



Electromagnetics:  
Electromagnetic Field Theory

# Math with Vectors



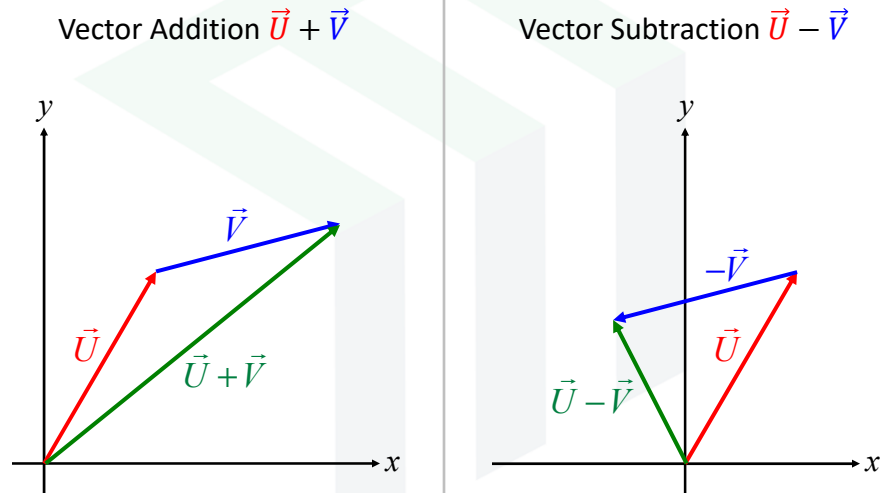
1

## Outline

- Vector Addition and Subtraction
- Distance, Length & Perimeter
- Dot Product & Projections
- Cross Product & Area
- Vector Algebra Rules
- Triple Products

2

## Visualization of Vector Addition & Subtraction



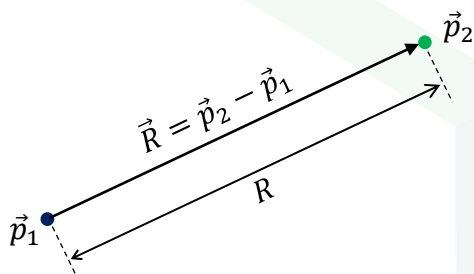
3

## Vector Addition & Subtraction

Cartesian	Cylindrical	Spherical
<b>Starting Vectors</b> $\vec{U} = U_x \hat{a}_x + U_y \hat{a}_y + U_z \hat{a}_z$ $\vec{V} = V_x \hat{a}_x + V_y \hat{a}_y + V_z \hat{a}_z$	<b>Starting Vectors</b> $\vec{U} = U_\rho \hat{a}_\rho + U_\phi \hat{a}_\phi + U_z \hat{a}_z$ $\vec{V} = V_\rho \hat{a}_\rho + V_\phi \hat{a}_\phi + V_z \hat{a}_z$	<b>Starting Vectors</b> $\vec{U} = U_r \hat{a}_r + U_\theta \hat{a}_\theta + U_\phi \hat{a}_\phi$ $\vec{V} = V_r \hat{a}_r + V_\theta \hat{a}_\theta + V_\phi \hat{a}_\phi$
<b>Addition</b> $\vec{U} + \vec{V} = (U_x + V_x) \hat{a}_x$ $+ (U_y + V_y) \hat{a}_y$ $+ (U_z + V_z) \hat{a}_z$	<b>Addition</b> $\vec{U} + \vec{V} = (U_\rho + V_\rho) \hat{a}_\rho$ $+ (U_\phi + V_\phi) \hat{a}_\phi$ $+ (U_z + V_z) \hat{a}_z$	<b>Addition</b> $\vec{U} + \vec{V} = (U_r + V_r) \hat{a}_r$ $+ (U_\theta + V_\theta) \hat{a}_\theta$ $+ (U_\phi + V_\phi) \hat{a}_\phi$
<b>Subtraction</b> $\vec{U} - \vec{V} = (U_x - V_x) \hat{a}_x$ $+ (U_y - V_y) \hat{a}_y$ $+ (U_z - V_z) \hat{a}_z$	<b>Subtraction</b> $\vec{U} - \vec{V} = (U_\rho - V_\rho) \hat{a}_\rho$ $+ (U_\phi - V_\phi) \hat{a}_\phi$ $+ (U_z - V_z) \hat{a}_z$	<b>Subtraction</b> $\vec{U} - \vec{V} = (U_r - V_r) \hat{a}_r$ $+ (U_\theta - V_\theta) \hat{a}_\theta$ $+ (U_\phi - V_\phi) \hat{a}_\phi$

4

## Vector and Distance Between Two Points



### Vector $\vec{R}$ Connecting Two Points

$$\vec{R} = \vec{p}_2 - \vec{p}_1$$

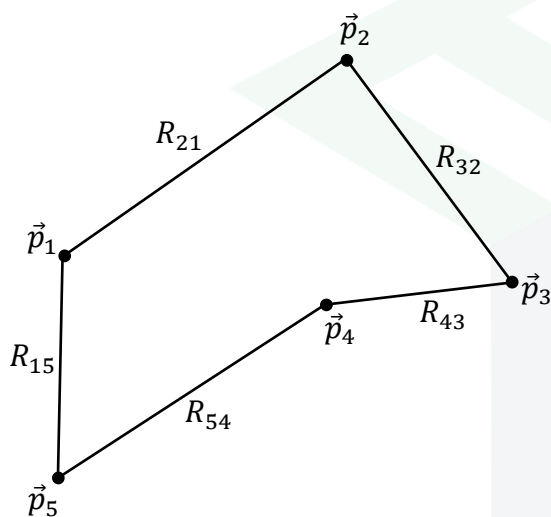
Observe the order of the subtraction.  
"End point minus start point."

### Distance $R$ Between Two Points

$$R = |\vec{R}| = |\vec{p}_2 - \vec{p}_1| = |\vec{p}_1 - \vec{p}_2|$$

5

## Perimeter of a Polygon



### Length of Each Side

$$R_{21} = |\vec{p}_2 - \vec{p}_1|$$

$$R_{32} = |\vec{p}_3 - \vec{p}_2|$$

$$R_{43} = |\vec{p}_4 - \vec{p}_3|$$

$$R_{54} = |\vec{p}_5 - \vec{p}_4|$$

$$R_{15} = |\vec{p}_1 - \vec{p}_5|$$

### Total Perimeter (or total length)

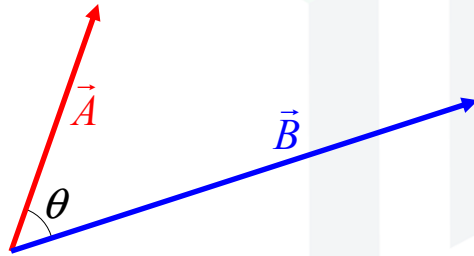
$$R = R_{21} + R_{32} + R_{43} + R_{54} + R_{15}$$

6

## The Dot Product $\vec{A} \cdot \vec{B}$

The dot product is all about projections. That is, calculating how much of one vector lies in the direction of another vector.

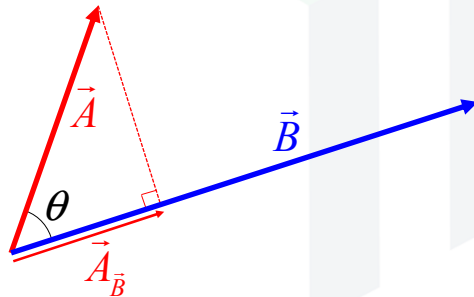
$$\begin{aligned}\vec{A} \cdot \vec{B} &= |\vec{A}| |\vec{B}| \cos \theta \\ &= A_x B_x + A_y B_y + A_z B_z\end{aligned}$$



7

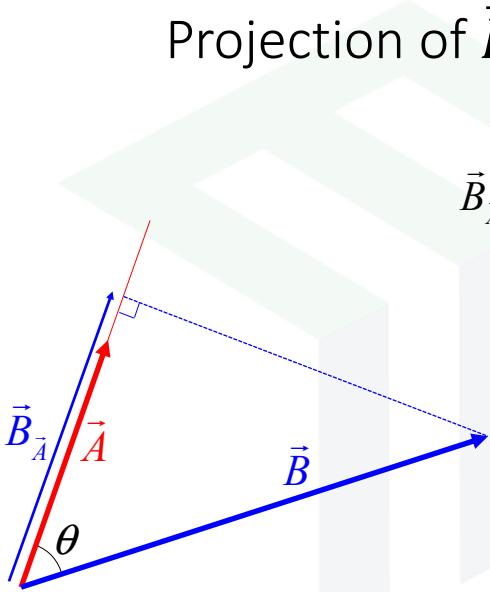
## Projection of $\vec{A}$ onto $\vec{B}$

$$\vec{A}_{\vec{B}} = \underbrace{\left( \frac{\vec{A} \cdot \vec{B}}{|\vec{B}|} \right)}_{\text{Magnitude } |\vec{A}_{\vec{B}}|} \underbrace{\left( \frac{\vec{B}}{|\vec{B}|} \right)}_{\text{Direction } \hat{B} = \frac{\vec{B}}{|\vec{B}|}} = \frac{\vec{A} \cdot \vec{B}}{|\vec{B}|^2} \vec{B}$$



8

## Projection of $\vec{B}$ onto $\vec{A}$




$$\vec{B}_A = \underbrace{\left( \frac{\vec{B} \cdot \vec{A}}{|\vec{A}|} \right)}_{\text{Magnitude } |\vec{B}_A|} \underbrace{\left( \frac{\vec{A}}{|\vec{A}|} \right)}_{\text{Direction } \hat{A} = \frac{\vec{A}}{|\vec{A}|}} = \frac{\vec{B} \cdot \vec{A}}{|\vec{A}|^2} \vec{A}$$

EMPossible Slide 9

9

## The Dot Product Test

The dot product can be used to test if two vectors are perpendicular. If they are, the component of one along the other must be zero so the dot product must be zero.

$$\vec{A} \cdot \vec{B} = 0 \quad \text{when } \vec{A} \perp \vec{B}$$


EMPossible Slide 10

10

## Sesquilinear Dot Product

Bilinear Dot Product

$$\vec{A} \bullet \vec{B} = \sum_i A_i B_i$$

$$\vec{A} \bullet \vec{B} = \vec{B} \bullet \vec{A}$$

$$\vec{A} \bullet \vec{A} > 0$$

$$\vec{A} \bullet [\epsilon] \vec{B} = \vec{B} \bullet [\epsilon]^T \vec{A}$$

Sesquilinear Dot Product

$$\vec{A} \bullet \vec{B} = \sum_i A_i B_i^*$$

$$\vec{A} \bullet \vec{B} = (\vec{B} \bullet \vec{A})^*$$

$$\vec{A} \bullet \vec{A} > 0$$

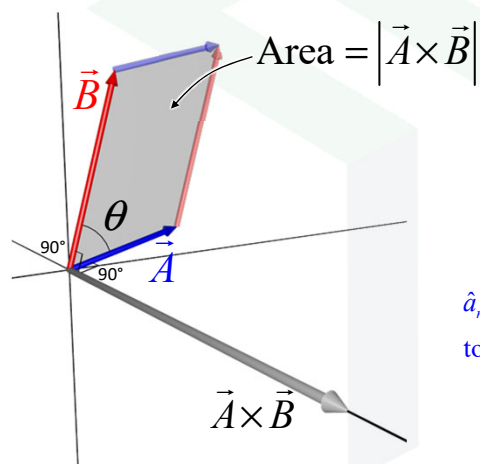
$$\vec{A} \bullet [\epsilon] \vec{B} = \vec{B} \bullet [\epsilon]^T \vec{A}^*$$

In frequency-domain electromagnetics, vectors can contain complex numbers. In this case, it is the sesquilinear dot product that should be used. For real-valued vectors, both dot products are equivalent.

11

## The Cross Product, $\vec{A} \times \vec{B}$

The cross product is all about area and calculating vectors that are perpendicular to  $\vec{A}$  and  $\vec{B}$ .

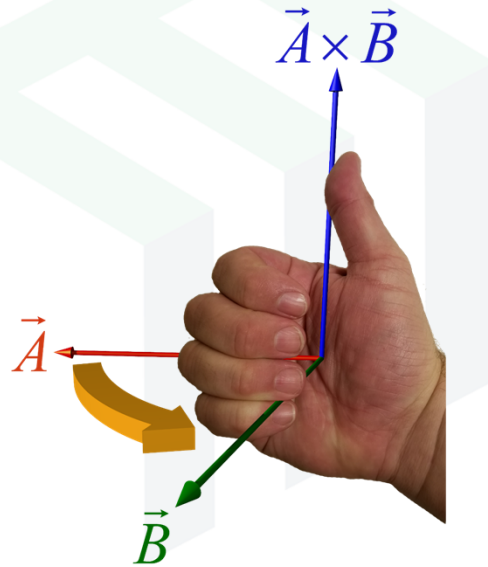


$$\vec{A} \times \vec{B} = |\vec{A}| |\vec{B}| \sin \theta \hat{a}_n$$

$\hat{a}_n$  is a unit vector perpendicular to the plane defined by  $\vec{A}$  and  $\vec{B}$ .

12

## Handedness of the Cross Product



13

## Calculating Cross Products (1 of 2)

Suppose we wish to calculate the cross product  $\vec{A} \times \vec{B}$ .

$$\vec{A} = A_x \hat{x} + A_y \hat{y} + A_z \hat{z} \quad \vec{B} = B_x \hat{x} + B_y \hat{y} + B_z \hat{z}$$

Step 1 – Construct an augmented matrix.

$$\left[ \begin{array}{ccc|cc} \hat{x} & \hat{y} & \hat{z} & \hat{x} & \hat{y} \\ A_x & A_y & A_z & A_x & A_y \\ B_x & B_y & B_z & B_x & B_y \end{array} \right]$$

First two columns are repeated outside of the matrix.

14

## Calculating Cross Products (2 of 2)

Step 2 – Multiply elements along the diagonals.

$$\begin{array}{c}
 \begin{array}{ccc|cc}
 \hat{x} & \hat{y} & \hat{z} & \hat{x} & \hat{y} \\
 A_x & A_y & A_z & A_x & A_y \\
 B_x & B_y & B_z & B_x & B_y
 \end{array} \\
 \begin{array}{l}
 -A_y B_x \hat{z} \\
 -A_z B_y \hat{x} \\
 -A_x B_z \hat{y}
 \end{array}
 \end{array}
 \begin{array}{l}
 A_x B_y \hat{z} \\
 A_z B_x \hat{y} \\
 A_y B_z \hat{x}
 \end{array}$$

Step 3 – Make left-hand side products negative.

Step 4 – Add up all of the products.

$$\vec{A} \times \vec{B} = (A_y B_z - A_z B_y) \hat{x} + (A_z B_x - A_x B_z) \hat{y} + (A_x B_y - A_y B_x) \hat{z}$$

## The Cross Product Test

The cross product can be used to test if two vectors are parallel. If they are, the cross product will be zero because the angle between the vectors is zero.

$$\vec{A} \times \vec{B} = 0 \quad \text{when } \vec{A} \parallel \vec{B}$$

$$\begin{array}{cc}
 |\vec{A} \times \vec{B}| = 0 & |\vec{A} \times \vec{B}| \neq 0 \\
 \begin{array}{c} \vec{A} \\ \vec{B} \end{array} & \begin{array}{c} \vec{A} \\ \vec{B} \end{array}
 \end{array}$$

## Vector Algebra Rules

### Commutative Laws

$$\vec{A} \cdot \vec{B} = \vec{B} \cdot \vec{A}$$

$$\vec{A} \times \vec{B} = -\vec{B} \times \vec{A}$$

### Associative Laws

$$\vec{A} \times (\vec{B} \times \vec{C}) \neq (\vec{A} \times \vec{B}) \times \vec{C}$$

### Distributive Laws

$$\vec{A} \cdot (\vec{B} + \vec{C}) = \vec{A} \cdot \vec{B} + \vec{A} \cdot \vec{C}$$

$$\vec{A} \times (\vec{B} + \vec{C}) = \vec{A} \times \vec{B} + \vec{A} \times \vec{C}$$

### Self-Product

$$\vec{A} \cdot \vec{A} = |\vec{A}|^2$$

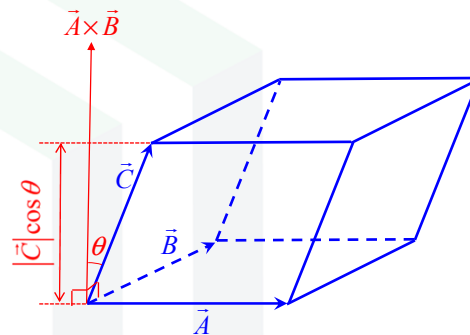
$$\vec{A} \times \vec{A} = 0$$

## Triple Products

### Scalar Triple Product

The scalar triple product is the volume of a parallelepiped.

$$(\vec{B} \times \vec{C}) \cdot \vec{A} = (\vec{C} \times \vec{A}) \cdot \vec{B} = (\vec{A} \times \vec{B}) \cdot \vec{C}$$



### Vector Triple Product

The vector triple product arises when deriving the wave equation.

$$\vec{A} \times (\vec{B} \times \vec{C}) = \vec{B}(\vec{A} \cdot \vec{C}) - \vec{C}(\vec{A} \cdot \vec{B})$$