Electromagnetics:
Electromagnetic Field Theory

Magnetostatic Devices

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Outline

- Preliminary concepts
- Energy in the magnetic field
- Recipe for analyzing inductors
- Example #1 – Solenoid
- Example #2 – McDonald's straw inductor
- Example #3 – Coaxial line
- Example #4 – RG-59 coaxial transmission line

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Preliminary Concepts

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Magnetic Flux Linkage

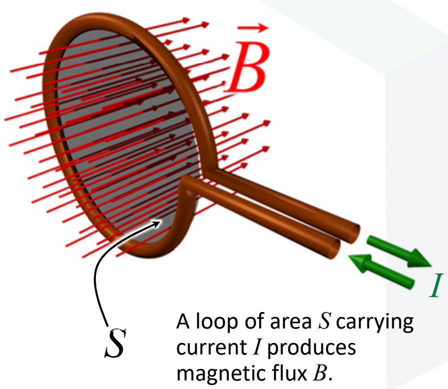
Magnetic Flux

$$\psi = \iint_S \vec{B} \bullet d\vec{s}$$

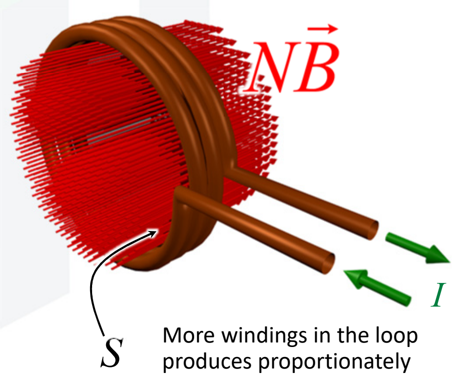
Magnetic Flux Linkage

$$\lambda = N\psi = N \iint_S \vec{B} \bullet d\vec{s}$$

$N \equiv$ number of turns



A loop of area S carrying current I produces magnetic flux B.



More windings in the loop produces proportionately more magnetic flux.

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Inductance

In linear materials, the magnetic flux linkage λ is directly proportional to the current I.

$$\lambda \propto I$$

Inductance L is the proportionality constant that makes this equation exact.

$$\lambda = LI$$

Inductance L is defined as the ratio of the magnetic flux linkage to the current I through the inductor.

$$\boxed{L = \frac{\lambda}{I}}$$

Inductance is a measure of how well a device stores magnetic energy.

Inductors store and release energy in a way that opposes any change in current.

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Energy Stored in an Inductor

From circuit theory, the magnetic energy (in joules) stored in an inductor is

$$W_{\mathrm{m}} = \frac{1}{2} LI^2$$

Solving this for L gives

$$L = \frac{2W_{\mathrm{m}}}{I^2}$$

Given that $L = \lambda / I$, another equation can be derived that shows flux linkage λ is a measure of how much energy W_{m} is stored for a given amount of current I.

$$\lambda = \frac{2W_{\mathrm{m}}}{I}$$

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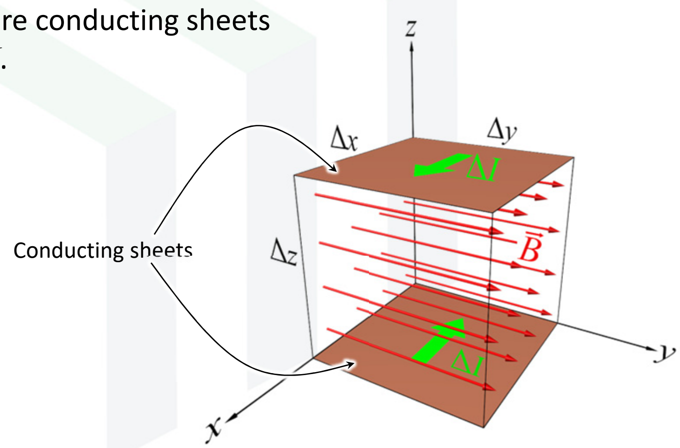
Energy in the Magnetic Field

Differential Volume of Magnetic Field

Let there be a differential volume in the presence of magnetic flux.

The top and bottom surfaces are conducting sheets carrying differential current ΔI.

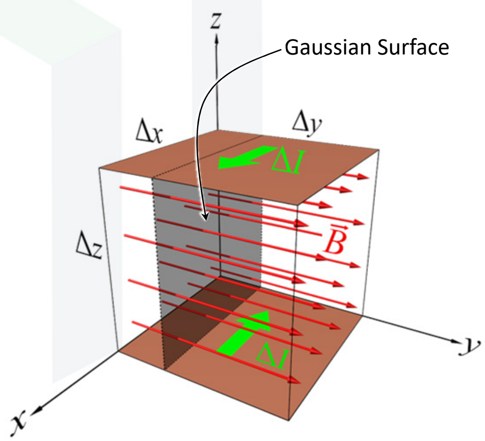


Differential Volume of Magnetic Field

Apply Gauss' law for magnetic fields to a surface in the x-z plane.

Since this is a differential surface, it calculates a differential flux.

$$\begin{aligned}\Delta\psi &= \iint_{\Delta x \Delta z} \vec{B} \bullet d\vec{s} \\ &= \int_0^{\Delta x}\int_0^{\Delta z} \left(B\hat{a}_y\right) \bullet \left(dxdz\hat{a}_y\right) \\ &= B\int_0^{\Delta x}\int_0^{\Delta z} dxdz \\ &= B\Delta x\Delta z\end{aligned}$$



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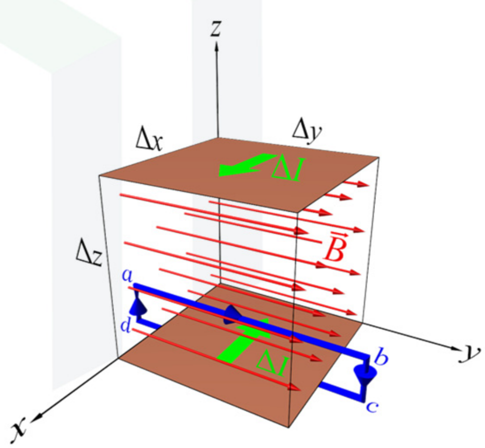
Differential Volume of Magnetic Field

Apply Ampere's circuit law around the bottom conductor.

Since this is a differential surface, it calculates a differential current.

$$\begin{aligned}\Delta I &= \int_L \vec{H} \bullet d\vec{\ell} \\ &= \int_a^b \vec{H} \bullet d\vec{\ell} - \int_b^c \vec{H} \bullet d\vec{\ell} - \int_c^d \vec{H} \bullet d\vec{\ell} + \int_d^a \vec{H} \bullet d\vec{\ell} \\ &= \int_a^b 2Hdy - \cancel{\int_b^c \vec{H} \bullet d\vec{\ell}} - \int_c^d Hdy + \cancel{\int_d^a \vec{H} \bullet d\vec{\ell}} \\ &= H\Delta y\end{aligned}$$



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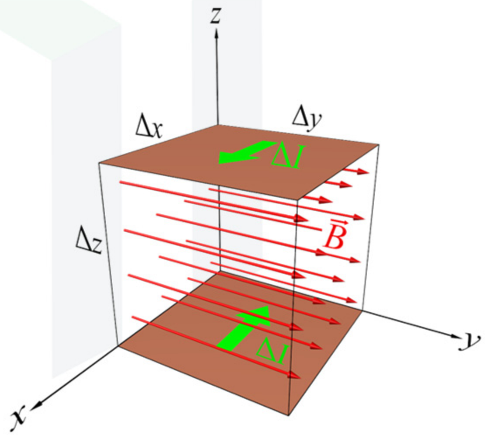
Energy in the Differential Volume

The differential inductance is then

$$\Delta L = \frac{\Delta \psi}{\Delta I} = \frac{B \Delta x \Delta z}{H \Delta y} = \mu \frac{\Delta x \Delta z}{\Delta y}$$

The differential energy stored in this differential inductor is

$$\Delta W_{\mathrm{m}} = \frac{1}{2}(\Delta L)(\Delta I)^2$$
$$= \frac{1}{2}\left(\mu \frac{\Delta x \Delta z}{\Delta y}\right)(H \Delta y)^2$$
$$= \frac{1}{2}\mu H^2 \Delta x \Delta y \Delta z$$
$$= \frac{1}{2}\mu H^2 \Delta v$$



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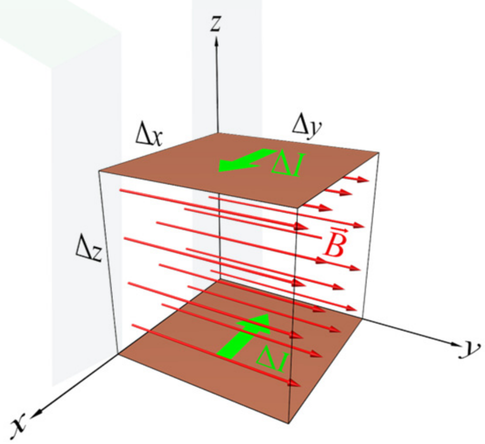
Magnetostatic Energy Density

The magnetostatic energy density (J/m^3) is defined as

$$w_{\mathrm{m}} = \lim_{\Delta v \to 0} \frac{\Delta W_{\mathrm{m}}}{\Delta v}$$
$$= \lim_{\Delta v \to 0} \frac{\frac{1}{2}\mu H^2 \Delta v}{\Delta v}$$

$$\boxed{w_{\mathrm{m}} = \frac{1}{2}\mu H^2}$$



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Total Magnetostatic Energy

The total magnetostatic energy is found by integrating w_{m}.

$$W_{\text{m}} = \iiint_{V} w_{\text{m}} dv \quad \Longrightarrow \quad W_{\text{m}} = \frac{1}{2}\iiint_{V} \mu H^2 dv$$

Recall from electrostatics

$$W_{\text{e}} = \frac{1}{2}\iiint_{V} \left(\vec{D} \bullet \vec{E}\right) dv = \frac{1}{2}\iiint_{V} \varepsilon \left|\vec{E}\right|^2 dv$$

This means for magnetostatics, the total energy is

$$\boxed{W_{\text{m}} = \frac{1}{2}\iiint_{V} \left(\vec{B} \bullet \vec{H}\right) dv}$$

General case

$$\boxed{W_{\text{m}} = \frac{1}{2}\iiint_{V} \mu \left|\vec{H}\right|^2 dv}$$

LHI Media

Recipe for Analyzing Inductors

Recipe to Analyze Inductors

1. Choose a suitable coordinate system.
2. Let the inductor carry current I_0.
3. Calculate magnetic field intensity $\vec{H}$.

   If the basic field configuration is obvious (i.e. if symmetry exists)

   $$I = \oint_L \vec{H} \bullet d\vec{\ell}$$

   Otherwise… $\vec{H} = \begin{cases} \int_L I d\vec{\ell} \times \hat{a}_R / 4\pi R^2 & \text{line current} \\ \iint_S \vec{K} ds \times \hat{a}_R / 4\pi R^2 & \text{surface current} \\ \iiint_V \vec{J} dv \times \hat{a}_R / 4\pi R^2 & \text{volume current} \end{cases}$

4. Calculate $\vec{B}$ from $\vec{H}$ using $\vec{B} = \mu\vec{H}$.
5. Calculate the magnetic flux ψ.

   $$\psi = \iint_S \vec{B} \bullet d\vec{s}$$

6. Calculate the inductance L.

   $$L = \frac{N\psi}{I_0}$$

Alternative to steps 4 – 6…

4. Calculate total magnetic energy W_m.

   $$W_m = \frac{1}{2} \iiint_V \mu \left|\vec{H}\right|^2 dv$$

5. Calculate inductance L.

   $$W_m = \frac{1}{2} L I_0^2 \quad \rightarrow \quad L = \frac{2W_m}{I_0^2}$$

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Example #1:

Solenoid

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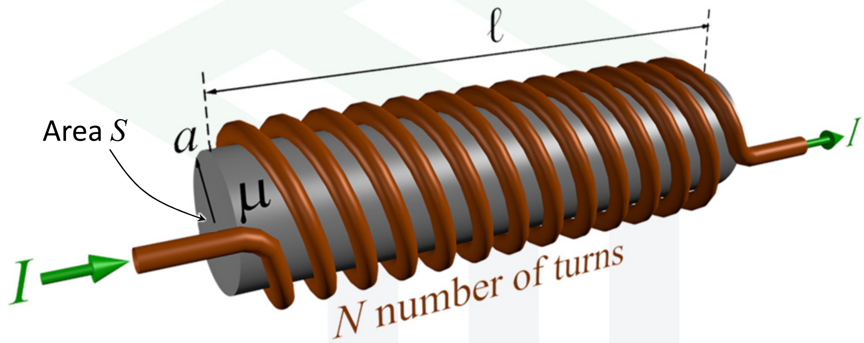
The Solenoidal Inductor



$$L = \frac{\mu N^2 S}{\ell} \quad \text{Henries}$$

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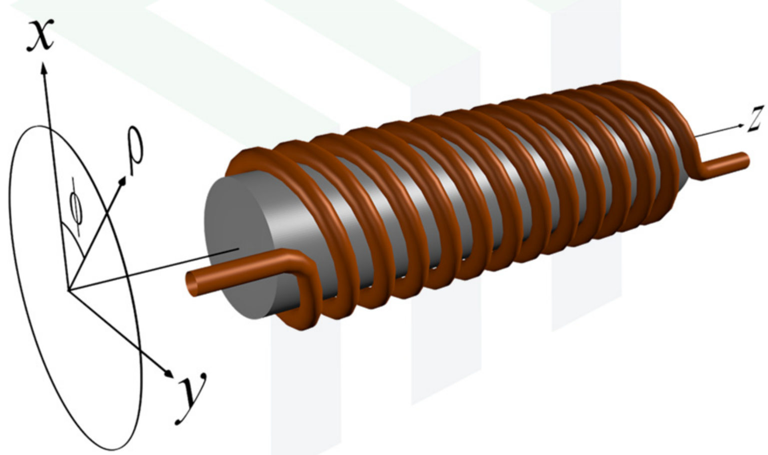
Step 1 – Choose a Coordinate System

Cylindrical coordinates (ρ, ϕ, z) seem appropriate.



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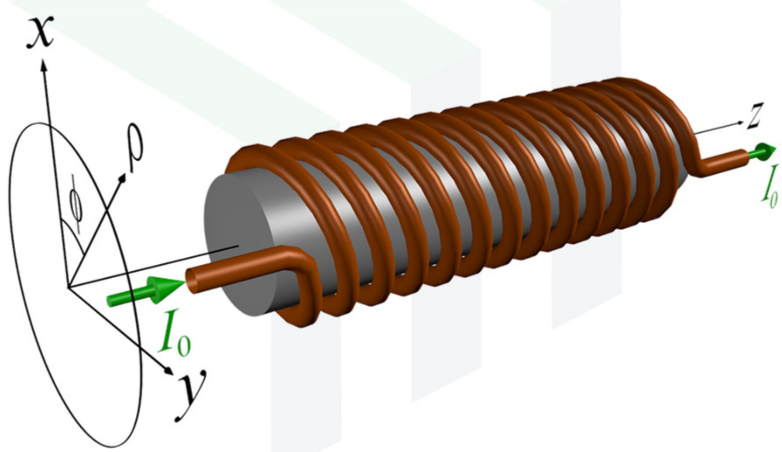
Step 2 – Let Inductor Carry Current I_0

That was easy!



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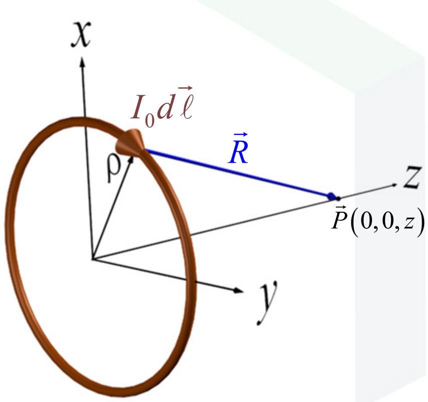
Step 3 – Calculate $\vec{H}$

Calculate the magnetic field $\vec{H}$ due to a single loop and then extend that answer to N loops.



Write the Biot-Savart law at point $(0,0,z)$.

$$d\vec{H} = \frac{I_0 d\vec{\ell} \times \vec{R}}{4\pi \left|\vec{R}\right|^3}$$

The terms in this equation are

$$d\vec{\ell} = a\,d\phi\,\hat{a}_\phi$$

$$\vec{R} = -a\hat{a}_\rho + z\hat{a}_z$$

$$\begin{aligned} d\vec{\ell} \times \vec{R} &= \left(a\,d\phi\,\hat{a}_\phi\right) \times \left(-a\hat{a}_\rho + z\hat{a}_z\right) \\ &= -a^2 d\phi\left(\hat{a}_\phi \times \hat{a}_\rho\right) + azd\phi\left(\hat{a}_\phi \times \hat{a}_z\right) \\ &= a^2 d\phi\,\hat{a}_z + azd\phi\,\hat{a}_\rho \end{aligned}$$

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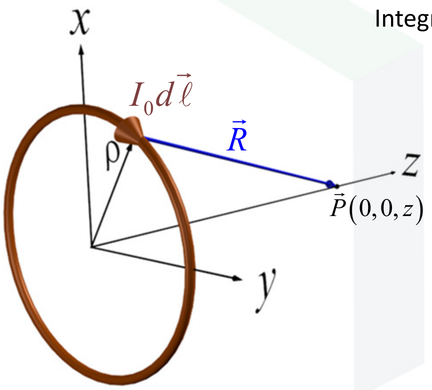
Step 3 – Calculate $\vec{H}$

Last, $|\vec{R}|^3$ is needed.

$$\vec{R} = -a\hat{a}_\rho + z\hat{a}_z$$

$$|\vec{R}|^3 = \left(a^2 + z^2\right)^{3/2}$$



Now a big ugly expression can be written for $d\vec{H}_1$.

$$d\vec{H}_1 = \frac{I_0 d\vec{\ell} \times \vec{R}}{4\pi|\vec{R}|^3} = \frac{I_0\left(a^2 d\phi\hat{a}_z + azd\phi\hat{a}_\rho\right)}{4\pi\left(a^2+z^2\right)^{3/2}}$$

Integrate this around the loop to find $\vec{H}_1$.

$$\vec{H}_1(z) = \int_0^{2\pi} d\vec{H}_1$$

$$= \int_0^{2\pi} \frac{I_0\left(a^2 d\phi\hat{a}_z + azd\phi\hat{a}_\rho\right)}{4\pi\left(a^2+z^2\right)^{3/2}}$$

$$= \int_0^{2\pi} \frac{I_0 a^2 d\phi\hat{a}_z}{4\pi\left(a^2+z^2\right)^{3/2}} + \int_0^{2\pi} \frac{I_0 azd\phi\hat{a}_\rho}{4\pi\left(a^2+z^2\right)^{3/2}}$$

Due to symmetry, the second integral equals zero.

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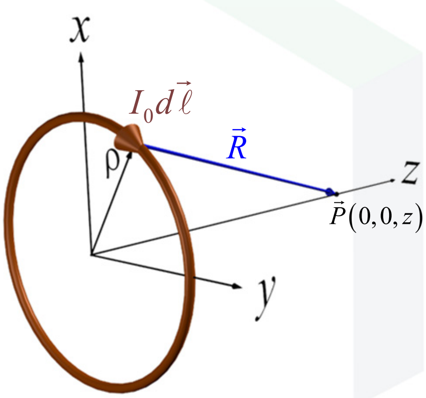
Step 3 – Calculate $\vec{H}$



Finally, the magnetic field along the z-axis is

$$\vec{H}_1 = \int_0^{2\pi} \frac{I_0 a^2 d\phi\hat{a}_z}{4\pi\left(a^2+z^2\right)^{3/2}} = \frac{I_0 a^2\hat{a}_z}{4\pi\left(a^2+z^2\right)^{3/2}} \underbrace{\int_0^{2\pi} d\phi}_{=2\pi}$$

$$= \frac{I_0 a^2\hat{a}_z}{4\pi\left(a^2+z^2\right)^{3/2}} 2\pi$$

$$= \frac{I_0 a^2}{2\left(a^2+z^2\right)^{3/2}}\hat{a}_z$$

Remember, this is just the magnetic field for a single loop.

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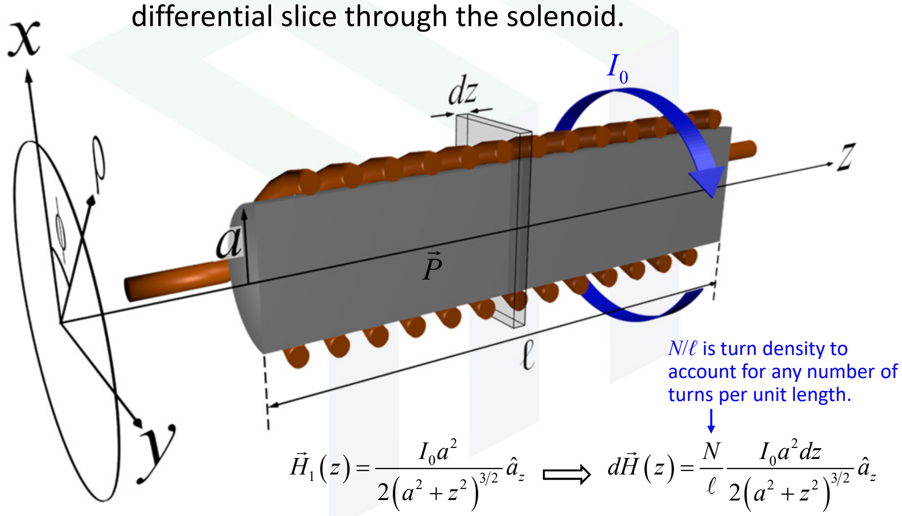
Step 3 – Calculate $\vec{H}$

Think of the expression we just derived as a differential $\vec{H}_1$ due to a differential slice through the solenoid.



N/ℓ is turn density to account for any number of turns per unit length.

$$\vec{H}_1(z)=\frac{I_0a^2}{2\left(a^2+z^2\right)^{3/2}}\hat{a}_z \Longrightarrow d\vec{H}(z)=\frac{N}{\ell}\frac{I_0a^2dz}{2\left(a^2+z^2\right)^{3/2}}\hat{a}_z$$

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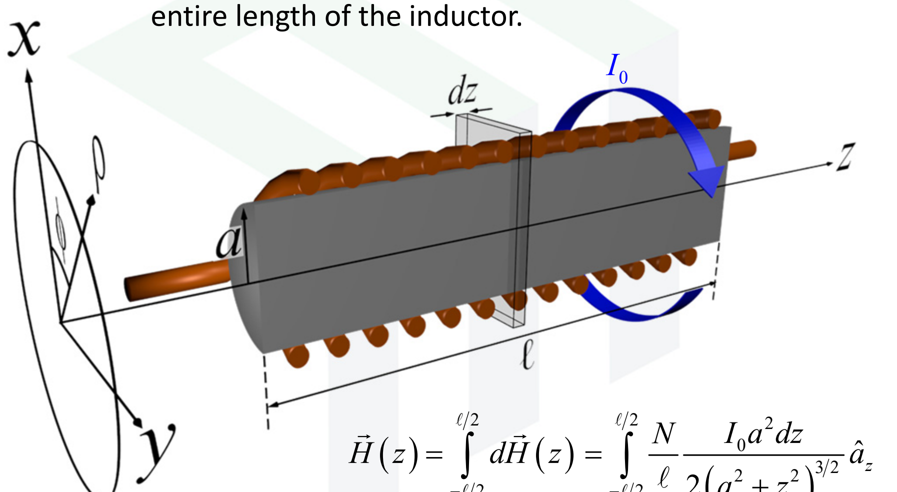
Step 3 – Calculate $\vec{H}$

The total magnetic field $\vec{H}$ is found by integrating $d\vec{H}_z$ through the entire length of the inductor.



$$\vec{H}(z)=\int_{-\ell/2}^{\ell/2}d\vec{H}(z)=\int_{-\ell/2}^{\ell/2}\frac{N}{\ell}\frac{I_0a^2dz}{2\left(a^2+z^2\right)^{3/2}}\hat{a}_z$$

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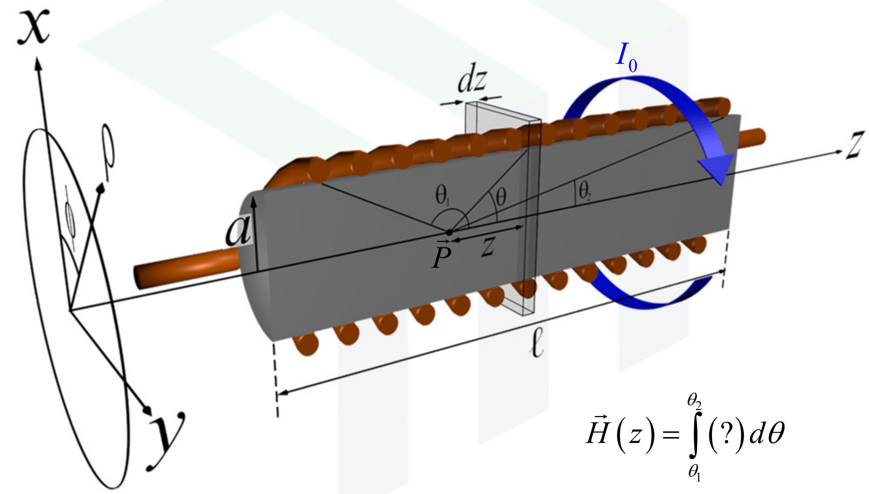
Step 3 – Calculate $\vec{H}$

It will be easier to integrate over angle θ instead of position z.

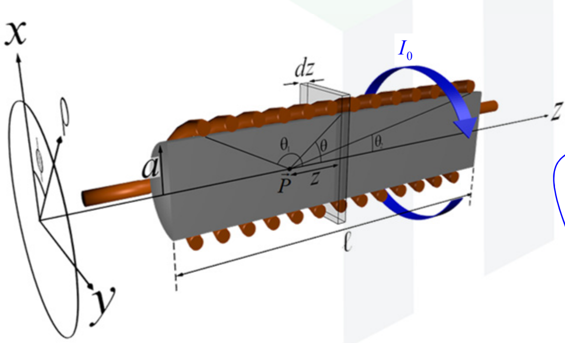


$$\vec{H}(z) = \int_{\theta_1}^{\theta_2} (?)\, d\theta$$

Step 3 – Calculate $\vec{H}$

Angle θ must be related to z. Observe that

$$\tan\theta = \frac{a}{z} \quad \rightarrow \quad z = \frac{a}{\tan\theta} \quad \rightarrow \quad dz = -\frac{a}{\sin^2\theta} d\theta = -\frac{a}{\sin^3\theta}\sin\theta\, d\theta$$



Now observe that

$$\sin\theta = \frac{a}{\sqrt{a^2 + z^2}}$$

$$\sin^3\theta = \frac{a^3}{\left(a^2 + z^2\right)^{3/2}}$$

This was done so that the $\left(a^2 + z^2\right)^{3/2}$ term be canceled out of the integral.

Step 3 – Calculate $\vec{H}$

The integral in terms of θ is…

$$\vec{H}(z)=\int_{-\ell/2}^{\ell/2}\frac{N}{\ell}\frac{I_0 a^2 dz}{2\left(a^2+z^2\right)^{3/2}}\hat{a}_z$$

$-\ell/2 \rightarrow \theta_1$

$\ell/2 \rightarrow \theta_2$

$dz \rightarrow -\dfrac{a}{\sin^3\theta}\sin\theta d\theta$

$\left(a^2+z^2\right)^{3/2} \rightarrow \dfrac{a^3}{\sin^3\theta}$

$$\vec{H}(z)=\int_{\theta_1}^{\theta_2}\frac{N}{\ell}\frac{I_0 a^2\left(-\dfrac{a}{\sin^3\theta}\sin\theta d\theta\right)}{2\left(\dfrac{a^3}{\sin^3\theta}\right)}\hat{a}_z=-\frac{NI_0\hat{a}_z}{2\ell}\int_{\theta_1}^{\theta_2}\sin\theta d\theta$$

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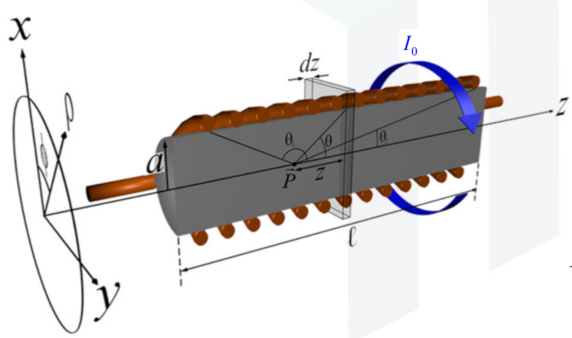
Step 3 – Calculate $\vec{H}$

Finish the integration to get

$$\vec{H}(z)=-\frac{NI_0\hat{a}_z}{2\ell}\int_{\theta_1}^{\theta_2}\sin\theta d\theta=\frac{NI_0\hat{a}_z}{2\ell}\left(\cos\theta_2-\cos\theta_1\right)$$



For an infinitely long solenoid

$\theta_1=180°\qquad\theta_2=0°$

The solution becomes

$$\vec{H}(z)=\frac{NI_0\hat{a}_z}{2\ell}\left(\cos 0°-\cos 180°\right)$$

$$\vec{H}(z)=\frac{NI_0}{\ell}\hat{a}_z$$

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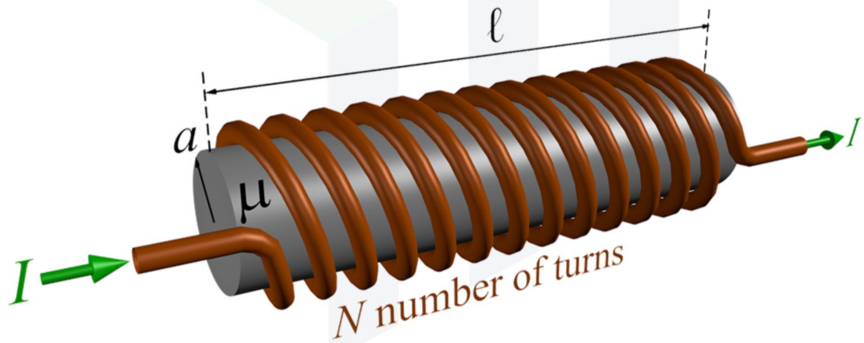
Step 4 – Calculate $\vec{B}$

Given the magnetic field intensity $\vec{H}$, the magnetic flux density $\vec{B}$ is found using the constitutive relation.

$$\vec{B} = \mu \vec{H} = \mu \frac{N I_0}{\ell} \hat{a}_z$$



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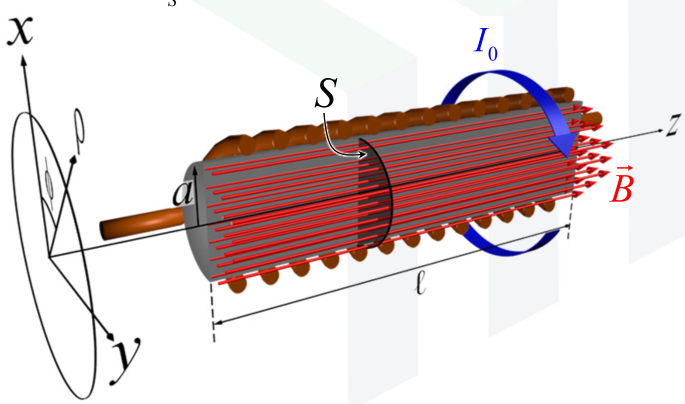
Step 5 – Calculate ψ

The magnetic flux ψ is calculated by integrating $\vec{B}$ in the cross section of the solenoid.

$$\psi = \iint_S \vec{B} \bullet d\vec{s} = B_z S = \frac{\mu N I_0}{\ell} \cdot S$$



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Step 6 – Calculate Inductance L

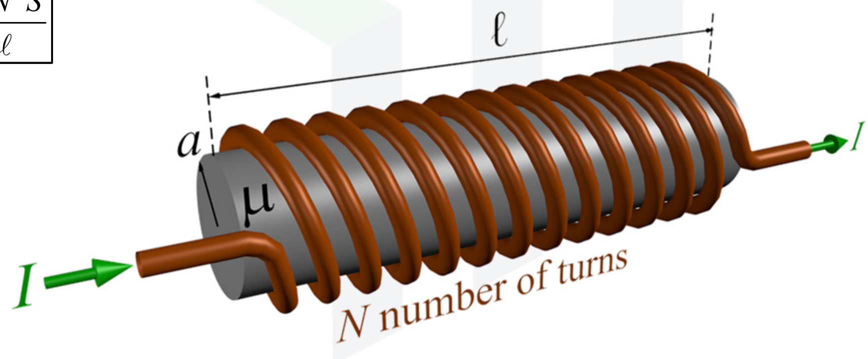
Finally, the inductance L is

$$L = \frac{N\psi}{I_0} = \frac{N\frac{\mu N I_0 S}{\ell}}{I_0} = \frac{\mu N^2 S}{\ell}$$

$$\boxed{L = \frac{\mu N^2 S}{\ell}}$$

This is often written as inductance per unit length.

$$\boxed{\frac{L}{\ell} = \mu S\left(\frac{N}{\ell}\right)^2}$$



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Example #2: McDonald's *Straw Inductor*

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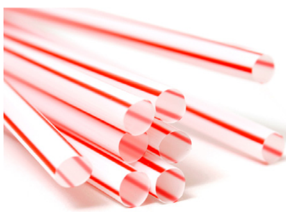
What is a McDonald's Straw Inductor?

It is possible to make an inductor with a reasonably accurate inductance, by wrapping magnet wire around a MacDonald's straw.

How can this be done? What is the design rule?

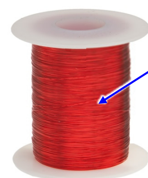
Solution

First, you need some magnet wire and a straw from MacDonald's. Magnet wire has a very thin insulating jacking so more windings can be fit.



The radius a of a MacDonald's straw is

$$a = 3.7 \text{ mm}$$



Let's say we have 22 AWG for this example. The diameter d of the coated wire is

$$d = 0.644 \text{ mm}$$

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Derivation of Design Rule

The inductance per unit length is

$$\frac{L}{\ell} = \mu S \left(\frac{N}{\ell} \right)^2$$

For this problem with an air core,

$$\mu = \mu_0 \mu_r = \mu_0 (1.0) = \mu_0$$

$$S = \pi a^2$$

The inductance per unit length becomes

$$\frac{L}{\ell} = \mu_0 \cdot \pi a^2 \cdot \left(\frac{N}{\ell} \right)^2 = \mu_0 \pi \left(a \frac{N}{\ell} \right)^2$$

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Derivation of Design Rule

If the inductor is wrapped tightly with only one layer of windings, the turn density is

$$\frac{N}{\ell}=\frac{1}{d}$$

The inductance per unit length becomes

$$\frac{L}{\ell}=\mu_0\pi\left(a\frac{N}{\ell}\right)^2=\mu_0\pi\left(\frac{a}{d}\right)^2=\left(1.2566\times10^{-6}\ \text{H/m}\right)\pi\left(\frac{3.7\ \text{mm}}{0.644\ \text{mm}}\right)^2$$

The design rule is

$$\frac{L}{\ell}\approx1.3\times10^{-4}\ \text{H/m}\quad\rightarrow\quad\boxed{\ell\ (\text{mm})\approx\frac{L\ (\text{nH})}{130}}$$

You can use this to easily make your own inductors!

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Example #3:

Coaxial Line

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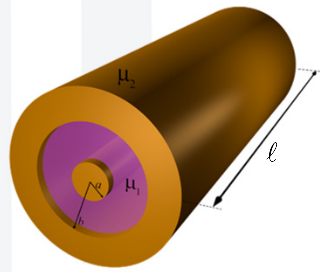
Problem Setup

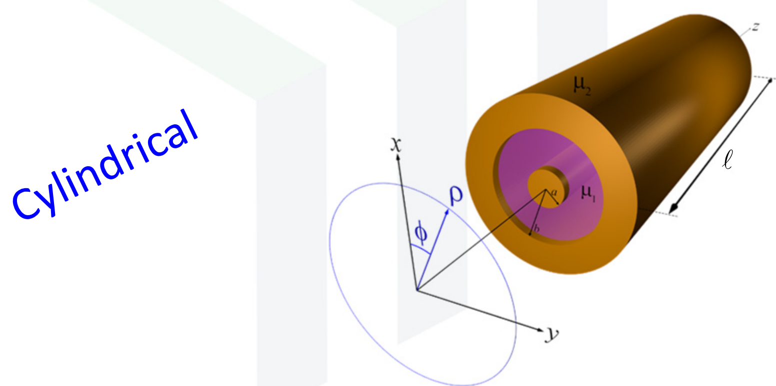
Derive an expression for the distributed inductance L/ℓ of coaxial line.



Problem Setup

Derive an expression for the distributed inductance L/ℓ of coaxial line.

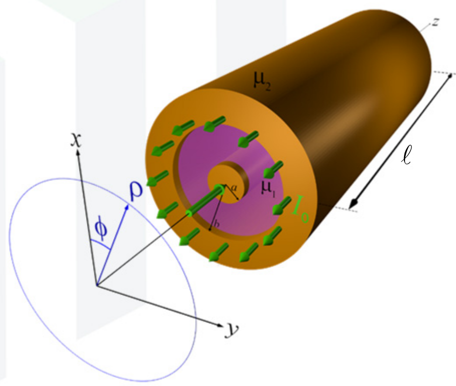
Step 1 – Choose a coordinate system.



Problem Setup

Step 2 – Let the inductor carry current I_0.

That was easy!



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Problem Setup

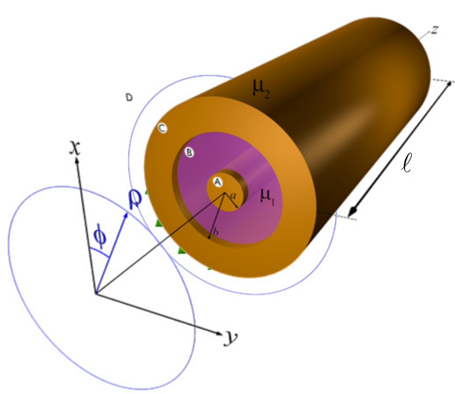
Step 3 – Calculate magnetic field intensity $\vec{H}$.

To do this, analyze the device in each of the four regions A, B, C, and D separately and then stitch together the answers.

A – inner conductor
B – dielectric region
C – outer conductor
D – outside of coax

Calculate the magnetic field using Ampere's circuit law.

$$I = \oint_L \vec{H} \bullet d\vec{\ell} = \iint_S \vec{J} \bullet d\vec{s}$$



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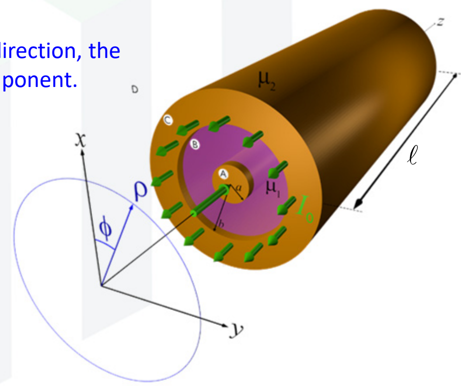
Problem Setup

Step 3 – Calculate magnetic field intensity $\vec{H}$.

Recall the Biot-Savart law which says the magnetic field will be perpendicular to the current and the direction of the observation point.

$$d\vec{H} = \frac{\vec{J}dv \times \hat{a}_R}{4\pi R^2}$$

1. Since the current is solely in the z direction, the magnetic field cannot have a z component.
2. Due to symmetry, the magnetic field will not have a ρ component.
3. The magnetic field will be oriented in the ϕ direction. This is consistent with the magnetic field circulating around currents.



$$\vec{H}(\rho,\phi,z) = H_\phi(\rho,\phi,z)\,\hat{a}_\phi$$

This is what must be found.

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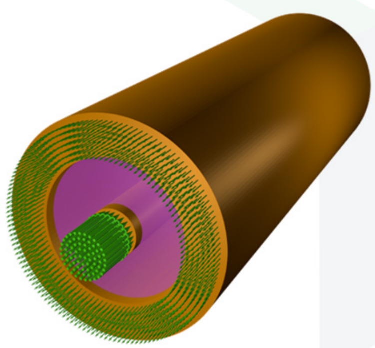
Region A – Inner Conductor

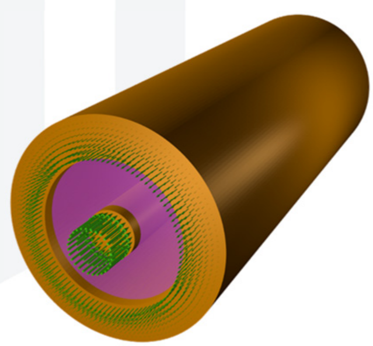
Step 3 – Calculate magnetic field intensity $\vec{H}$.

Assume the current is uniformly distributed throughout the conductors.
This is perfectly valid for magnetostatics, but is a bad approximation at high frequency due to the skin effect.

Current distribution at low frequency



Current distribution at high frequency



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Region A – Inner Conductor

Step 3 – Calculate magnetic field intensity $\vec{H}$.

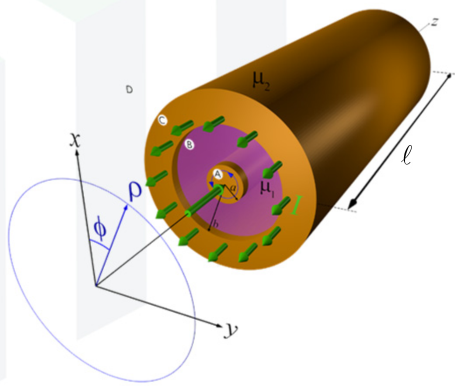
Given this approximation, the current density in the inner conductor is

$$\vec{J}_{\text{inner}} = \frac{I_0}{\pi a^2}\hat{a}_z$$

The total current enclosed within radius ρ is

$$I_{\text{A}}(\rho) = \left(\frac{\text{Area enclosed by } \rho}{\text{Area of inner conductor}}\right) I_0$$

$$= \left(\frac{\pi\rho^2}{\pi a^2}\right) I_0 = I_0\left(\frac{\rho}{a}\right)^2$$



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Region A – Inner Conductor

Step 3 – Calculate magnetic field intensity $\vec{H}$.

Applying Ampere's circuit law, the current is related to the magnetic field as

$$I_{\text{A}}(\rho) = \oint_L \vec{H}\bullet d\vec{\ell} = \int_0^{2\pi}\left(H_\phi\hat{a}_\phi\right)\bullet\left(\rho d\phi\hat{a}_\phi\right) = \rho H_\phi\int_0^{2\pi} d\phi = 2\pi\rho H_\phi$$

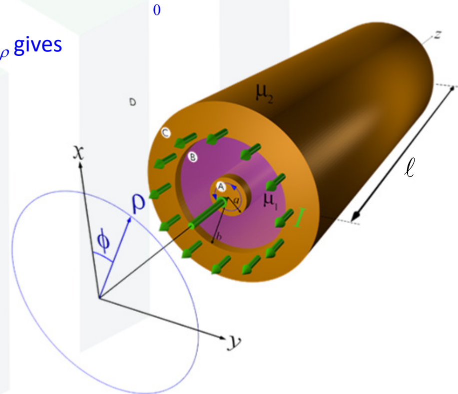
Applying our previous expression for $I_{<\rho}$ gives

$$I_0\left(\frac{\rho}{a}\right)^2 = 2\pi\rho H_\phi$$

Solve this for H_ϕ to get

$$H_\phi = \frac{I_0\rho}{2\pi a^2}$$

$$\vec{H}_{\text{A}}(\rho) = \frac{I_0\rho}{2\pi a^2}\hat{a}_\phi$$



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Region B – Dielectric

Step 3 – Calculate magnetic field intensity $\vec{H}$.

Applying Ampere's circuit law, the current is related to the magnetic field as

$$I_{\mathrm{B}}(\rho)=\oint_L \vec{H}\bullet d\vec{\ell}=\int_0^{2\pi}\left(H_\phi \hat{a}_\phi\right)\bullet\left(\rho d\phi \hat{a}_\phi\right)=\rho H_\phi \int_0^{2\pi} d\phi=2\pi\rho H_\phi$$

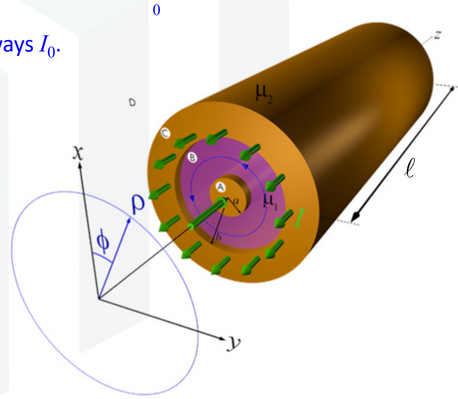
In Region B, the enclosed current is always I_0.

$$I_0=2\pi\rho H_\phi$$

Solve this for H_ϕ to get

$$H_\phi=\frac{I_0}{2\pi\rho}$$

$$\vec{H}_{\mathrm{B}}(\rho)=\frac{I_0}{2\pi\rho}\hat{a}_\phi$$



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Region C – Outer Conductor

Step 3 – Calculate magnetic field intensity $\vec{H}$.

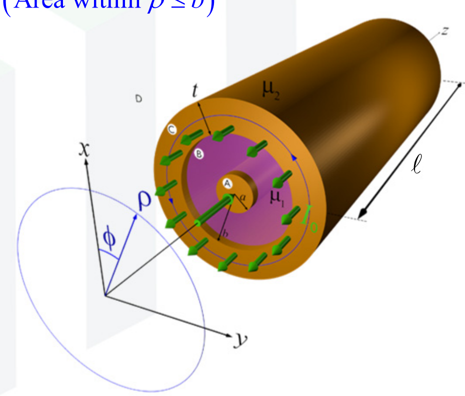
The current density in the outer conductor is

$$\vec{J}_{\text{outer}}=\frac{-I_0}{(\text{Area within } \rho\le b+t)-(\text{Area within } \rho\le b)}\hat{a}_z$$

$$=\frac{-I_0}{\pi(b+t)^2-\pi b^2}\hat{a}_z$$

$$=-\frac{I_0}{\pi}\frac{1}{b^2+2bt+t^2-b^2}\hat{a}_z$$

$$=-\frac{I_0}{\pi}\frac{1}{t^2+2bt}\hat{a}_z$$



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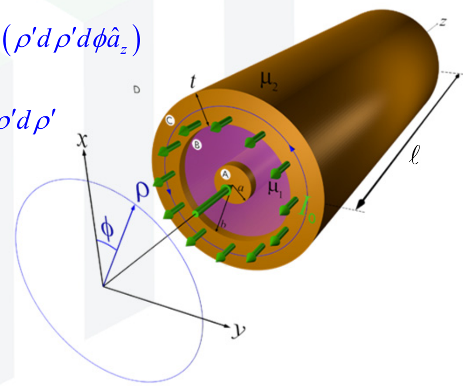
Region C – Outer Conductor

Step 3 – Calculate magnetic field intensity $\vec{H}$.

The total current enclosed by radius ρ is

$$
\begin{aligned}
I_C(\rho) &= I_0 + \iint_C \vec{J}_{\text{outer}} \bullet d\vec{s} \\
&= I_0 - \int_0^{2\pi}\int_b^{\rho}\left(\frac{I_0}{\pi}\frac{1}{t^2+2bt}\hat{a}_z\right)\bullet\left(\rho' d\rho' d\phi \hat{a}_z\right) \\
&= I_0 - \frac{I_0}{\pi}\frac{1}{t^2+2bt}\int_b^{\rho}\left(\int_0^{2\pi} d\phi\right)\rho' d\rho' \\
&= I_0 - \frac{I_0}{\pi}\frac{2\pi}{t^2+2bt}\int_b^{\rho}\rho' d\rho' \\
&= I_0 - \frac{2I_0}{t^2+2bt}\left.\frac{\rho'^2}{2}\right|_b^{\rho} \\
&= I_0\left(1-\frac{\rho^2-b^2}{t^2+2bt}\right)
\end{aligned}
$$



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Region C – Outer Conductor

Step 3 – Calculate magnetic field intensity $\vec{H}$.

Applying Ampere's circuit law, the current is related to the magnetic field as

$$I_C(\rho) = \oint_L \vec{H} \bullet d\vec{\ell} = \int_0^{2\pi}\left(H_\phi \hat{a}_\phi\right)\bullet\left(\rho d\phi \hat{a}_\phi\right) = \rho H_\phi \int_0^{2\pi} d\phi = 2\pi\rho H_\phi$$

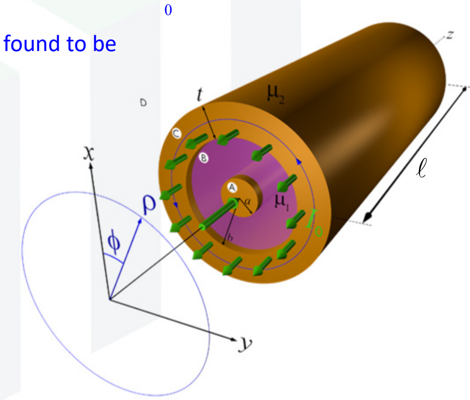
In Region C, the enclosed current was found to be

$$I_C(\rho) = I_0\left(1-\frac{\rho^2-b^2}{t^2+2bt}\right)$$

Putting these expressions together and solving for H_ϕ gives

$$I_0\left(1-\frac{\rho^2-b^2}{t^2+2bt}\right) = 2\pi\rho H_\phi$$

$$H_\phi(\rho) = \frac{I_0}{2\pi\rho}\left(1-\frac{\rho^2-b^2}{t^2+2bt}\right)$$



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Region D – Outside Coax

Step 3 – Calculate magnetic field intensity $\vec{H}$.

Applying Ampere's circuit law, the current is related to the magnetic field as

$$I_{\mathrm{D}}(\rho) = \oint_L \vec{H} \bullet d\vec{\ell} = \int_0^{2\pi} \left(H_\phi \hat{a}_\phi\right) \bullet \left(\rho d\phi \hat{a}_\phi\right) = \rho H_\phi \int_0^{2\pi} d\phi = 2\pi\rho H_\phi$$

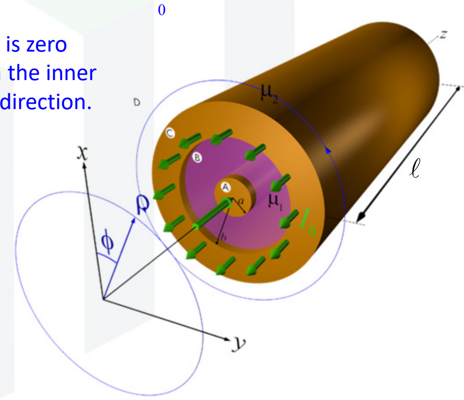
In Region D, the total enclosed current is zero because the same current is present in the inner and outer conductors, but in opposite direction.

$$I_{\mathrm{D}}(\rho) = 0$$

Putting these together shows that

$$0 = 2\pi\rho H_\phi$$

$$H_\phi(\rho) = 0$$



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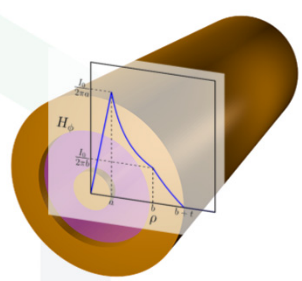
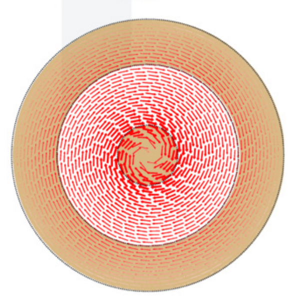
All Together

Step 3 – Calculate magnetic field intensity $\vec{H}$.

All together, the magnetic field intensity is

$$\vec{H}(\rho) = \begin{cases} \dfrac{I_0 \rho}{2\pi a^2} \hat{a}_\phi & 0 \le \rho \le a \\ \dfrac{I_0}{2\pi\rho} \hat{a}_\phi & a \le \rho \le b \\ \dfrac{I_0}{2\pi\rho}\left(1 - \dfrac{\rho^2 - b^2}{t^2 + 2bt}\right)\hat{a}_\phi & b \le \rho \le b+t \\ 0 & \rho \ge b+t \end{cases}$$





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Remaining Steps…

Step 4(alternate) – Calculate total magnetic energy W_m.

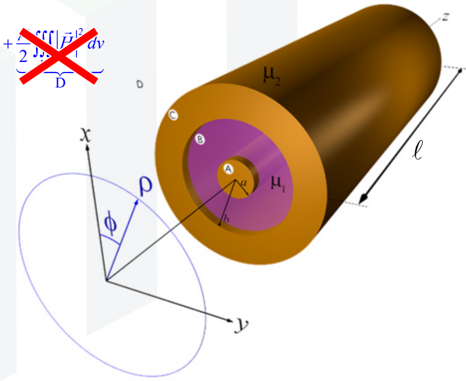
The total magnetic energy is

$$W_\mathrm{m} = \frac{1}{2}\iiint_V \mu\left|\vec{H}\right|^2 dv$$

$$= \underbrace{\frac{\mu}{2}\iiint_V \left|\vec{H}\right|^2 dv}_{\mathrm{A}} + \underbrace{\frac{\mu}{2}\iiint_V \left|\vec{H}\right|^2 dv}_{\mathrm{B}} + \underbrace{\frac{\mu}{2}\iiint_V \left|\vec{H}\right|^2 dv}_{\mathrm{C}} + \underbrace{\frac{\mu}{2}\iiint_V \left|\vec{H}\right|^2 dv}_{\mathrm{D}}$$

There is no magnetic field outside of the coax.

If the outer conductor is very thin, we can ignore the magnetic energy here.



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Remaining Steps…

Step 4(alternate) – Calculate total magnetic energy W_m.

$$W_\mathrm{m} = \underbrace{\frac{\mu}{2}\iiint_V \left|\vec{H}\right|^2 dv}_{\mathrm{A}} + \underbrace{\frac{\mu}{2}\iiint_V \left|\vec{H}\right|^2 dv}_{\mathrm{B}}$$

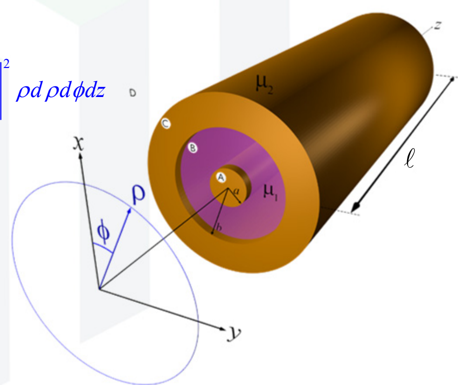
The first term is

$$W_\mathrm{A} = \underbrace{\frac{\mu}{2}\iiint_V \left|\vec{H}\right|^2 dv}_{0\le\rho\le a} = \frac{\mu}{2}\int_{z=0}^{\ell}\int_{\phi=0}^{2\pi}\int_{\rho=0}^{a}\left|\frac{I_0\rho}{2\pi a^2}\hat{a}_\phi\right|^2 \rho d\rho d\phi dz$$

$$= \frac{\mu I_0^2}{8\pi^2 a^4}\int_{z=0}^{\ell}\int_{\phi=0}^{2\pi}\int_{\rho=0}^{a}\rho^3 d\rho d\phi dz$$

$$= \frac{\mu I_0^2}{8\pi^2 a^4}\int_{\rho=0}^{a}\left(\int_{\phi=0}^{2\pi} d\phi\right)\left(\int_{z=0}^{\ell} dz\right)\rho^3 d\rho$$

$$= \frac{\mu I_0^2}{8\pi^2 a^4}\int_{\rho=0}^{a}(2\pi)(\ell)\rho^3 d\rho$$



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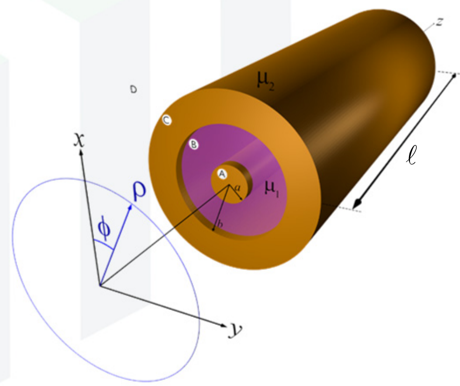
Remaining Steps…

Step 4(alternate) – Calculate total magnetic energy W_m.

$$W_\mathrm{m} = \underbrace{\frac{\mu}{2}\iiint_V \left|\vec{H}\right|^2 dv}_{\mathrm{A}} + \underbrace{\frac{\mu}{2}\iiint_V \left|\vec{H}\right|^2 dv}_{\mathrm{B}}$$

The first term continued…

$$\begin{aligned} W_\mathrm{A} &= \frac{\mu I_0^2}{8\pi^2 a^4}\int_{\rho=0}^{a}(2\pi)(\ell)\rho^3 d\rho \\ &= \frac{\mu I_0^2 \ell}{4\pi a^4}\int_{\rho=0}^{a}\rho^3 d\rho \\ &= \frac{\mu I_0^2 \ell}{4\pi a^4}\left(\left.\frac{\rho^4}{4}\right|_0^a\right) \\ &= \frac{\mu I_0^2 \ell}{4\pi a^4}\left(\frac{a^4}{4} - \frac{0}{4}\right) \\ &= \frac{\mu I_0^2 \ell}{16\pi} \end{aligned}$$



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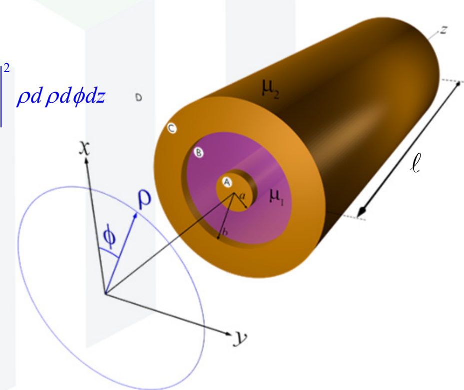
Remaining Steps…

Step 4(alternate) – Calculate total magnetic energy W_m.

$$W_\mathrm{m} = \underbrace{\frac{\mu}{2}\iiint_V \left|\vec{H}\right|^2 dv}_{\mathrm{A}} + \underbrace{\frac{\mu}{2}\iiint_V \left|\vec{H}\right|^2 dv}_{\mathrm{B}}$$

The second term is

$$\begin{aligned} W_\mathrm{B} &= \underbrace{\frac{\mu}{2}\iiint_V \left|\vec{H}\right|^2 dv}_{a<\rho\le b} = \frac{\mu}{2}\int_{z=0}^{\ell}\int_{\phi=0}^{2\pi}\int_{\rho=a}^{b}\left|\frac{I_0}{2\pi\rho}\hat{a}_\phi\right|^2 \rho d\rho d\phi dz \\ &= \frac{\mu I_0^2}{8\pi^2}\int_{z=0}^{\ell}\int_{\phi=0}^{2\pi}\int_{\rho=a}^{b}\frac{1}{\rho}d\rho d\phi dz \\ &= \frac{\mu I_0^2}{8\pi^2}\int_{\rho=a}^{b}\left(\int_{\phi=0}^{2\pi}d\phi\right)\left(\int_{z=0}^{\ell}dz\right)\frac{1}{\rho}d\rho \\ &= \frac{\mu I_0^2}{8\pi^2}\int_{\rho=a}^{b}(2\pi)(\ell)\frac{1}{\rho}d\rho \end{aligned}$$



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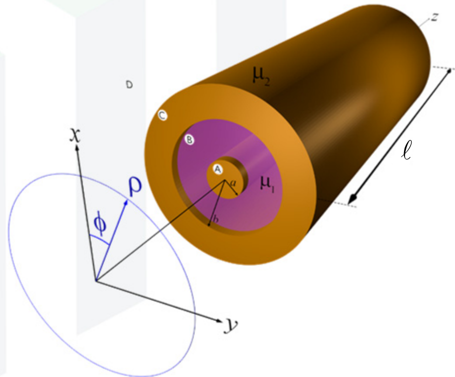
Remaining Steps…

Step 4(alternate) – Calculate total magnetic energy W_m.

$$W_m = \underbrace{\frac{\mu}{2}\iiint_V \left|\vec{H}\right|^2 dv}_{A} + \underbrace{\frac{\mu}{2}\iiint_V \left|\vec{H}\right|^2 dv}_{B}$$

The second term continued…

$$W_B = \frac{\mu I_0^2}{8\pi^2}\int_{\rho=a}^{b}(2\pi)(\ell)\frac{1}{\rho}d\rho$$
$$= \frac{\mu I_0^2 \ell}{4\pi}\int_{\rho=a}^{b}\frac{1}{\rho}d\rho$$
$$= \frac{\mu I_0^2 \ell}{4\pi}\left(\ln\rho\Big|_a^b\right)$$
$$= \frac{\mu I_0^2 \ell}{4\pi}(\ln b - \ln a)$$
$$= \frac{\mu I_0^2 \ell}{4\pi}\ln\left(\frac{b}{a}\right)$$



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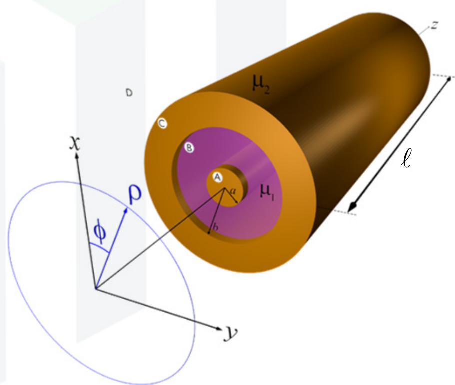
Remaining Steps…

Step 4(alternate) – Calculate total magnetic energy W_m.

$$W_m = \underbrace{\frac{\mu}{2}\iiint_V \left|\vec{H}\right|^2 dv}_{A} + \underbrace{\frac{\mu}{2}\iiint_V \left|\vec{H}\right|^2 dv}_{B}$$

The total magnetic energy in the coax is

$$W_m = W_A + W_B$$
$$= \frac{\mu I_0^2 \ell}{16\pi} + \frac{\mu I_0^2 \ell}{4\pi}\ln\left(\frac{b}{a}\right)$$
$$= \frac{\mu I_0^2 \ell}{4\pi}\left[\frac{1}{4} + \ln\left(\frac{b}{a}\right)\right]$$



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Remaining Steps…

Step 5(alternate) – Calculate inductance L.

The inductance is

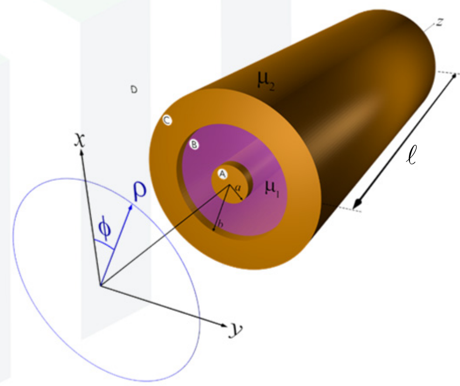
$$L = \frac{2W_m}{I_0^2}$$

$$= \frac{2\dfrac{\mu I_0^2 \ell}{4\pi}\left[\dfrac{1}{4}+\ln\left(\dfrac{b}{a}\right)\right]}{I_0^2}$$

$$\boxed{L = \frac{\mu \ell}{2\pi}\left[\frac{1}{4}+\ln\left(\frac{b}{a}\right)\right]}$$

The inductance per unit length is

$$\boxed{\frac{L}{\ell} = \frac{\mu}{2\pi}\left[\frac{1}{4}+\ln\left(\frac{b}{a}\right)\right]}$$



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Example #4:

RG-59 Coaxial Transmission Line

Transmission Line Impedance

We have actually performed a very sophisticated analysis.

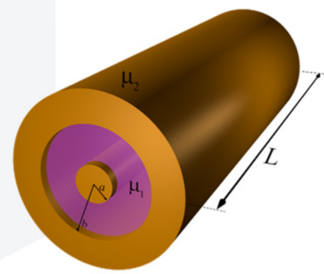
At the end of electrostatics, we derived the distributed capacitance.

$$\frac{C}{\ell}=\frac{2\pi\varepsilon}{\ln\left(\frac{a}{b}\right)}$$

At the end of magnetostatics, we derived the distributed inductance.

$$\frac{L}{\ell}=\frac{\mu}{2\pi}\left[\frac{1}{4}+\ln\left(\frac{b}{a}\right)\right]$$

We will now be able to calculate the characteristic impedance Z_0 of a coaxial transmission line!



$$Z_0=\sqrt{\frac{L/\ell}{C/\ell}}$$

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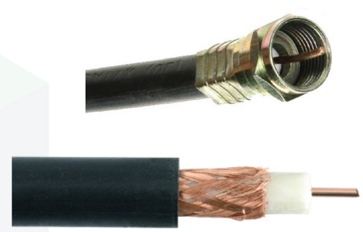
Example #7 – RG-59 Coax

A standard RG-59 coax has

Inner conductor diameter:	0.81 mm (20 AWG)
Outer conductor diameter:	3.66 mm
Dielectric constant:	2.1
Specified capacitance:	86.9 pF/m



$$\frac{C}{\ell}=\frac{2\pi\left(8.854\times10^{-12}\ \text{F/m}\right)(2.1)}{\ln\left(3.66\ \text{mm}/0.81\ \text{mm}\right)}=7.746\times10^{-11}\ \text{F/m}=77.46\ \text{pF/m}$$

$$\frac{L}{\ell}=\frac{1.2566\times10^{-6}\ \text{H/m}}{2\pi}\left[\frac{1}{4}+\ln\left(\frac{3.66}{0.81}\right)\right]=3.52\times10^{-7}\ \text{H/m}=352\ \text{nH/m}$$

$$Z_0=\sqrt{\frac{3.52\times10^{-7}\ \text{H/m}}{7.746\times10^{-11}\ \text{F/m}}}=64.7\ \Omega$$

The specified impedance is 75 Ω.

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