



Electromagnetics:
Electromagnetic Field Theory

Forces on Current Elements

1

Outline

- Force on a current element
- Force between two current elements
- Force between two current loops
- Magnetic torque and moment
- Magnetic dipole

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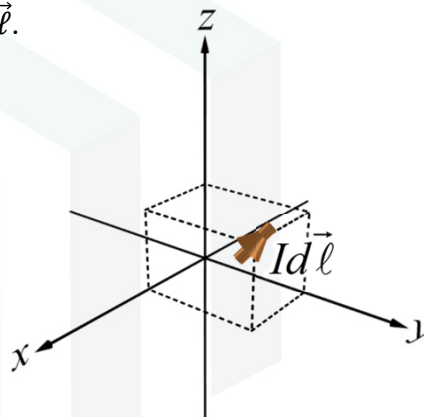
Force on a Current Element

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Setup

Suppose there exists a current element $I d\vec{\ell}$.



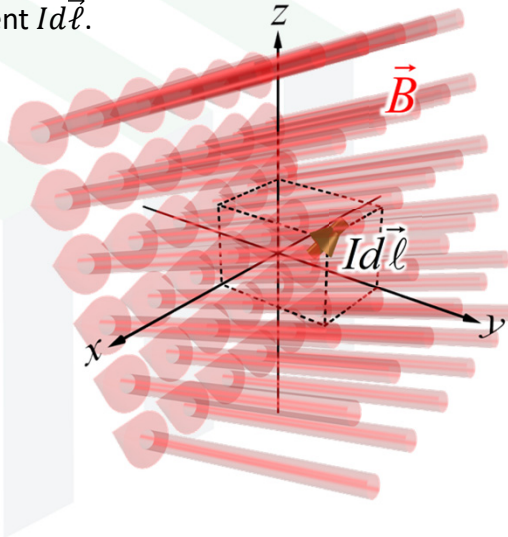
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Setup

Suppose there exists a current element $I d\vec{\ell}$.

...and a magnetic field described by \vec{B} .



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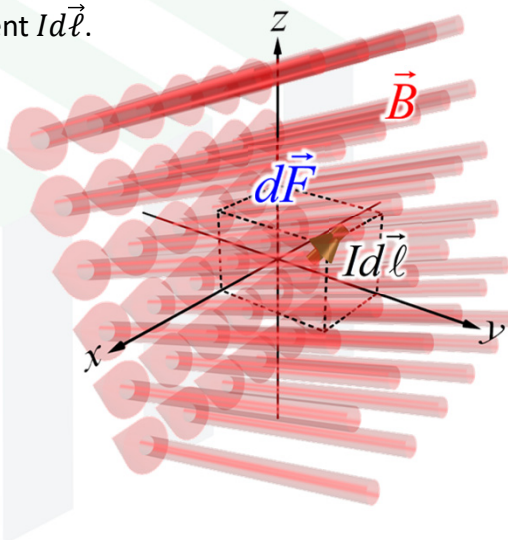
Setup

Suppose there exists a current element $I d\vec{\ell}$.

...and a magnetic field described by \vec{B} .

The magnetic field \vec{B} will put a force \vec{F} on the current element $I d\vec{\ell}$ according to

$$d\vec{F} = (I d\vec{\ell}) \times \vec{B}$$



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Derivation of Differential Force

Start with the Lorentz force law for magnetic fields

$$\vec{F} = Q\vec{u} \times \vec{B}$$

Differentiate this equation to get

$$d\vec{F} = dQ\vec{u} \times \vec{B}$$

What is $dQ\vec{u}$ → It is a moving differential charge.

Recall our differential currents

$$\begin{aligned} Id\vec{\ell} &= \vec{K}ds = \vec{J}dv \\ &= (\rho_v\vec{u})dv && \text{Because } \vec{J} = \rho_v\vec{u} \\ &= (\rho_v dv)\vec{u} && \text{Swap terms} \\ &= (dQ)\vec{u} && \text{Because } dQ = \rho_v dv \end{aligned} \quad \left. \vphantom{\begin{aligned} Id\vec{\ell} &= \vec{K}ds = \vec{J}dv \\ &= (\rho_v\vec{u})dv \\ &= (\rho_v dv)\vec{u} \\ &= (dQ)\vec{u} \end{aligned}} \right\} Id\vec{\ell} = dQ\vec{u}$$

Derivation of Differential Force

$$Id\vec{\ell} = dQ\vec{u}$$

This shows that a differential charge dQ moving at velocity \vec{u} (convection current) is equivalent to conduction current $Id\vec{\ell}$.

$$d\vec{F} = (dQ\vec{u}) \times \vec{B}$$

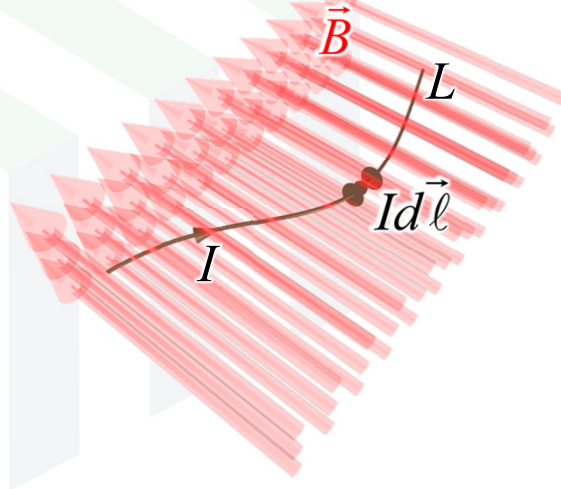
$$\boxed{d\vec{F} = (Id\vec{\ell}) \times \vec{B}}$$

Total Force on a Line Current

The total force on a line current is obtained by integrating the differential force over the length of the wire.

$$\vec{F} = \int_L d\vec{F}$$

$$\vec{F} = \int_L (I d\vec{\ell}) \times \vec{B}$$



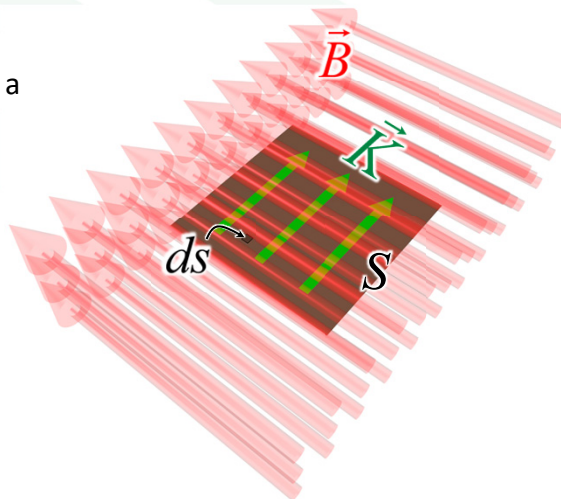
Total Force on a Surface Current

Recall the current density expressions.

$$I d\vec{\ell} = \vec{K} ds = \vec{J} dv$$

It is no surprise that the total force on a surface current is

$$\vec{F} = \iint_S (\vec{K} ds) \times \vec{B}$$



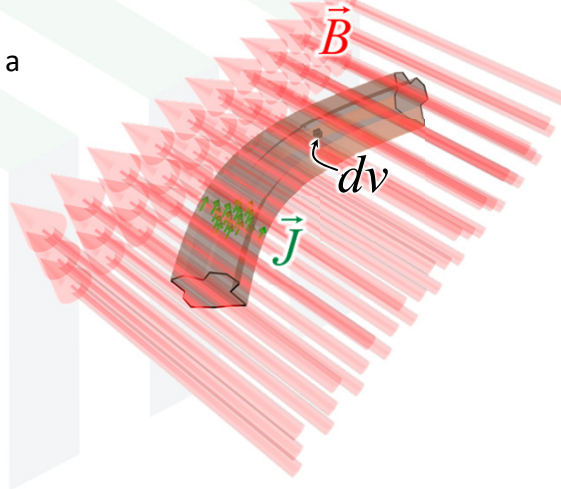
Total Force on a Volume Current

Recall the current density expressions.

$$I d\vec{\ell} = \vec{K} ds = \vec{J} dv$$

It is no surprise that the total force on a volume current is

$$\vec{F} = \iiint_V (\vec{J} dv) \times \vec{B}$$



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Example #1 – Force on a Wire

What is the maximum force that a kitchen magnet can put a 1.0 inch long wire carrying 1 A?

Solution

The average kitchen magnet has a strength on the order of 5 mT.

$$\vec{B} = 5 \text{ mT} = 0.005 \text{ Wb/m}^2$$

The maximum force occurs when the magnetic field is perpendicular to the current. When this is the case,

$$\begin{aligned} \vec{F} &= \int_L (I d\vec{\ell}) \times \vec{B} = \int_0^L IB dz = IBL \\ &= (1 \text{ A}) \left(0.005 \frac{\text{Wb}}{\text{m}^2} \right) (1 \text{ in}) \left(\frac{2.54 \text{ cm}}{1 \text{ in}} \right) \left(\frac{1 \text{ m}}{100 \text{ cm}} \right) = \boxed{127 \mu\text{N}} \end{aligned}$$

This is roughly 1/100 of the weight of a penny.



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Force Between Two Current Elements

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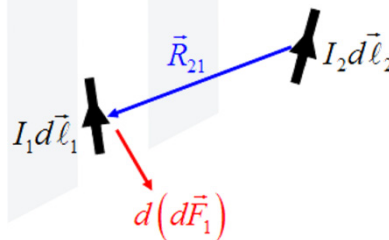
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Problem Setup

Consider two current elements, $I_1 d\vec{\ell}_1$ and $I_2 d\vec{\ell}_2$.

Each produces a magnetic field that puts a force on the other.

Let's calculate the force $d(d\vec{F}_1)$ on $I_1 d\vec{\ell}_1$ due to $I_2 d\vec{\ell}_2$.



EMPossible

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Derivation (1 of 2)

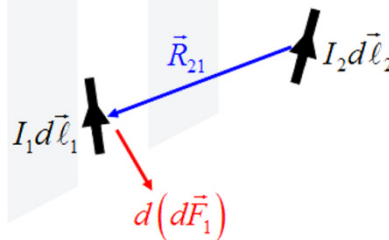
The magnetic field produced by $I_2 d\vec{\ell}_2$ is written using the Biot-Savart law.

$$d\vec{H}_2 = \frac{I_2 d\vec{\ell}_2 \times \hat{a}_{21}}{4\pi R_{21}^2}$$

Write this in terms of \vec{B}_2 using the constitutive relation.

$$d\left(\frac{\vec{B}_2}{\mu}\right) = \frac{I_2 d\vec{\ell}_2 \times \hat{a}_{21}}{4\pi R_{21}^2}$$

$$d\vec{B}_2 = \frac{\mu I_2 d\vec{\ell}_2 \times \hat{a}_{21}}{4\pi R_{21}^2}$$



Derivation (2 of 2)

The force on $I_1 d\vec{\ell}_1$ is

$$d\vec{F}_1 = (I_1 d\vec{\ell}_1) \times \vec{B}_2$$

Differentiate this to get

$$d(d\vec{F}_1) = (I_1 d\vec{\ell}_1) \times d\vec{B}_2$$

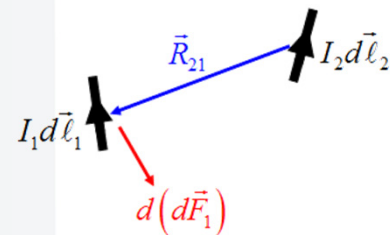
Combine these expressions to get

$$d(d\vec{F}_1) = (I_1 d\vec{\ell}_1) \times \frac{\mu I_2 d\vec{\ell}_2 \times \hat{a}_{21}}{4\pi R_{21}^2}$$

$$d(d\vec{F}_1) = \frac{\mu}{4\pi} \frac{(I_1 d\vec{\ell}_1) \times (I_2 d\vec{\ell}_2) \times \hat{a}_{21}}{R_{21}^2} = \frac{\mu I_1 I_2}{4\pi} \frac{d\vec{\ell}_1 \times d\vec{\ell}_2 \times \hat{a}_{21}}{R_{21}^2}$$

We just derived $d\vec{B}_2$.

$$d\vec{B}_2 = \frac{\mu I_2 d\vec{\ell}_2 \times \hat{a}_{21}}{4\pi R_{21}^2}$$

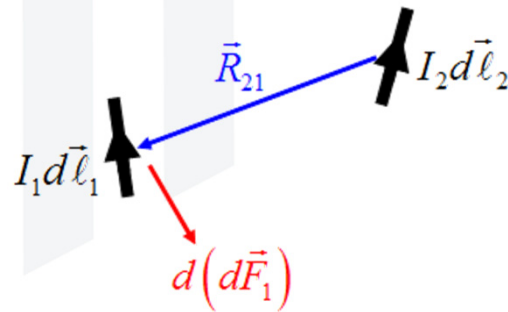


Interpretation

$$d(d\vec{F}_1) = \frac{\mu}{4\pi} \frac{(I_1 d\vec{\ell}_1) \times (I_2 d\vec{\ell}_2) \times \hat{a}_{21}}{R_{21}^2} = \frac{\mu I_1 I_2}{4\pi} \frac{d\vec{\ell}_1 \times d\vec{\ell}_2 \times \hat{a}_{21}}{R_{21}^2}$$

This equation is analogous to Coulomb's law in electrostatics.

The double differential means we have to integrate over the length of both wires to get the total force.



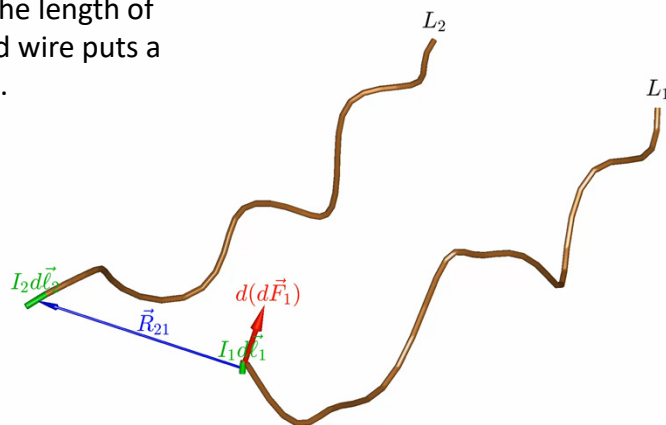
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Total Force Between Two Line Currents

To calculate the total force between two line currents, we must integrate over the length of each wire. Each part of the second wire puts a force on each part of the first wire.

$$\vec{F}_1 = \int_{L_1} \int_{L_2} d(d\vec{F}_1)$$

$$\vec{F}_1 = \frac{\mu I_1 I_2}{4\pi} \int_{L_1} \int_{L_2} \frac{d\vec{\ell}_1 \times d\vec{\ell}_2 \times \hat{a}_{21}}{R_{21}^2}$$



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Example #2 – Force Between Two Wires

Two parallel wires carrying currents I_1 and I_2 are a distance s apart.

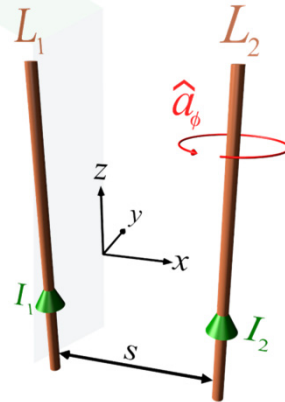
What is the force per meter between these two wires?

Solution

We will solve this by calculating the magnetic field induced by the second wire and then use this result to calculate the force on the first wire.

Using the infinite wire approximation, the magnetic field around the first wire was found to be

$$\vec{H}_2 = \frac{I_2}{2\pi\rho} \hat{a}_\phi$$



Example #2 – Force Between Two Wires

Use the constitutive relation to find \vec{B} around the second wire.

$$\vec{H}_2 = \frac{I_2}{2\pi\rho} \hat{a}_\phi \rightarrow \frac{\vec{B}_2}{\mu} = \frac{I_2}{2\pi\rho} \hat{a}_\phi$$

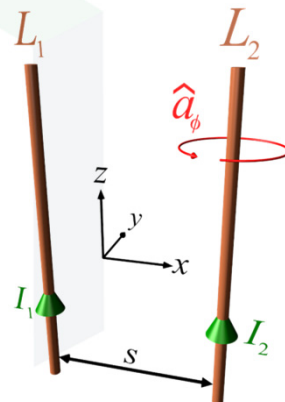
$$\rightarrow \vec{B}_2 = \frac{\mu I_2}{2\pi\rho} \hat{a}_\phi$$

The force this field puts on the first wire is

$$\vec{F}_1 = \int_{L_1} (I_1 d\vec{l}_1) \times \vec{B}_2 = \int_0^L (I_1 dz \hat{a}_z) \times \left(\frac{\mu I_2}{2\pi\rho} \hat{a}_\phi \right)$$

The coordinate systems have been mixed here.

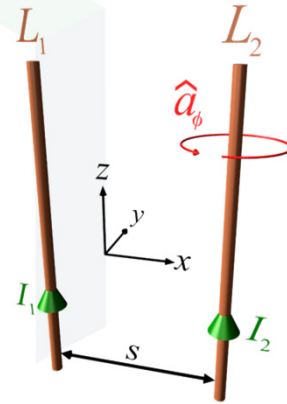
At the first wire $\hat{a}_\phi = -\hat{a}_y$ and $\rho = s$.



Example #2 – Force Between Two Wires

The equation for force becomes

$$\begin{aligned}\vec{F}_1 &= \int_0^L (I_1 dz \hat{a}_z) \times \left(-\frac{\mu I_2}{2\pi s} \hat{a}_y \right) \\ &= \frac{\mu I_1 I_2}{2\pi s} \int_0^L (\hat{a}_z) \times (-\hat{a}_y) dz \\ &= \frac{\mu I_1 I_2}{2\pi s} \int_0^L (\hat{a}_x) dz \\ &= \frac{\mu I_1 I_2}{2\pi s} \hat{a}_x \int_0^L dz \\ &= \frac{\mu I_1 I_2}{2\pi s} \hat{a}_x L\end{aligned}$$



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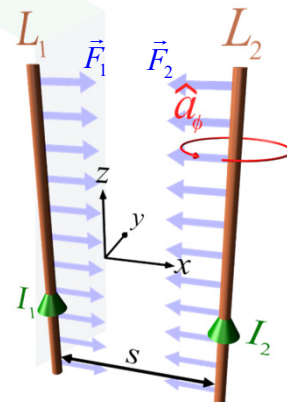
Example #2 – Force Between Two Wires

The force per unit length is

$$\vec{F}_1 = \frac{\mu I_1 I_2}{2\pi s} \hat{a}_x L$$

$$\boxed{\frac{\vec{F}_1}{L} = \frac{\mu I_1 I_2}{2\pi s} \hat{a}_x}$$

Observe that the wires are being attracted toward each other. This is called the *pinch effect*.



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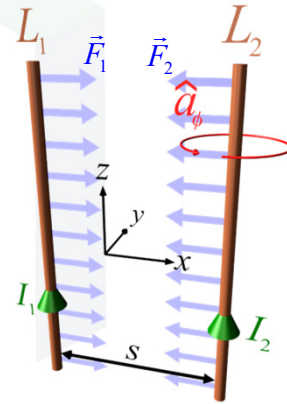
Example #3 – The Numbers

If the wires both carry 1 A, are 1 m long, and are 1 m apart, what is the total force between them?

Approximate Solution

We plug these numbers into the equation we just derived.

$$\begin{aligned}\vec{F}_1 &= \frac{\mu_0 \mu_r I_1 I_2 L}{2\pi s} \hat{a}_x \\ &= \frac{(1.2566 \times 10^{-6} \frac{\text{H}}{\text{m}})(1.0)(1 \text{ A})(1 \text{ A})(1 \text{ m})}{2\pi(1 \text{ m})} \hat{a}_x \\ &= \boxed{2.0 \times 10^{-7} \hat{a}_x \text{ N} = 200 \hat{a}_x \text{ nN}}\end{aligned}$$



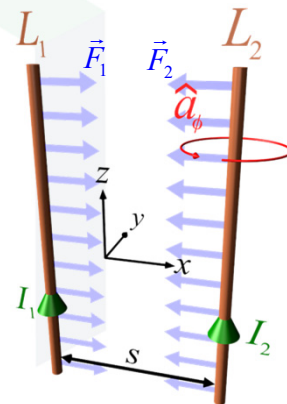
Example #4 – Exact Numbers

If the wires both carry 1 A, are 1 m long, and are 1 m apart, what is the total force between them?

Exact Solution

The rigorous equation to calculate the force between these two wires is

$$\begin{aligned}\vec{F}_1 &= \frac{\mu I_1 I_2}{4\pi} \int_{L_1} \int_{L_2} \frac{d\vec{\ell}_1 \times d\vec{\ell}_2 \times \hat{a}_{21}}{R_{21}^2} \\ &= \frac{\mu I_1 I_2}{4\pi} \int_{L_1} \int_{L_2} d\vec{\ell}_1 \times d\vec{\ell}_2 \times \frac{\vec{R}_{21}}{|\vec{R}_{21}|^3}\end{aligned}$$



Example #4 – Exact Numbers

Determine expressions for each term in the force equation.

$$\vec{F}_1 = \frac{\mu I_1 I_2}{4\pi} \int_{L_1} \int_{L_2} d\vec{\ell}_1 \times d\vec{\ell}_2 \times \frac{\vec{R}_{21}}{|\vec{R}_{21}|^3}$$

$$\mu = \mu_0$$

$$I_1 = 1 \text{ A}$$

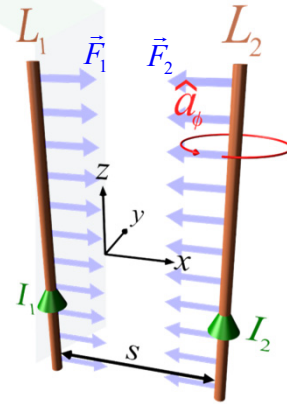
$$I_2 = 1 \text{ A}$$

$$d\vec{\ell}_1 = dz_1 \hat{a}_z$$

$$d\vec{\ell}_2 = dz_2 \hat{a}_z$$

$$\vec{R}_{21} = -s\hat{a}_x + (z_2 - z_1)\hat{a}_z$$

$$\frac{\vec{R}_{21}}{|\vec{R}_{21}|^3} = \frac{-s\hat{a}_x + (z_2 - z_1)\hat{a}_z}{[s^2 + (z_2 - z_1)^2]^{3/2}}$$



Example #4 – Exact Numbers

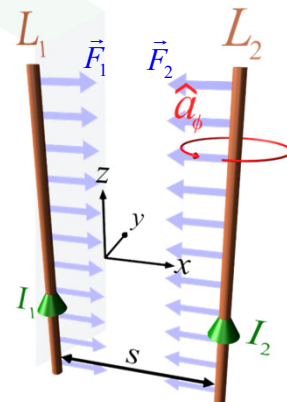
The force equation becomes

$$\vec{F}_1 = \frac{\mu I_1 I_2}{4\pi} \int_{L_1} \int_{L_2} d\vec{\ell}_1 \times d\vec{\ell}_2 \times \frac{\vec{R}_{21}}{|\vec{R}_{21}|^3}$$

$$= \frac{\mu I_1 I_2}{4\pi} \int_0^L \int_0^L dz_1 \hat{a}_z \times \left(dz_2 \hat{a}_z \times \frac{-s\hat{a}_x + (z_2 - z_1)\hat{a}_z}{[s^2 + (z_2 - z_1)^2]^{3/2}} \right)$$

$$= \frac{\mu I_1 I_2}{4\pi} \int_0^L \int_0^L dz_1 \hat{a}_z \times \left(\frac{-s dz_2}{[s^2 + (z_2 - z_1)^2]^{3/2}} \hat{a}_y \right)$$

$$= \frac{\mu s I_1 I_2}{4\pi} \hat{a}_x \int_0^L \int_0^L \frac{dz_1 dz_2}{[s^2 + (z_2 - z_1)^2]^{3/2}}$$



Example #4 – Exact Numbers

If we $L \rightarrow \infty$, we get the approximate result from the last example.

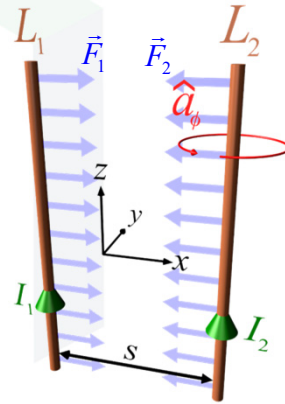
$$\vec{F}_1 = \frac{\mu_0 \mu_r I_1 I_2 L}{2\pi s} \hat{a}_x \rightarrow (200 \text{ nN}) \hat{a}_x$$

Performing the integration gives us the exact result of

$$\vec{F}_1 = (82.8 \text{ nN}) \hat{a}_x$$

If the first wire were 1 m long and the second was 10 m long, the force would be

$$\vec{F}_1 = (196 \text{ nN}) \hat{a}_x$$

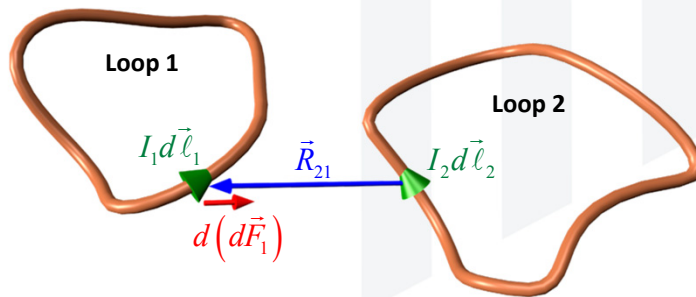


Force Between Two Current Loops

Setup

To calculate the net force between two loops, use the same integral derived for two line currents, but the line integrals are now closed-contour integrals. The force on loop 1 is

$$\vec{F}_1 = \frac{\mu I_1 I_2}{4\pi} \oint_{L_1} \oint_{L_2} \frac{d\vec{\ell}_1 \times d\vec{\ell}_2 \times \hat{a}_{21}}{R_{21}^2}$$



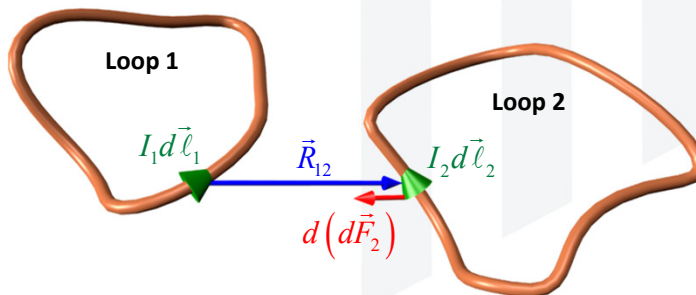
Setup

Write the same integral to calculate the force on loop 2.

$$\vec{F}_2 = \frac{\mu I_2 I_1}{4\pi} \oint_{L_1} \oint_{L_2} \frac{d\vec{\ell}_2 \times d\vec{\ell}_1 \times \hat{a}_{12}}{R_{12}^2}$$

Comparing this to the equation for \vec{F}_1 , we see that

$$\vec{F}_2 = -\vec{F}_1 \quad \text{Forces are equal, but opposite directions.}$$



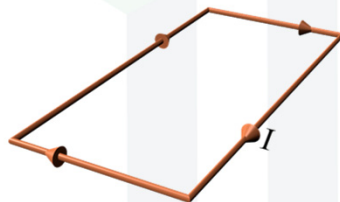
Magnetic Torque & Moment

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Force on a Wire Loop

Suppose there is a wire loop carrying current I .



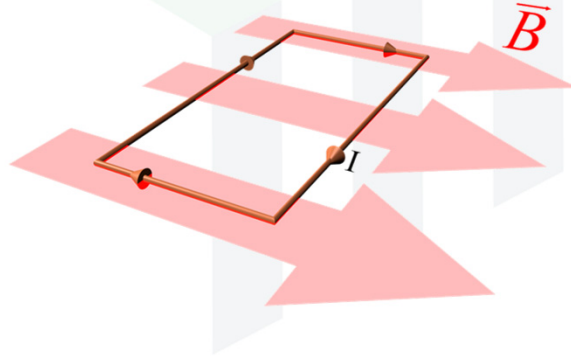
EM Possible

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Force on a Wire Loop

Then apply a magnetic field \vec{B} oriented in the plane of the loop.

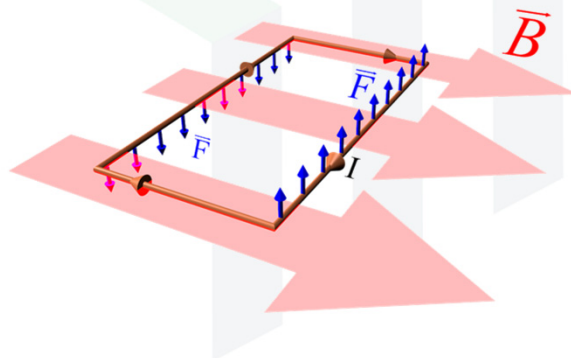


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Force on a Wire Loop

Each part of the wire loop can experience a force in a different direction.

The net force is zero so the loop will not translate its position.

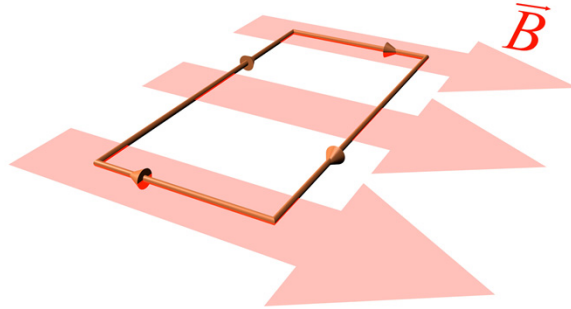


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Force on a Wire Loop

However, the forces being different on different parts of the loop will create a torque on the that will make the loop rotate.

The loop will rotate in a way that makes the cross section of the loop perpendicular to \vec{B} .



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Torque \vec{T} (N·m)

Define torque \vec{T} (or mechanical moment of force) is the cross product of the force \vec{F} and moment arm \vec{r} .

$$\vec{T} = \vec{r} \times \vec{F}$$

N·m meters Newtons

The vector moment arm \vec{r} is defined as

$$\vec{r} = \rho \hat{a}_n$$

Right-Hand Rule for Torque



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Vector Moment Arm \vec{r} for a Loop

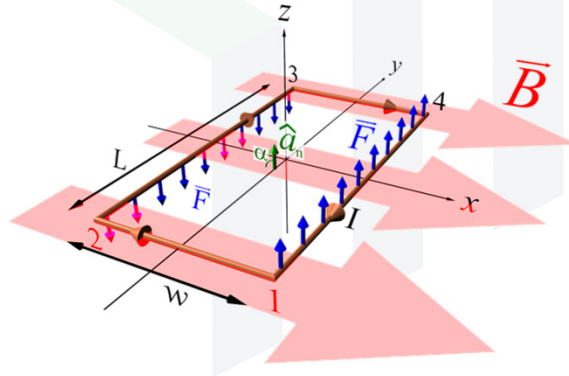
The vector moment arm \vec{r} for a wire loop is defined as

$$\vec{r} = w\hat{a}_n$$

Magnitude of \vec{r} is proportional to how easily the loop can be rotated.

Direction of \vec{r} is perpendicular to the cross section of the loop.

Note that \vec{r} is proportional to twice the radius because force is applied at both sides of the loop.



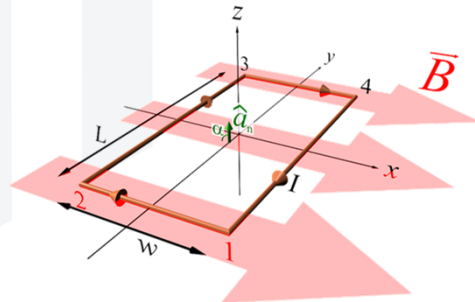
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Net Force on Loop is Zero

The net force on the loop is

$$\begin{aligned}\vec{F} &= \oint_L (I d\vec{\ell}) \times \vec{B} \\ &= -\int_1^2 I dx \hat{a}_x \times B \hat{a}_x + \int_2^3 I dy \hat{a}_y \times B \hat{a}_x + \int_3^4 I dx \hat{a}_x \times B \hat{a}_x - \int_4^1 I dy \hat{a}_y \times B \hat{a}_x \\ &= -\int_2^3 I dy B (\hat{a}_y \times \hat{a}_x) + \int_4^1 I B dy (\hat{a}_y \times \hat{a}_x) \\ &= IB \hat{a}_z \int_2^3 dz - IB \hat{a}_z \int_4^1 dz \\ &= IBL \hat{a}_z - IBL \hat{a}_z \\ &= 0\end{aligned}$$

$\hat{a}_x \times \hat{a}_x = 0$



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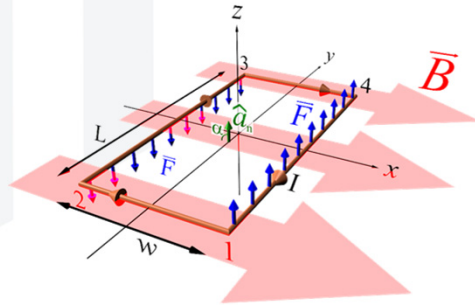
Net Force on Loop in Zero

The net force on the loop is

$$\begin{aligned}\vec{F} &= \oint_L (I d\vec{\ell}) \times \vec{B} \\ &= -\int_1^2 I dx \hat{a}_x \times B \hat{a}_x + \int_2^3 I dy \hat{a}_y \times B \hat{a}_x + \int_3^4 I dx \hat{a}_x \times B \hat{a}_x - \int_4^1 I dy \hat{a}_y \times B \hat{a}_x \\ &= -\int_2^3 I dy B (\hat{a}_y \times \hat{a}_x) + \int_4^1 I B dy (\hat{a}_y \times \hat{a}_x) \\ &= IB \hat{a}_z \int_2^3 dz - IB \hat{a}_z \int_4^1 dz \\ &= IBL \hat{a}_z - IBL \hat{a}_z \\ &= 0\end{aligned}$$

$\hat{a}_x \times \hat{a}_x = 0$

Opposite sign on these forces indicates there will be torque.



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Magnetic Dipole Moment \vec{m}

It is convenient to define the magnetic dipole moment \vec{m} so that torque on a loop can be calculated directly from the magnetic flux \vec{B} .

$$\vec{T} = \vec{m} \times \vec{B}$$

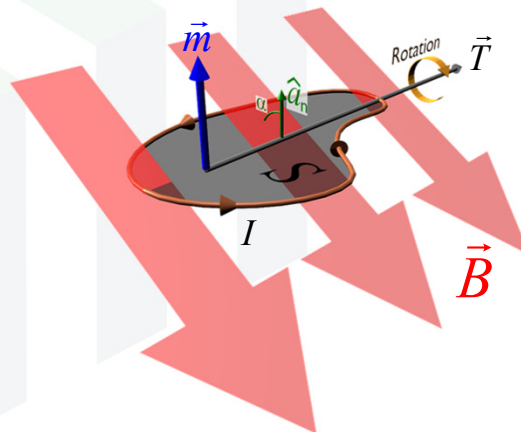
For any planar loop, \vec{m} is

$$\vec{m} = IS \hat{a}_n \quad (\text{A} \cdot \text{m}^2)$$

This parameter lumps together everything about the loop in order to calculate how the loop will respond to a magnetic field \vec{B} .

The dependence on alignment angle α is

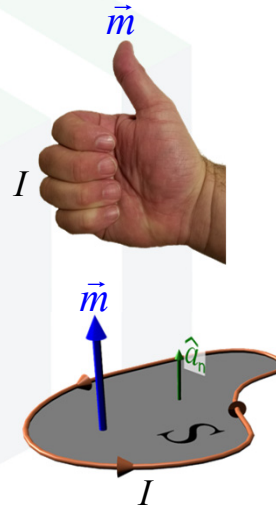
$$|\vec{T}| = BIS \sin \alpha$$



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Handedness of the Magnetic Dipole Moment

$$\vec{m} = IS\hat{a}_n \quad (\text{A} \cdot \text{m}^2)$$



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Example #5 – Magnetic Moment

Determine the magnetic dipole moment formed by the triangular loop show below.

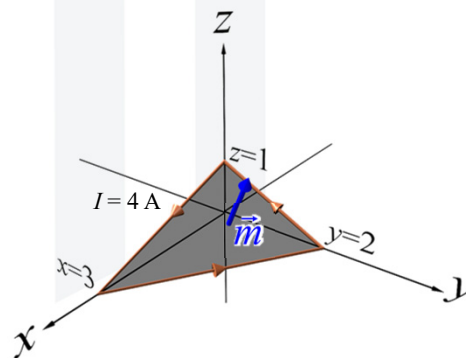
Solution

The magnetic dipole moment \vec{m} is defined as

$$\vec{m} = IS\hat{a}_n$$

The current I is given in the figure to be

$$I = 4 \text{ A}$$



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Example #5 – Magnetic Moment

Area S of the loop is calculated using the cross product.

$$S = \frac{1}{2} |\vec{b} \times \vec{a}| = \frac{1}{2} |(-3, 2, 0) \times (-3, 0, 1)| = \frac{1}{2} |(2, 3, 6)| = 3.5$$

$$\vec{a} = (0, 0, 1) - (3, 0, 0) = (-3, 0, 1)$$

$$\vec{b} = (0, 2, 0) - (3, 0, 0) = (-3, 2, 0)$$

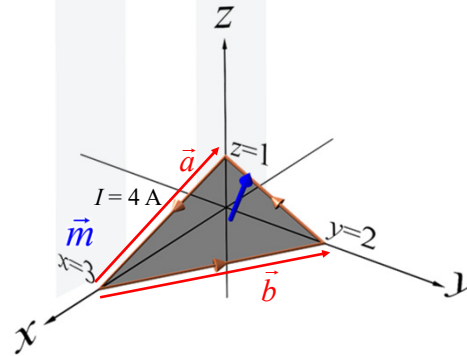
Surface normal \hat{a}_n is

$$\hat{a}_n = \frac{\vec{b} \times \vec{a}}{|\vec{b} \times \vec{a}|} = \frac{(2, 3, 6)}{|(2, 3, 6)|} = \left(\frac{2}{7}, \frac{3}{7}, \frac{6}{7} \right)$$

Altogether, \vec{m} is

$$\begin{aligned} \vec{m} &= IS\hat{a}_n \\ &= (4 \text{ A})(3.5 \text{ m}^2) \left(\frac{2}{7}, \frac{3}{7}, \frac{6}{7} \right) \\ &= (4, 6, 12) \end{aligned}$$

$$\vec{m} = 4\hat{a}_x + 6\hat{a}_y + 12\hat{a}_z \quad \text{A} \cdot \text{m}^2$$



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Magnetic Dipole

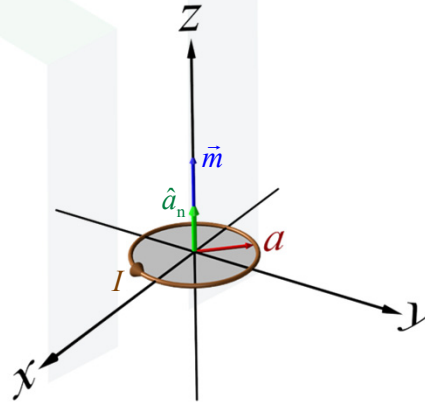
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Definition

A magnetic dipole is a closed-loop of electric current in the limit as the size of the loop is reduced to zero.

The magnetic dipole moment for a small loop of radius a is

$$\vec{m} = \pi a^2 I \hat{a}_n$$



Visualization of Magnetic Field Lines

The magnetic field around the dipole in spherical coordinates is

$$\vec{B} = \frac{\mu |\vec{m}|}{4\pi r^3} (2 \cos \theta \hat{a}_r + \sin \theta \hat{a}_\theta)$$

