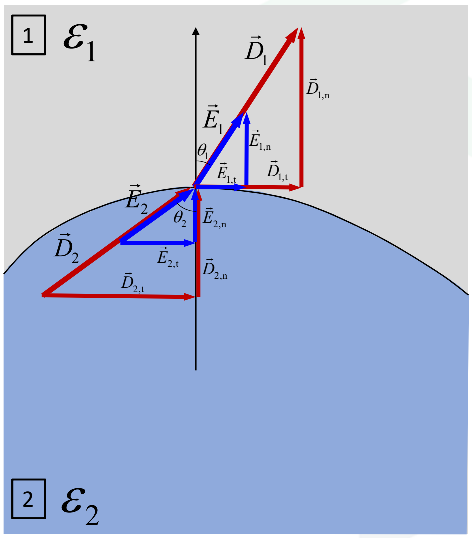
Derivation of Refraction Law



We now have

$$E_1 \sin\theta_1 = E_2 \sin\theta_2$$
$$\varepsilon_1 E_1 \cos\theta_1 = \varepsilon_2 E_2 \cos\theta_2$$

Divide these equations to get

$$\frac{E_1 \sin\theta_1}{\varepsilon_1 E_1 \cos\theta_1} = \frac{E_2 \sin\theta_2}{\varepsilon_2 E_2 \cos\theta_2}$$

Simplify

$$\boxed{\frac{\tan\theta_1}{\varepsilon_1} = \frac{\tan\theta_2}{\varepsilon_2}}$$

This is NOT Snell's law.

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Boundary Conditions for Dielectric-Conductor Interface

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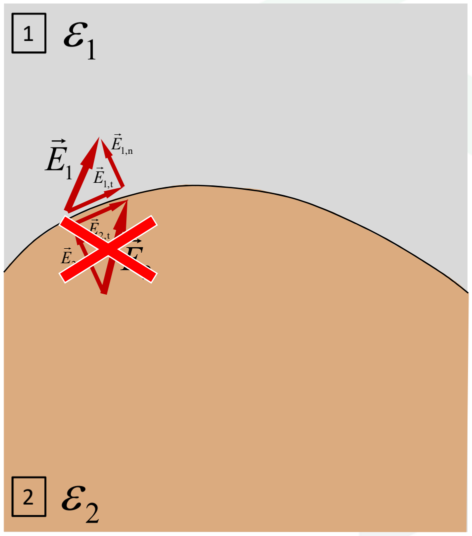
Analysis Setup



We start like we did for the dielectric-dielectric interface.

Assume the conductor is perfect.

$$\sigma \to \infty$$

Recall Ohm's law

$$\vec{J} = \sigma \vec{E}$$

In order for $\vec{J}$ not to be infinite, $\vec{E} = 0$ inside the conductor.

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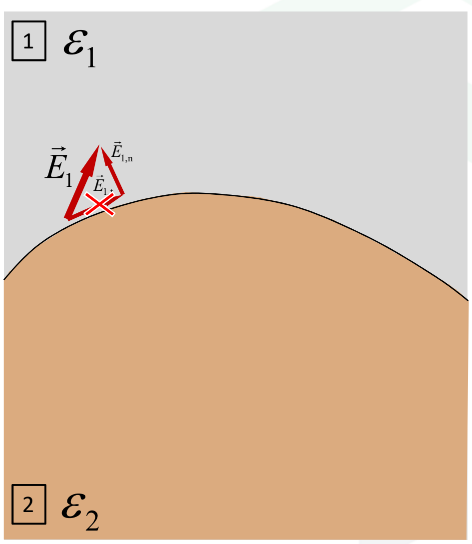
Analysis Setup



If $E_{2,t}$ is zero, then

$$E_{1,t} = 0$$

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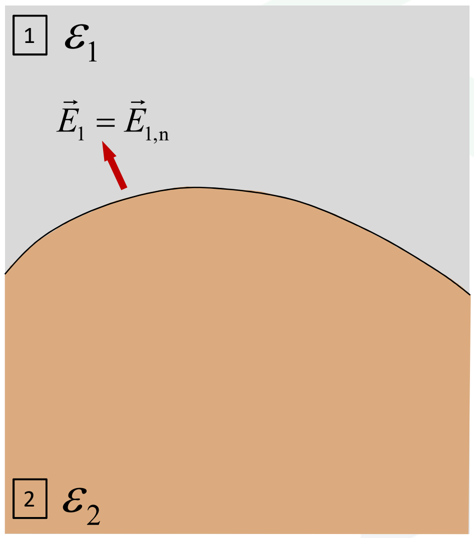
Analysis Setup



There can only be a normal component for the electric field at the interface with a perfect conductor.

$$\vec{E}_1 = E_{1,n}\hat{a}_n$$

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Notes About Perfect Conductors

- No electric field can exist inside of a perfect conductor (i.e. $\vec{E} = 0$).
- Electric potential V is constant throughout a perfect conductor (i.e. $\nabla^2 V = 0$).
- The electric field at the boundary has no tangential component. The electric field can only be normal at the interface to a metal.

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Examples

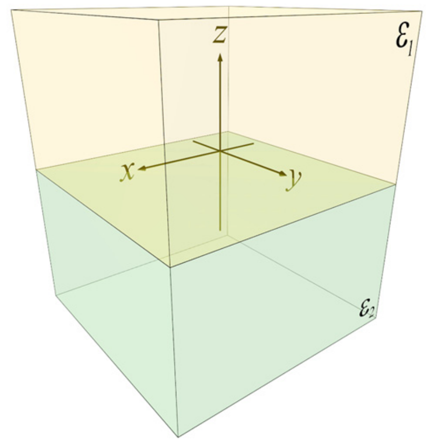
Example #1

Let there be an interface between two semi-infinite media in the x-y plane. The dielectric constant of the first medium is 2.0 and the second medium is 4.4.

1. Given that the electric flux density in medium 1 is $\vec{D}_1 = 2.1\hat{a}_x + 0.7\hat{a}_y + 1.5\hat{a}_z$, calculate the electric flux density in medium 2, $\vec{D}_2$.
2. Calculate the angle $\vec{D}_1$ makes with the interface.
3. Using the law of refraction, calculate the angle $\vec{D}_2$ makes with the interface.

Example #1 – Problem Setup

Start by visualizing the problem and setting up the coordinates.

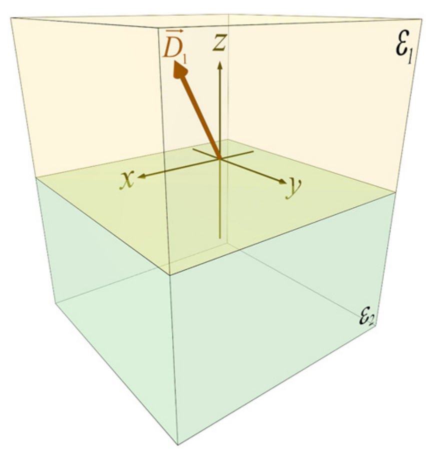


Example #1 – Problem Setup

We start by visualizing the problem and setting up the coordinates.

Plot $\vec{D}_1$

$$\vec{D}_1 = 2.1\hat{a}_x + 0.7\hat{a}_y + 1.5\hat{a}_z$$

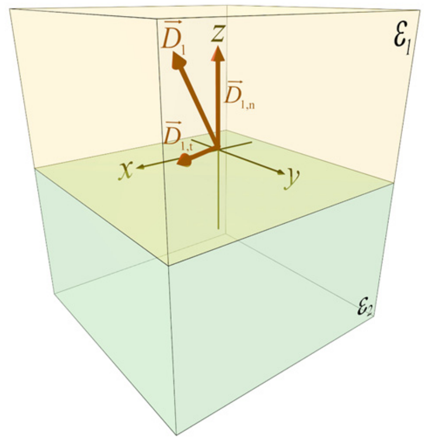


Example #1 – Part 1

Separate $\vec{D}_1$ into tangential and normal components.

$$\vec{D}_1 = \underbrace{2.1\hat{a}_x + 0.7\hat{a}_y}_{\text{Tangential}} + \underbrace{1.5\hat{a}_z}_{\text{Normal}}$$

$\vec{D}_{1,t} = 2.1\hat{a}_x + 0.7\hat{a}_y$ $\vec{D}_{1,n} = 1.5\hat{a}_z$



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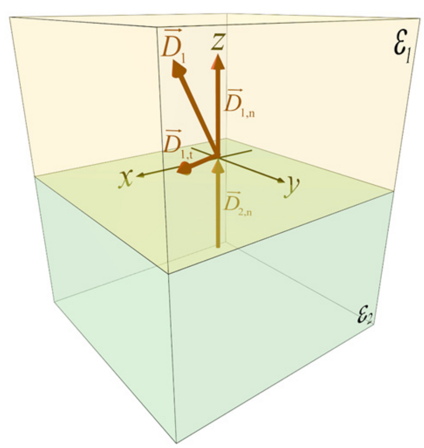
Example #1 – Part 1

Apply boundary condition for normal component.

$$\vec{D}_{1,n} = \vec{D}_{2,n}$$

$$1.5\hat{a}_z = \vec{D}_{2,n}$$



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Example #1 – Part 1

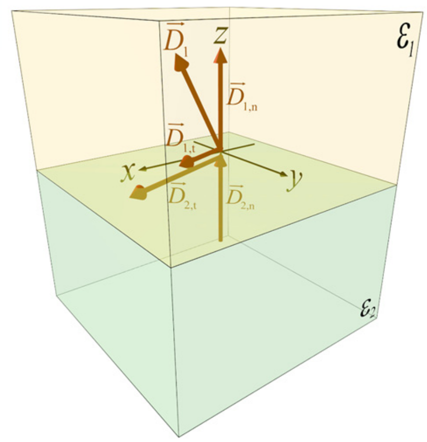
Apply boundary condition for tangential component.

$$\frac{\vec{D}_{1,\mathrm{t}}}{\varepsilon_1} = \frac{\vec{D}_{2,\mathrm{t}}}{\varepsilon_2}$$

$$\vec{D}_{2,\mathrm{t}} = \frac{\varepsilon_2}{\varepsilon_1} \vec{D}_{1,\mathrm{t}}$$

$$\vec{D}_{2,\mathrm{t}} = \frac{4.4}{2.0}\left(2.1\hat{a}_x + 0.7\hat{a}_y\right)$$

$$\vec{D}_{2,\mathrm{t}} = 4.62\hat{a}_x + 1.54\hat{a}_y$$



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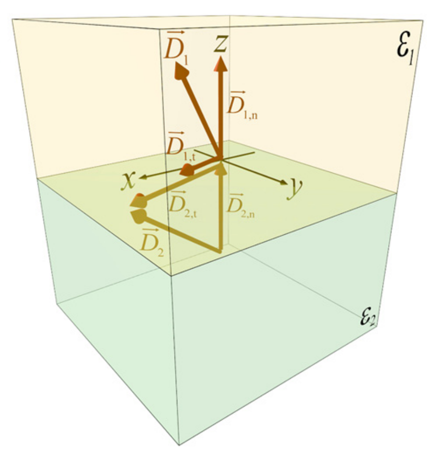
Example #1 – Part 1

Gather both components to get overall $\vec{D}_2$.

$$\vec{D}_2 = \vec{D}_{2,\mathrm{t}} + \vec{D}_{2,\mathrm{n}}$$

$$\vec{D}_2 = \left(4.62\hat{a}_x + 1.54\hat{a}_y\right) + \left(1.5\hat{a}_z\right)$$

$$\boxed{\vec{D}_2 = 4.62\hat{a}_x + 1.54\hat{a}_y + 1.5\hat{a}_z}$$



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Example #1 – Part 2

Calculate the angle θ_1 of $\vec{D}_1$.

Recall the property of dot products.

$$\vec{A} \bullet \vec{B} = |\vec{A}||\vec{B}|\cos\theta_{AB}$$

We can calculate θ_1 by letting

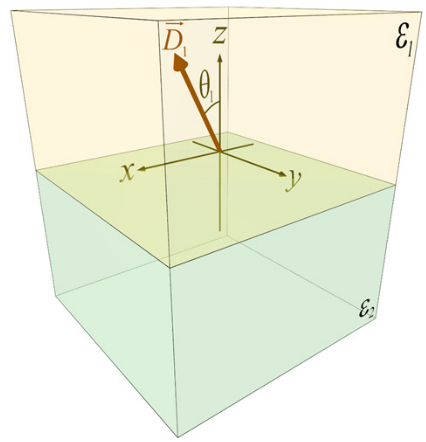
$$\vec{A} = \vec{D}_1$$

$$\vec{B} = \hat{a}_z$$

$$\theta_{AB} = \theta_1$$

Our dot product becomes

$$\vec{D}_1 \bullet \hat{a}_z = |\vec{D}_1||\hat{a}_z|\cos\theta_1$$



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Example #1 – Part 2

Continued…

Solve the dot product equation for θ_1.

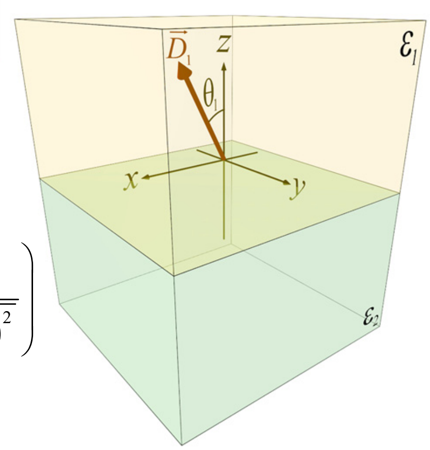
$$\vec{D}_1 \bullet \hat{a}_z = |\vec{D}_1||\hat{a}_z|\cos\theta_1$$

$$D_z = |\vec{D}_1|\cos\theta_1$$

$$\theta_1 = \cos^{-1}\left(\frac{D_z}{|\vec{D}_1|}\right)$$

$$\theta_1 = \cos^{-1}\left(\frac{1.5}{\sqrt{(2.1)^2 + (0.7)^2 + (1.5)^2}}\right)$$

$$\boxed{\theta_1 = 55.9^\circ}$$



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Example #1 – Part 2

Calculate the angle θ_2 of $\vec{D}_2$.

The law of refraction is

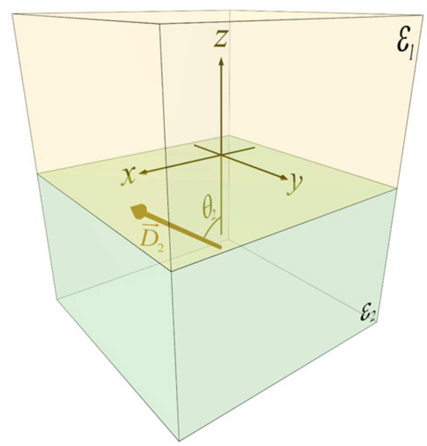
$$\frac{\tan\theta_1}{\varepsilon_1} = \frac{\tan\theta_2}{\varepsilon_2}$$

Solving this for θ_2 gives

$$\theta_2 = \tan^{-1}\left(\frac{\varepsilon_2}{\varepsilon_1}\tan\theta_1\right)$$

$$\theta_2 = \tan^{-1}\left(\frac{4.4}{2.0}\tan 55.9°\right)$$

$$\boxed{\theta_2 = 72.9°}$$



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