Electromagnetics:
Electromagnetic Field Theory

Impedance Transformation and Parameter Relations

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Lecture Outline

- Input Impedance, Z_{in}
- Parameter Relations
- Special Cases of Terminated Transmission Lines
  - Shorted line ($Z_L = 0$)
  - Open-circuit line ($Z_L = \infty$)
  - Matched line ($Z_L = Z_0$)

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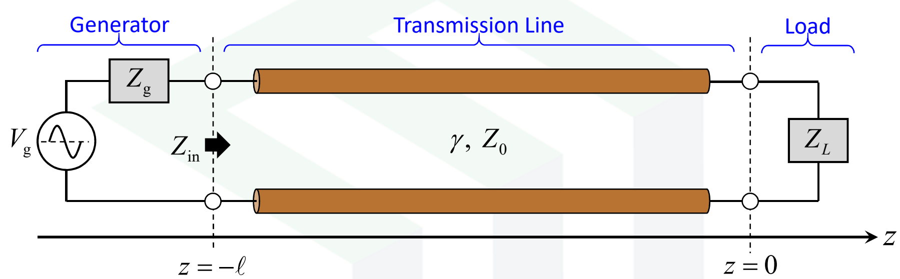
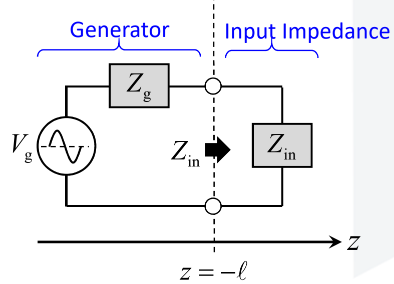
Input Impedance, Z_{in}

Problem Setup

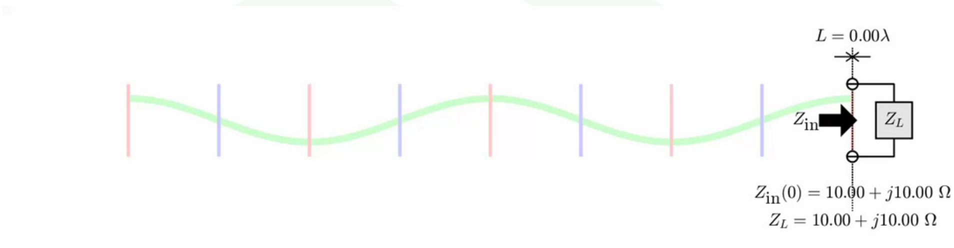




The input impedance Z_{in} is the impedance observed by the generator.

The input impedance Z_{in} is NOT necessarily the line's characteristic impedance Z_0 or the load impedance Z_L.

Animation of Impedance Transformation



$$Z_{\text{in}}\left(\frac{\lambda}{4}+m\frac{\lambda}{2}\right)=\frac{Z_0^2}{Z_{\text{L}}}$$

Input impedance inverts

$$Z_{\text{in}}=125-j125\ \Omega$$

$$Z_{\text{in}}\left(m\frac{\lambda}{2}\right)=Z_{\text{L}}$$

Input impedance repeats

$$Z_{\text{in}}=10+j10\ \Omega$$

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Derivation of Input Impedance, Z_{in} (1 of 2)

The reflection coefficient at any point z from the load is

Backward Wave

$$\Gamma(z)=\frac{V_0^- e^{\gamma z}}{V_0^+ e^{-\gamma z}}=\frac{V_0^-}{V_0^+}e^{2\gamma z}$$

Forward Wave

This means that from the perspective of the generator, the reflection going into the transmission line will change depending on the length of the transmission line. This can only happen of the input impedance to the transmission line is changing.

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Derivation of Input Impedance, Z_{in} (2 of 2)

We define the impedance of the line at position z to be

$$Z(z)=\frac{V(z)}{I(z)}$$

We previously wrote $V(z)$ and $I(z)$ as

$$V(z)=V_0^+\left(e^{-\gamma z}+\Gamma_L e^{\gamma z}\right)$$

$$I(z)=\frac{V_0^+}{Z_0}\left(e^{-\gamma z}-\Gamma_L e^{\gamma z}\right)$$

$$\Gamma_L=\frac{V_0^-}{V_0^+}=\frac{Z_L-Z_0}{Z_L+Z_0}$$

Substituting in our expressions for $V(z)$ and $I(z)$ gives

$$Z(z)=\frac{\cancel{V_0^+}\left(e^{-\gamma z}+\Gamma_L e^{\gamma z}\right)}{\frac{\cancel{V_0^+}}{Z_0}\left(e^{-\gamma z}-\Gamma_L e^{\gamma z}\right)}=Z_0\frac{e^{-\gamma z}+\Gamma_L e^{\gamma z}}{e^{-\gamma z}-\Gamma_L e^{\gamma z}}$$

It makes sense that the impedance is not a function of voltage in a linear system.

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Sanity Check: Input Impedance at Load

The input impedance at the load can be determined by setting $z = 0$ in the previous equation.

$$Z_{\text{in}}(0)=Z_0\frac{\cancel{e^0}+\Gamma_L\cancel{e^0}}{\cancel{e^0}-\Gamma_L\cancel{e^0}}$$

$e^{-\gamma\cdot 0}=1$

$$=Z_0\frac{1+\Gamma_L}{1-\Gamma_L}$$

Cancel e^0 from the equation.

$$=Z_0\frac{1+\frac{Z_L-Z_0}{Z_L+Z_0}}{1-\frac{Z_L-Z_0}{Z_L+Z_0}}$$

$\Gamma_L=\frac{Z_L-Z_0}{Z_L+Z_0}$

$$=Z_0\frac{Z_L+\cancel{Z_0}+Z_L-\cancel{Z_0}}{\cancel{Z_L}+Z_0-\cancel{Z_L}+Z_0}$$

Multiply numerator and denominator by Z_L+Z_0.

$$=Z_0\frac{2Z_L}{2Z_0}$$

Cancel Z_0 in numerator and Z_L in denominator.

$$=Z_L$$

We got the answer we were expecting!

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Input Impedance at $z = -\ell$

The input impedance at location $z = -\ell$ is

$$Z_{\text{in}}(-\ell) = Z_0 \frac{e^{-\gamma(-\ell)} + \Gamma_L e^{\gamma(-\ell)}}{e^{-\gamma(-\ell)} - \Gamma_L e^{\gamma(-\ell)}} = Z_0 \frac{e^{\gamma\ell} + \Gamma_L e^{-\gamma\ell}}{e^{\gamma\ell} - \Gamma_L e^{-\gamma\ell}}$$

A Note About Sign: Backing away from the load, z becomes negative. However, we defined $z = -\ell$ so ℓ stays positive in this equation and for equations that follow.

Impedance Transformation Formula (1 of 2)

Recall that

$$Z(-\ell) = Z_0 \frac{e^{\gamma\ell} + \Gamma_L e^{-\gamma\ell}}{e^{\gamma\ell} - \Gamma_L e^{-\gamma\ell}} \qquad \Gamma_L = \frac{Z_L - Z_0}{Z_L + Z_0}$$

We can eliminate Γ_L from the input impedance equation by substituting in our expression for Γ_L.

$$\cosh(z) = \frac{e^z + e^{-z}}{2} \qquad \sinh(z) = \frac{e^z - e^{-z}}{2}$$

$$Z_{\text{in}}(-\ell) = Z_0 \frac{e^{\gamma\ell} + \dfrac{Z_L - Z_0}{Z_L + Z_0} e^{-\gamma\ell}}{e^{\gamma\ell} - \dfrac{Z_L - Z_0}{Z_L + Z_0} e^{-\gamma\ell}} = Z_0 \frac{Z_L\left(e^{\gamma\ell} + e^{-\gamma\ell}\right) + Z_0\left(e^{\gamma\ell} - e^{-\gamma\ell}\right)}{Z_L\left(e^{\gamma\ell} - e^{-\gamma\ell}\right) + Z_0\left(e^{\gamma\ell} + e^{-\gamma\ell}\right)}$$

$$= Z_0 \frac{Z_L \cdot 2\cosh(\gamma\ell) + Z_0 \cdot 2\sinh(\gamma\ell)}{Z_L \cdot 2\sinh(\gamma\ell) + Z_0 \cdot 2\cosh(\gamma\ell)}$$

Impedance Transformation Formula (2 of 2)

Recall that

$$\tanh(z) = \frac{\sinh(z)}{\cosh(z)}$$

This lets us write the input impedance expression as

$$Z_{\text{in}}(-\ell) = Z_0 \frac{Z_L \cdot 2\cosh(\gamma\ell) + Z_0 \cdot 2\sinh(\gamma\ell)}{Z_L \cdot 2\sinh(\gamma\ell) + Z_0 \cdot 2\cosh(\gamma\ell)} = Z_0 \frac{Z_L + Z_0 \dfrac{\sinh(\gamma\ell)}{\cosh(\gamma\ell)}}{Z_L \dfrac{\sinh(\gamma\ell)}{\cosh(\gamma\ell)} + Z_0}$$

$$\boxed{Z_{\text{in}}(-\ell) = Z_0 \frac{Z_L + Z_0 \tanh(\gamma\ell)}{Z_0 + Z_L \tanh(\gamma\ell)}}$$

Input Transformation for Lossless Line

The lossless line has

$$\alpha = 0$$
$$\gamma = j\beta$$

Putting these values into our impedance transformation formula gives

$$Z_{\text{in}}(-\ell) = Z_0 \frac{Z_L + Z_0 \tanh(j\beta\ell)}{Z_0 + Z_L \tanh(j\beta\ell)}$$

Recognizing that $\tanh(jz) = j\tan(z)$, the expression for lossless lines becomes

$$\boxed{Z_{\text{in}}(-\ell) = Z_0 \frac{Z_L + jZ_0 \tan(\beta\ell)}{Z_0 + jZ_L \tan(\beta\ell)}}$$

Input Impedance Repeats for Lossless Lines

For lossless lines, the $\tan(\beta\ell)$ function in the impedance transformation equation tells us that the function is periodic and repeats.

The function repeats every integer multiple of π.

$$\beta\ell \pm m\pi \qquad m = -\infty,\ldots,-3,-2,-1,0,1,2,3,\ldots,\infty$$

Recognizing that $\beta = 2\pi/\lambda$, the above expression leads to

$$\ell \pm m\frac{\lambda}{2}$$

Note: λ is the wavelength in the transmission line, not the free space wavelength λ_0.

This means the input impedance repeats for every half-wavelength long the transmission line is.

We will revisit this when we cover Smith charts, which will give you a way to visualize the impedance transformation phenomenon.

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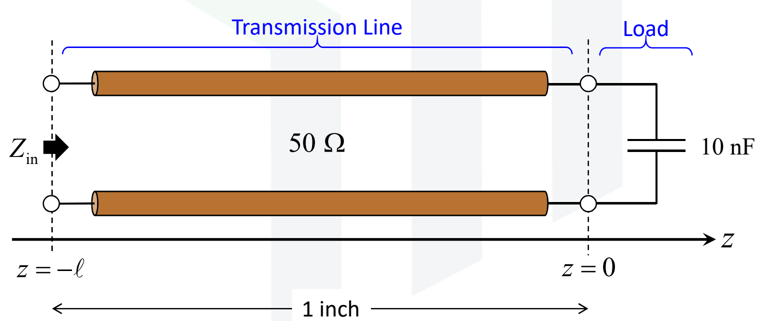
Example: Impedance Transformation (1 of 3)

A transmission line with 50 Ω characteristic impedance is connected to a 10 nF capacitor as the load. If the phase constant of the transmission line is $\beta = 60\ \mathrm{m}^{-1}$, what is the input impedance Z_{in} of a 1 inch section of line operating at 4.0 GHz? What equivalent circuit would the source see?



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Example: Impedance Transformation (2 of 3)

Loss was not specified so a lossless transmission line is assumed. The impedance transformation equation is therefore

$$Z_{\text{in}}(-\ell) = Z_0 \frac{Z_L + jZ_0 \tan(\beta\ell)}{Z_0 + jZ_L \tan(\beta\ell)}$$

The variables in this equation are

$$Z_0 = 50\ \Omega$$

$$\beta\ell = \left(60\ \text{m}^{-1}\right)(1\ \text{inch})\left(\frac{2.54\ \text{cm}}{1\ \text{inch}}\right)\left(\frac{1\ \text{m}}{100\ \text{cm}}\right) = 1.524$$

$$Z_L = \frac{1}{j\omega C} = \frac{1}{j2\pi fC} = \frac{1}{j2\pi\left(4.0\times10^{9}\ \text{s}^{-1}\right)\left(10\times10^{-9}\ \text{F}\right)} = -j0.004\ \Omega$$

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Example: Impedance Transformation (3 of 3)

Substituting these values into the impedance transformation equation gives

$$Z_{\text{in}} = (50\ \Omega)\left[\frac{(-j0.004\ \Omega) + j(50\ \Omega)\tan(60\cdot0.0254)}{(50\ \Omega) + j(-j0.004\ \Omega)\tan(60\cdot0.0254)}\right] = \boxed{j1.07\times10^{3}\ \Omega}$$

The input impedance is purely imaginary and positive. Thus, the input impedance looks like an inductor to the generator.

$$Z_{\text{in}} = j\omega L_{\text{eq}}$$

$$L_{\text{eq}} = \frac{Z_{\text{in}}}{j\omega} = \frac{Z_{\text{in}}}{j2\pi f} = \frac{j1.07\times10^{3}\ \Omega}{j2\pi\left(4.0\times10^{9}\ \text{s}^{-1}\right)} = 4.24\times10^{-8}\ \text{H} = \boxed{42.4\ \text{nH}}$$

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Parameter Relations

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V_{max}, V_{min}, I_{max} & I_{min} in Terms of VSWR

V_{max} and V_{min}

$$V_{\text{max}} = \left|V_0^+\right|\left(1+\left|\Gamma_L\right|\right) = \left|V_0^+\right|\frac{2\,\text{VSWR}}{\text{VSWR}+1}$$

$$V_{\text{min}} = \left|V_0^+\right|\left(1-\left|\Gamma_L\right|\right) = \left|V_0^+\right|\frac{2}{\text{VSWR}+1}$$

I_{max} and I_{min}

$$I_{\text{max}} = \frac{\left|V_0^+\right|}{Z_0}\left(1+\left|\Gamma_L\right|\right) = \frac{\left|V_0^+\right|}{Z_0}\frac{2\,\text{VSWR}}{\text{VSWR}+1}$$

$$I_{\text{min}} = \frac{\left|V_0^+\right|}{Z_0}\left(1-\left|\Gamma_L\right|\right) = \frac{\left|V_0^+\right|}{Z_0}\frac{2}{\text{VSWR}+1}$$

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Z_0 in Terms of VSWR

The characteristic impedance Z_0 can be calculated from $V_{\max}$ and $I_{\max}$ or $V_{\min}$ and $I_{\min}$.

$$Z_0 = \frac{V_{\max}}{I_{\max}} = \frac{V_{\min}}{I_{\min}}$$

The input impedance Z_{in} repeats as you back away from the load. We can calculate the maximum and minimum impedance as

$$\max\left[Z_{\text{in}}\right] = \frac{V_{\max}}{I_{\min}} = Z_0 \cdot \text{VSWR}$$

$$\min\left[Z_{\text{in}}\right] = \frac{V_{\min}}{I_{\max}} = \frac{Z_0}{\text{VSWR}}$$

$$\min\left[Z_{\text{in}}\right] \le Z_{\text{in}} \le \max\left[Z_{\text{in}}\right]$$

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Example (1 of 3)

A 50 Ω impedance transmission line is connected to an antenna with a 72 Ω input impedance. A source provides an input signal of 24 V peak-to-peak.

What is the reflection coefficient at the antenna?

$$\Gamma_L = \frac{Z_L - Z_0}{Z_L + Z_0} = \frac{(72\ \Omega)-(50\ \Omega)}{(72\ \Omega)+(50\ \Omega)} = \boxed{0.1803}$$

In this case, the antenna is the load.

What fraction of the input power is delivered to the antenna?

$$R = \left|\Gamma_L\right|^2 = \left|0.1803\right|^2 = 0.0325$$

$$T = 1 - R = 1 - 0.0325 = 0.9675 = \boxed{96.7\%}$$

Despite the mismatch, almost all power is still delivered to the antenna. This still does not mean the antenna will radiate!

What is the VSWR on the line feeding the antenna?

$$\text{VSWR} = \frac{1+\left|\Gamma_L\right|}{1-\left|\Gamma_L\right|} = \frac{1+\left|0.1803\right|}{1-\left|0.1803\right|} = \boxed{1.44}$$

$$\text{VSWR}_{\text{dB}} = 20\log_{10}\left(\text{VSWR}\right) = 20\log_{10}\left(1.44\right) = \boxed{3.17\ \text{dB}}$$

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Example (2 of 3)

What is the minimum and maximum voltage on the line?

First, we need to convert voltage peak-to-peak $V_{\text{p-p}}$ to voltage magnitude V_0.

$$V_0^+ = \frac{V_{\text{p-p}}}{2} = \frac{24\text{ V}}{2} = 12\text{ V}$$

Now we are in a position to calculate $V_{\min}$ and $V_{\max}$.

$$\text{V}_{\min} = \left|V_0^+\right|\left(1-\left|\Gamma_L\right|\right) = \left|12\text{ V}\right|\left(1-\left|0.1803\right|\right) = 9.84\text{ V}$$

$$\text{V}_{\max} = \left|V_0^+\right|\left(1+\left|\Gamma_L\right|\right) = \left|12\text{ V}\right|\left(1+\left|0.1803\right|\right) = \boxed{14.16\text{ V}}$$

When we are utilizing high voltages, we want to be sure $V_{\max}$ will not cause arcing or any other breakdown problems.

What is the minimum and maximum current on the line?

$$\text{I}_{\min} = \frac{\text{V}_{\min}}{Z_0} = \frac{9.84\text{ V}}{50\ \Omega} = 0.1967\text{ A}$$

$$\text{I}_{\max} = \frac{\text{V}_{\max}}{Z_0} = \frac{14.16\text{ V}}{50\ \Omega} = \boxed{0.2833\text{ A}}$$

At high power, we want to be sure $I_{\max}$ will not cause heating problems.

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Example (3 of 3)

What is the total range of input impedances a source could see?

$$\min\left[Z_{\text{in}}\right] = \frac{V_{\min}}{I_{\max}} = \frac{9.84\text{ V}}{0.2833\text{ A}} = 34.72\ \Omega$$

$$\max\left[Z_{\text{in}}\right] = \frac{V_{\max}}{I_{\min}} = \frac{14.16\text{ V}}{0.1967\text{ A}} = 72\ \Omega$$

$$\min\left[Z_{\text{in}}\right] \le Z_{\text{in}} \le \max\left[Z_{\text{in}}\right]$$

$$\boxed{34.72\ \Omega \le Z_{\text{in}} \le 72\ \Omega}$$

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Special Cases of Terminated Transmission Lines

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Shorted Line, $Z_L = 0$

Reflection from Load

$$\Gamma_L = -1$$

V_{min} and V_{max}

$$V_{min} = 0$$
$$V_{max} = 2\left|V_0^+\right|$$

Input Impedance

$$Z_{in}(-\ell) = \begin{cases} Z_0 \tanh \gamma\ell & \text{lossy} \\ jZ_0 \tan \beta\ell & \text{lossless} \end{cases}$$

Note 1: Z_{in} for the lossless line is purely imaginary. This means it is purely reactive and no dissipation occurs in the line. The input impedance alternates between being capacitive and inductive as you back away from the load.

Voltage Standing Wave Ratio

$$\text{VSWR} = \infty$$

There exists some z where $V(z) = 0$.

I_{min} and I_{max}

$$I_{min} = 0$$
$$I_{max} = \frac{2\left|V_0^+\right|}{Z_0}$$

$\min[Z_{in}]$ and $\max[Z_{in}]$

$\min[Z_{in}] = 0$ short circuit

$\max[Z_{in}] = \infty$ open circuit

Note 2: The shorted line behaves much the same way as the open-circuit line. We also observe that

$$Z_{in,short} Z_{in,open} = Z_0^2$$

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Open-Circuit Line, $Z_L = \infty$

Reflection from Load

$$\Gamma_L = +1$$

Voltage Standing Wave Ratio

$$\text{VSWR} = \infty$$

There exists some z where $V(z) = 0$.

V_{min} and V_{max}

$$V_{min} = 0$$
$$V_{max} = 2\left|V_0^+\right|$$

I_{min} and I_{max}

$$I_{min} = 0$$
$$I_{max} = \frac{2\left|V_0^+\right|}{Z_0}$$

Input Impedance

$$Z_{in}(-\ell) = \begin{cases} Z_0 \coth \gamma\ell & \text{lossy} \\ -jZ_0 \cot \beta\ell & \text{lossless} \end{cases}$$

Note 1: Z_{in} for the lossless line is purely imaginary. This means it is purely reactive and no dissipation occurs in the line. The input impedance alternates between being capacitive and inductive as you back away from the load.

$\min[Z_{in}]$ and $\max[Z_{in}]$

$$\min[Z_{in}] = 0 \quad \text{short circuit}$$
$$\max[Z_{in}] = \infty \quad \text{open circuit}$$

Note 2: The open-circuit line behaves much the same way as the shorted line. We also observe that

$$Z_{in,short} Z_{in,open} = Z_0^2$$

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Matched Line, $Z_L = Z_0$

Reflection from Load

$$\Gamma_L = 0$$

Voltage Standing Wave Ratio

$$\text{VSWR} = 1$$

because $V_{max} = V_{min}$

V_{min} and V_{max}

$$V_{min} = V_{max} = \left|V_0^+\right|$$

I_{min} and I_{max}

$$I_{min} = I_{max} = \left|V_0^+\right| / Z_0$$

Input Impedance

$$Z_{in}(-\ell) = Z_0$$

$\min[Z_{in}]$ and $\max[Z_{in}]$

$$\min[Z_{in}] = \max[Z_{in}] = Z_0$$

Note: For the matched line, there are no reflections and all of the power is delivered to the load.

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