Electromagnetics:
Electromagnetic Field Theory

Magnetostatic Boundary Conditions

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Outline

- Boundary conditions for tangential fields
- Boundary conditions for normal fields
- Law of refraction for magnetic flux

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Deriving Boundary Conditions

Just like was done for electrostatic fields, the boundary conditions for magnetostatic fields will be derived using Maxwell's equations in integral form.

$$I = \oint_L \vec{H} \bullet d\vec{\ell} \quad \Longrightarrow$$

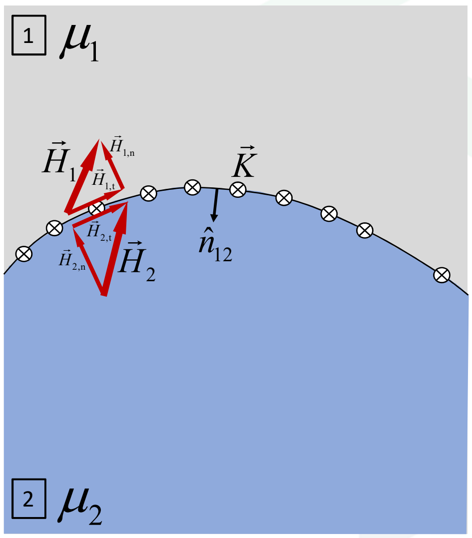
Boundary conditions for tangential magnetic fields.

$$0 = \oiint_S \vec{B} \bullet d\vec{s} \quad \Longrightarrow$$

Boundary conditions for normal magnetic fields.

Boundary Conditions for Tangential Components

Analysis Setup



Let's examine the interface between two materials with a surface current at the interface.

We wish to examine the relation between magnetic fields on either side of the interface, so that if one is known the other can be calculated.

It will be useful to separate the field on either side of the interface into tangential and normal components.

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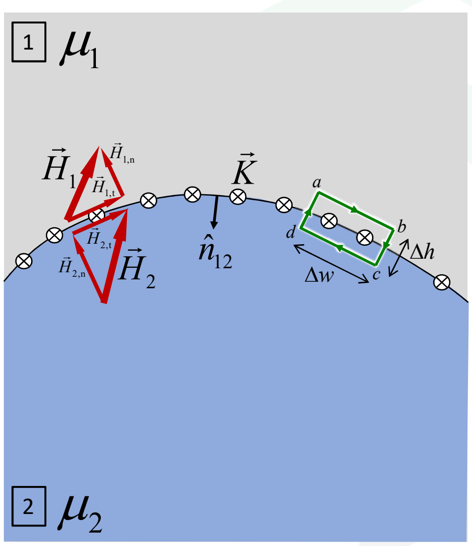
Derivation of Tangential BCs



Apply the following integral to a closed path spanning some section of the interface.

$$I = \oint_L \vec{H} \bullet d\vec{\ell}$$

$$\begin{aligned} K\Delta w &= \int_a^b \vec{H} \bullet d\vec{\ell} + \int_b^0 \vec{H} \bullet d\vec{\ell} + \int_0^c \vec{H} \bullet d\vec{\ell} \\ &+ \int_c^d \vec{H} \bullet d\vec{\ell} + \int_d^0 \vec{H} \bullet d\vec{\ell} + \int_0^a \vec{H} \bullet d\vec{\ell} \\ &= H_{1,t}\Delta w - H_{1,n}\frac{\Delta h}{2} - H_{2,n}\frac{\Delta h}{2} \\ &- H_{2,t}\Delta w + H_{2,n}\frac{\Delta h}{2} + H_{1,n}\frac{\Delta h}{2} \end{aligned}$$

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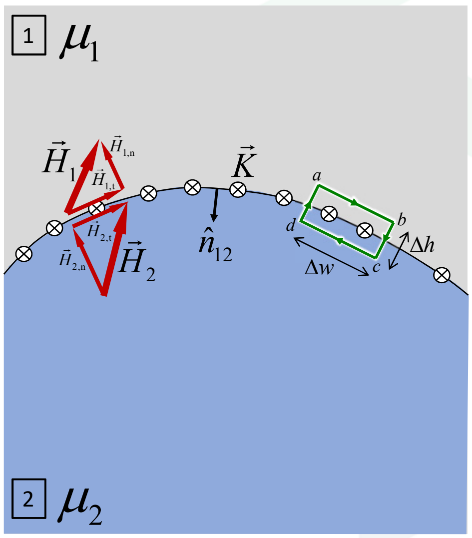
Derivation of Tangential BCs



Cancel like terms with opposite sign.

$$K\Delta w = H_{1,t}\Delta w - \cancel{H_{1,n}\frac{\Delta h}{2}} - \cancel{H_{2,n}\frac{\Delta h}{2}}$$

$$-H_{2,t}\Delta w + \cancel{H_{2,n}\frac{\Delta h}{2}} + \cancel{H_{1,n}\frac{\Delta h}{2}}$$

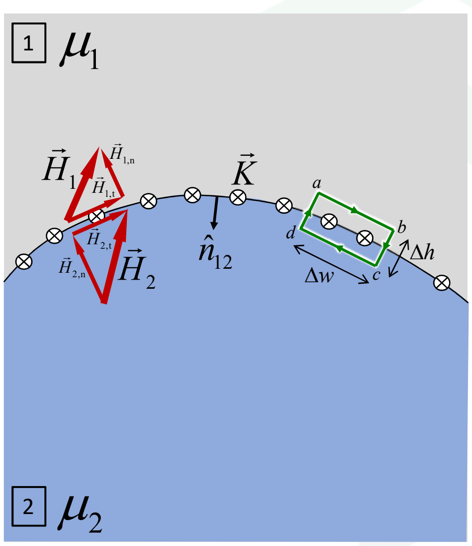
$$= H_{1,t}\Delta w - H_{2,t}\Delta w$$

$$K = H_{1,t} - H_{2,t}$$

From this, it is concluded that

$$\boxed{\left(\vec{H}_1 - \vec{H}_2\right)\times\hat{n}_{12} = \vec{K}}$$

Derivation of Tangential BCs

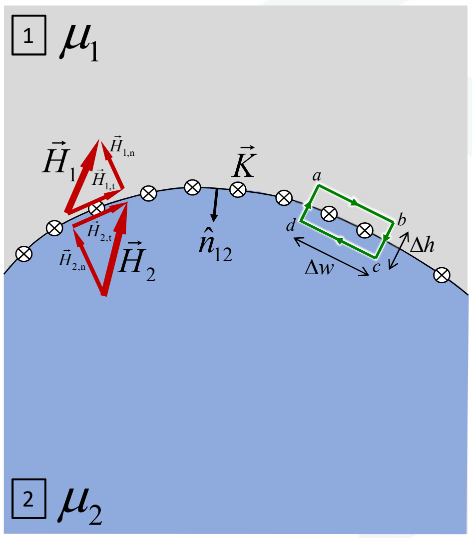


Apply the constitutive relation to get the boundary condition for $\vec{B}$.

$$\left(\vec{H}_1 - \vec{H}_2\right)\times\hat{n}_{12} = \vec{K}$$

$$\boxed{\left(\frac{\vec{B}_1}{\mu_1} - \frac{\vec{B}_2}{\mu_2}\right)\times\hat{n}_{12} = \vec{K}}$$

Derivation of Tangential BCs



In the absence of a surface current $\vec{K}$, the boundary conditions reduce to

$$\vec{H}_{1,t} = \vec{H}_{2,t}$$

$$\frac{\vec{B}_{1,t}}{\mu_1} = \frac{\vec{B}_{2,t}}{\mu_2}$$

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Boundary Conditions for Normal Components

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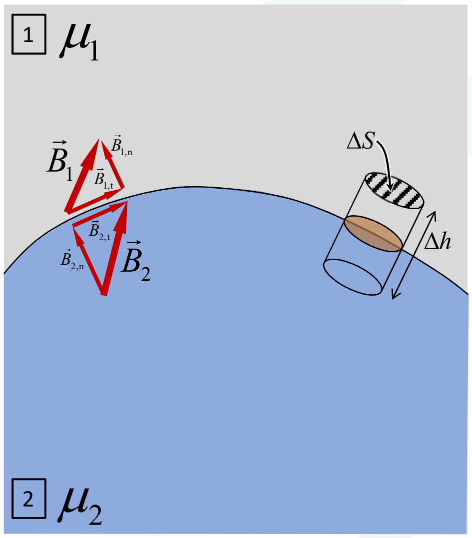
Derivation of Normal BCs



Apply the following surface integral to a pillbox spanning the interface.

$$0 = \oiint_S \vec{B} \bullet d\vec{s}$$

Separate the closed-surface integral into three separate surface integrals.

$$0 = \iint_{\text{top}} \vec{B} \bullet d\vec{s} + \iint_{\text{bottom}} \vec{B} \bullet d\vec{s} + \iint_{\text{sides}} \vec{B} \bullet d\vec{s}$$

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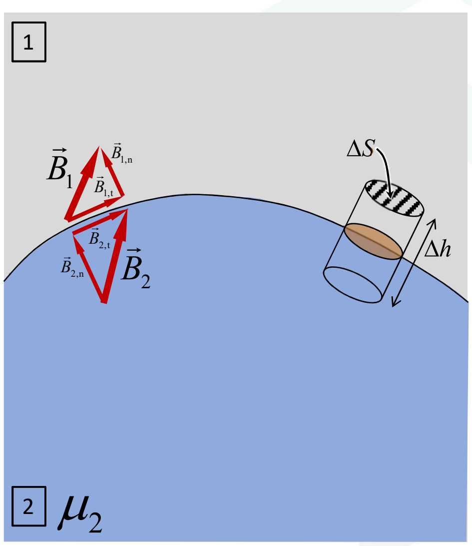
Derivation of Normal BCs



In the limit as $\Delta h \rightarrow 0$

$$0 = \iint_{\text{top}} \vec{B} \bullet d\vec{s} + \iint_{\text{bottom}} \vec{B} \bullet d\vec{s} + \cancel{\iint_{\text{sides}} \vec{B} \bullet d\vec{s}}$$

$$= B_{1,\text{n}} \Delta S - B_{2,\text{n}} \Delta S$$

The boundary condition is that the normal component of $\vec{B}$ is continuous across the interface.

$$\boxed{\vec{B}_{1,\text{n}} = \vec{B}_{2,\text{n}}}$$

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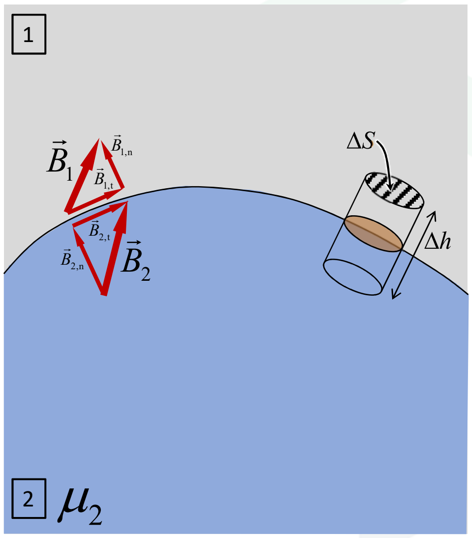
Derivation of Normal BCs



Apply the constitutive relation to get the boundary condition for $\vec{H}$.

$$\vec{B}_{1,n} = \vec{B}_{2,n}$$

$$\boxed{\mu_1 \vec{H}_{1,n} = \mu_2 \vec{H}_{2,n}}$$

The normal component of $\vec{H}$ is NOT continuous across the interface, but the product $\mu\vec{H}$ is.

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Law of Refraction for Magnetic Flux

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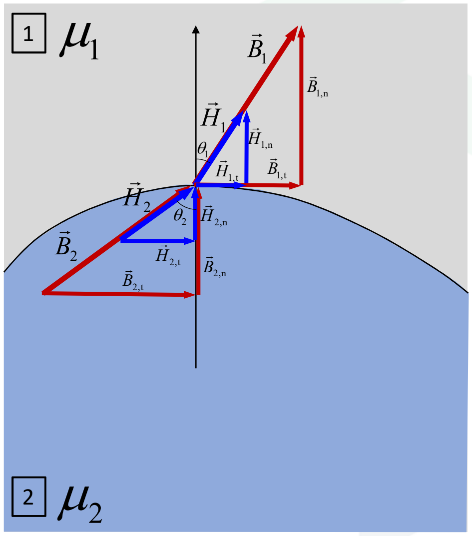
Analysis Setup



It is desired to have a single equation that relates θ_1, θ_2, μ_1, and μ_2 without any field quantities in the equation.

Given the angles θ_1 and θ_2, the field components can be written as

$$\vec{H}_1 = H_{1,t}\hat{a}_t + H_{1,n}\hat{a}_n = (H_1 \sin\theta_1)\hat{a}_t + (H_1 \cos\theta_1)\hat{a}_n$$

$$\vec{H}_2 = H_{2,t}\hat{a}_t + H_{2,n}\hat{a}_n = (H_2 \sin\theta_2)\hat{a}_t + (H_2 \cos\theta_2)\hat{a}_n$$

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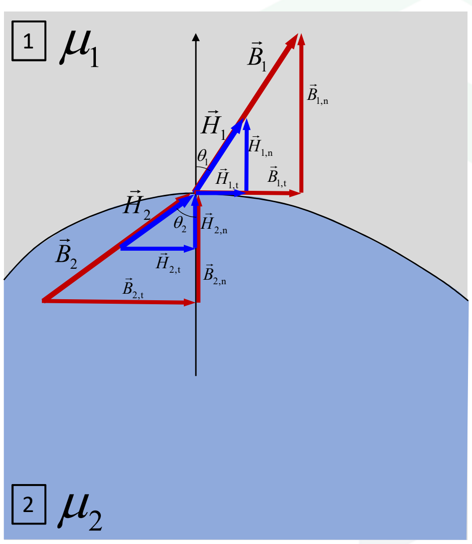
Derivation of Refraction Law



Apply the boundary conditions for tangential components.

$$H_{1,t} = H_{2,t}$$

$$H_1 \sin\theta_1 = H_2 \sin\theta_2$$

Apply the boundary conditions for normal components.

$$\mu_1 H_{1,n} = \mu_2 H_{2,n}$$

$$\mu_1 H_1 \cos\theta_1 = \mu_2 H_2 \cos\theta_2$$

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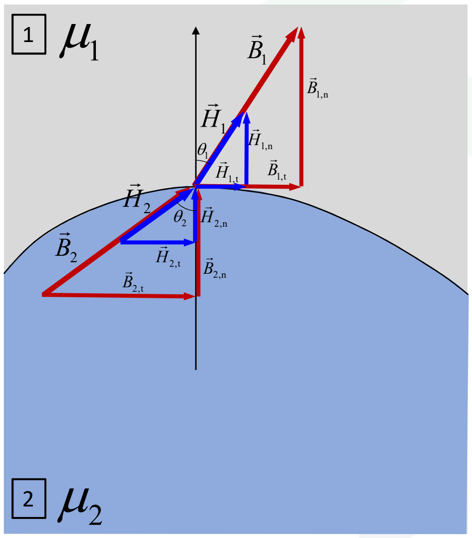
Derivation of Refraction Law



Given the following two equations

$$H_1 \sin\theta_1 = H_2 \sin\theta_2$$
$$\mu_1 H_1 \cos\theta_1 = \mu_2 H_2 \cos\theta_2$$

Divide these equations to get

$$\frac{H_1 \sin\theta_1}{\mu_1 H_1 \cos\theta_1} = \frac{H_2 \sin\theta_2}{\mu_2 H_2 \cos\theta_2}$$

Simplify

$$\boxed{\frac{\tan\theta_1}{\mu_1} = \frac{\tan\theta_2}{\mu_2}}$$

This is NOT Snell's law.

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