Electromagnetics:
Electromagnetic Field Theory

The Parallel Plate Waveguide

Lecture Outline

- What is a parallel plate waveguide?
- TEM Analysis
- TM Analysis
- TE Analysis
- Conclusions

What is a Parallel Plate Waveguide?

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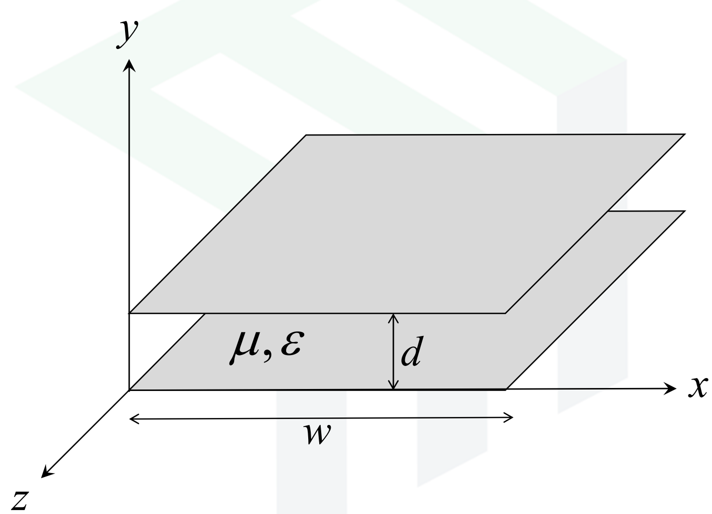
Geometry of Parallel Plate Waveguide



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TEM Analysis

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Starting Point for TEM Analysis

Assuming the parallel plate waveguide has an LHI dielectric between the plates, start with the homogeneous Laplaces' equation.

$$\nabla^2 V(x,y,z) = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0$$

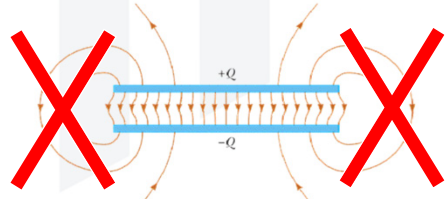
The parallel plate waveguide is uniform in the x and z directions so the governing equation $\nabla^2 V = 0$ reduces to

$$\cancel{\frac{\partial^2 V}{\partial x^2}} + \frac{\partial^2 V}{\partial y^2} + \cancel{\frac{\partial^2 V}{\partial z^2}} = 0$$

$$\frac{\partial^2 V}{\partial y^2} = 0$$

$$\frac{d^2 V}{d y^2} = 0$$

Note, by assuming the field is uniform in the x direction, the fringing fields at the edges are ignored.



(a)

The derivative becomes ordinary because y is the only independent variable left.

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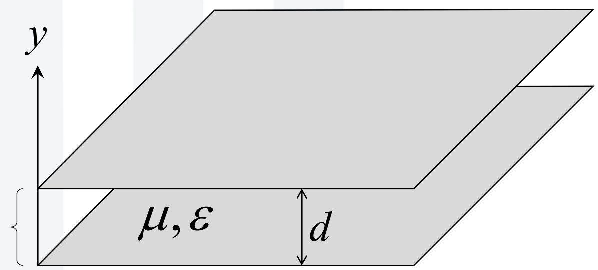
How to Interpret the Governing Equation

The governing equation is now

$$\frac{d^2V}{dy^2} = 0$$

The solution to this will give $V(y)$.

$V(y) \quad 0 \le y \le d$



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Boundary Conditions

Boundary conditions are needed to solve the differential equation.

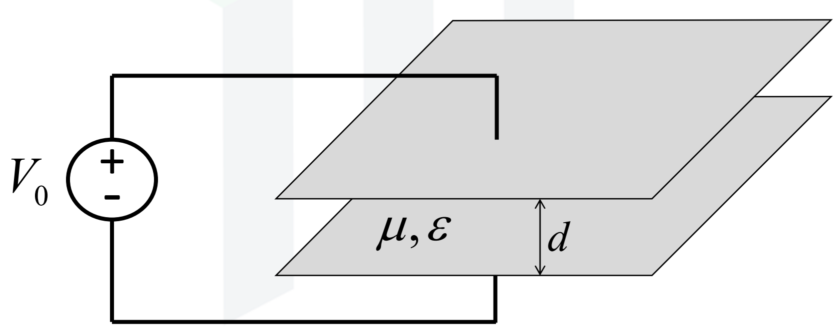
$V(0) = ?$

$V(d) = ?$

Apply a voltage V_0 across the plates and the boundary conditions will be

$V(0) = 0$

$V(d) = V_0$



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General Solution to Differential Equation

The differential equation with boundary conditions is

$$\frac{d^2V}{dy^2} = 0 \qquad 0 \le y \le d \qquad\qquad V(0) = 0 \text{ and } V(d) = V_0$$

This is solved after integrating with respect to y twice.

$$\frac{d^2V}{dy^2} = 0$$

$$\frac{dV}{dy} = A$$

$$V(y) = Ay + B$$

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Apply Boundary Conditions

The general solution is now

$$V(y) = Ay + B$$

Apply the boundary condition at $y = 0$.

$$V(0) = 0$$

$$A \cdot 0 + B = 0$$

$$B = 0$$

Apply the boundary condition at $y = d$.

$$V(d) = V_0$$

$$A \cdot d + \cancel{B} = V_0$$

$$A \cdot d = V_0$$

$$A = V_0 / d$$

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The Solution (1 of 2)

The final solution to the governing equation is

$$V(y) = \frac{V_0}{d} y$$

The analysis is still not finished because nothing was learned about the waveguide.

The electric field is calculated from the electric potential to be

$$\vec{E} = -\nabla V = -\hat{a}_x \frac{\partial V}{\partial x} - \hat{a}_y \frac{\partial V}{\partial y} - \hat{a}_z \frac{\partial V}{\partial z}$$

$$\vec{E} = -\hat{a}_x \frac{\partial}{\partial x}\left(\frac{V_0}{d} y\right) - \hat{a}_y \frac{\partial}{\partial y}\left(\frac{V_0}{d} y\right) - \hat{a}_z \frac{\partial}{\partial z}\left(\frac{V_0}{d} y\right)$$

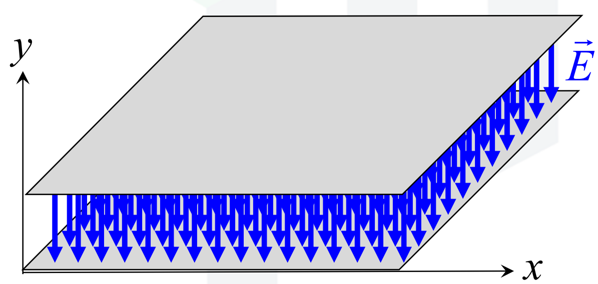
$$\vec{E} = -\cancel{\hat{a}_x \frac{\partial}{\partial x}\left(\frac{V_0}{d} y\right)} - \hat{a}_y \frac{V_0}{d} - \cancel{\hat{a}_z \frac{\partial}{\partial z}\left(\frac{V_0}{d} y\right)}$$

$$\vec{E} = -\hat{a}_y \frac{V_0}{d}$$

The Solution (2 of 2)

If the fringing fields are ignored outside of the waveguide, the electric field is expressed as

$$\vec{E}(x,y) = \begin{cases} -\hat{a}_y \dfrac{V_0}{d} & \text{for } 0 \le x \le w \text{ and } 0 \le y \le d \\ 0 & \text{otherwise} \end{cases}$$



The Wave Solution

The TEM wave was derived by way of an electrostatic analysis. This ignores the wave propagating nature of a TEM wave. To account for propagation in the z direction, $e^{-j\beta z}$ must be incorporated to account for accumulation of phase in the z direction.

$$\vec{E}(x,y,z)=\begin{cases}-\hat{a}_y \dfrac{V_0}{d} e^{-j\beta z} & \text{for } 0 \le x \le w \text{ and } 0 \le y \le d \\ 0 & \text{otherwise}\end{cases}$$

It follows that the magnetic field component is

$$\vec{H}(x,y,z)=\frac{\hat{a}_z \times \vec{E}}{\eta}=\frac{\hat{a}_z \times \left(-\hat{a}_y \frac{V_0}{d} e^{-j\beta z}\right)}{\eta}=-\left(\hat{a}_z \times \hat{a}_y\right)\frac{V_0}{\eta d} e^{-j\beta z}=\hat{a}_x \frac{V_0}{\eta d} e^{-j\beta z}$$

$$\vec{H}(x,y,z)=\begin{cases}\hat{a}_x \dfrac{V_0}{\eta d} e^{-j\beta z} & \text{for } 0 \le x \le w \text{ and } 0 \le y \le d \\ 0 & \text{otherwise}\end{cases}$$

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Impedance from Wave Solution (1 of 2)

The impedance Z_{TEM} of the TEM wave is defined as

$$Z_{\text{TEM}}=\frac{V_0}{I}$$

The current term I must be determined. Recall the magnetic field $\vec{H}$ above an infinite current sheet is

$$\vec{H}_{1\text{ sheet}}=\frac{\vec{K}\times\hat{n}}{2}$$

$\vec{K} \equiv$ surface current density (A/m) $\hat{n}=-\hat{a}_y$

Using this equation for the parallel plate waveguide ignores fringing fields at the edges.

It follows that the field between two current sheets (i.e. in the parallel plate waveguide) is

$$\vec{H}_{2\text{ sheets}}=2\vec{H}_{1\text{ sheet}}=\vec{K}\times\hat{n}$$

Solving this for the surface current $\vec{K}$ yields

$$\vec{K}=\hat{n}\times\vec{H}=\left(-\hat{a}_y\right)\times\vec{H}=\vec{H}\times\hat{a}_y$$

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Impedance from Wave Solution (2 of 2)

Find the total current I by integrating the surface current across the plate.

$$I = \int_0^w \left(\vec{K} \bullet \hat{a}_z \right) dx = \int_0^w \left[\left(\vec{H} \times \hat{a}_y \right) \bullet \hat{a}_z \right] dx = \int_0^w H_x dx$$

Let $z = 0$ and the magnetic field $\vec{H}$ reduces to

$$H_x \left(z = 0 \right) = \frac{V_0}{\eta d}$$

Substituting $H_x(z = 0)$ into the equation for I leads to

$$I = \int_0^w \left(\frac{V_0}{\eta d} \right) dx = \frac{V_0}{\eta d} \int_0^w dx = \frac{V_0}{\eta d} w = \frac{w}{d} \frac{V_0}{\eta}$$

The characteristic impedance Z_{TEM} is found by substituting this into the original definition.

$$Z_{\text{TEM}} = \frac{V_0}{I} = \frac{V_0}{\frac{w}{d} \frac{V_0}{\eta}} = \eta \frac{d}{w}$$

$$\boxed{Z_{\text{TEM}} = \eta \frac{d}{w}}$$

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Propagation Constant β_{TEM}

The phase constant β_{TEM} cannot be calculated from the present solution because it was analyzed using an electrostatic approximation where there is no propagation.

Recall for TEM modes that $\beta = k$. This implies that TEM waves propagate with the same speed as a plane wave in an infinite medium composed of the dielectric that resides between the plates.

$$\boxed{\beta_{\text{TEM}} \cong \omega \sqrt{\mu \varepsilon}}$$

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Distributed Inductance and Capacitance

The characteristic impedance Z_{TEM} of the parallel plate transmission line is

$$Z_{\text{TEM}} = \eta \frac{d}{w}$$

The distributed capacitance C can be estimated by looking at the transmission like a parallel plate capacitor.

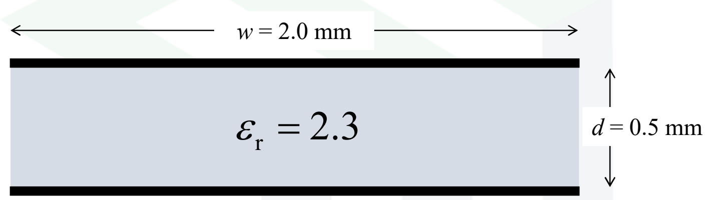
$$C = \varepsilon \frac{w}{d}$$

It follows that the distributed inductance L is

$$Z_{\text{TEM}} = \eta \frac{d}{w} = \sqrt{\frac{L}{C}} = \sqrt{\frac{L}{\varepsilon \left(w/d \right)}} \qquad L = \mu \frac{d}{w}$$

Example #1 (1 of 3)

Given the following parallel plate waveguide...



What is the characteristic impedance Z_{TEM}?

What value of w would make this transmission line 50 Ω?

Example #1 (2 of 3)

The equation for characteristic impedance Z_{TEM} is

$$Z_{\text{TEM}} = \eta \frac{d}{w}$$

The impedance η of the dielectric is

$$\eta = \eta_0 \sqrt{\frac{\mu_r}{\varepsilon_r}} = (376.73\ \Omega)\sqrt{\frac{1.0}{2.3}} = 248.4\ \Omega$$

The characteristic impedance Z_{TEM} is therefore

$$Z_{\text{TEM}} = \frac{\eta d}{w} = \frac{(248.4\ \Omega)(0.5\ \text{mm})}{(2.0\ \text{mm})} = \boxed{62.1\ \Omega}$$

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Example #1 (3 of 3)

Solve the equation for characteristic impedance for w.

$$Z_{\text{TEM}} = \eta \frac{d}{w} \quad \rightarrow \quad w = \frac{\eta d}{Z_{\text{TEM}}}$$

To get 50 Ω, w must be

$$w = \frac{\eta d}{Z_0} = \frac{(248.4\ \Omega)(0.5\ \text{mm})}{(50\ \Omega)} = \boxed{2.48\ \text{mm}}$$

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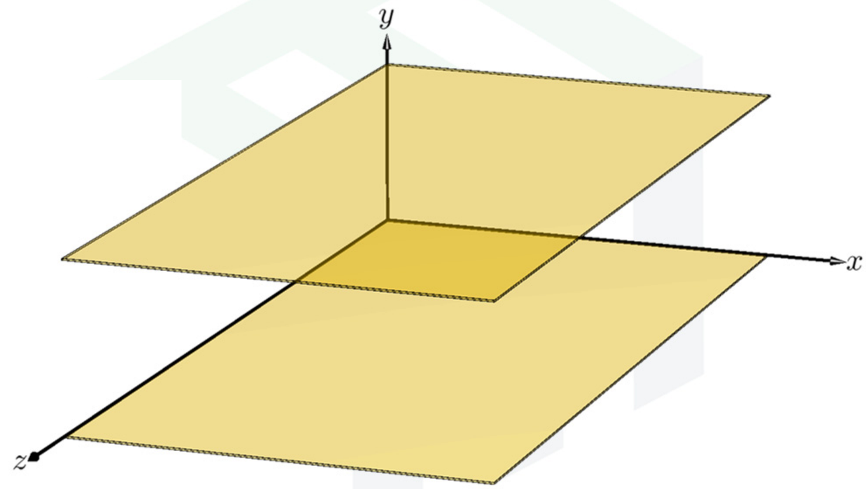
Visualization of TEM Mode



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Summary of TEM Analysis

Field Solution

$E_x(x,y,z)=0$

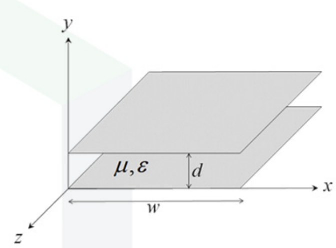
$E_y(x,y,z)=-\frac{V_0}{d}e^{-j\beta z}$

$E_z(x,y,z)=0$

$H_x(x,y,z)=\frac{V_0}{\eta d}e^{-j\beta z}$

$H_y(x,y,z)=0$

$H_z(x,y,z)=0$



- TEM has no cutoff frequency
- TEM is the TM_0 mode.

Phase Constant

$\beta_{\text{TEM}} \cong \omega\sqrt{\mu\varepsilon}$

Same as plane wave.

Cutoff Frequency

$f_c = 0$

No cutoff frequency.
Mode supported at DC.

Characteristic Impedance

$Z_{\text{TEM}} = \eta\frac{d}{w}$

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TE Analysis

Recall the Starting Point

The governing equation for TE analysis (i.e. $E_{0,z} = 0$) is

$$\frac{\partial^2 H_{0,z}}{\partial x^2} + \frac{\partial^2 H_{0,z}}{\partial y^2} - k_c^2 H_{0,z} = 0 \qquad k_c^2 = k^2 - \beta^2$$

After a solution is obtained, the remaining field components are calculated according to

$$H_{0,x} = -\frac{j\beta}{k_c^2}\frac{\partial H_{0,z}}{\partial x} \qquad E_{0,x} = -\frac{j\omega\mu}{k_c^2}\frac{\partial H_{0,z}}{\partial y}$$

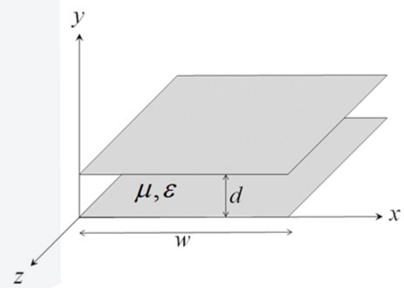
$$H_{0,y} = -\frac{j\beta}{k_c^2}\frac{\partial H_{0,z}}{\partial y} \qquad E_{0,y} = \frac{j\omega\mu}{k_c^2}\frac{\partial H_{0,z}}{\partial x}$$

$$E_{0,z} = 0$$

Simplify Governing Equation

Assuming the waveguide is uniform in the direction of x

$$\frac{\partial}{\partial x} = 0 \quad \text{and} \quad \frac{\partial^2}{\partial x^2} = 0$$



The governing equation reduces to

$$\cancel{\frac{\partial^2 H_{0,z}}{\partial x^2}} + \frac{\partial^2 H_{0,z}}{\partial y^2} - k_c^2 H_{0,z} = 0 \quad \rightarrow \quad \frac{d^2 H_{0,z}}{dy^2} - k_c^2 H_{0,z} = 0$$

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General Solution

The general solution to the governing equation is

$$\frac{d^2 H_{0,z}}{dy^2} - k_c^2 H_{0,z} = 0 \quad \rightarrow \quad H_{0,z} = A\sin(k_c y) + B\cos(k_c y)$$

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Boundary Conditions (1 of 2)

The tangential component of the electric field $\vec{E}$ must be zero at the plates.

The solution, however, is in terms of the magnetic field $H_{0,z}$.

The solution must be written in terms of an electric field that is tangential to the plates.

The only component of the electric field tangential to the plates is $E_{0,x}$.

$$E_{0,x} = -\frac{j\omega\mu}{k_c^2}\frac{\partial H_{0,z}}{\partial y} = -\frac{j\omega\mu}{k_c^2}\frac{\partial}{\partial y}\left[A\sin(k_c y) + B\cos(k_c y)\right]$$

$$= -\frac{j\omega\mu}{k_c}\left[A\cos(k_c y) - B\sin(k_c y)\right]$$

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Boundary Conditions (2 of 2)

The first boundary condition is

$$E_{0,x}(x,0) = 0 \qquad E_{0,x}(x,0) = -\frac{j\omega\mu}{k_c}\left[A\cos(0) - B\sin(0)\right] = -\frac{j\omega\mu}{k_c}A = 0 \quad \rightarrow \quad A = 0$$

The second boundary condition is

$$E_{0,x}(x,d) = 0 \qquad E_{0,x}(x,d) = -\frac{j\omega\mu}{k_c}\left[-B\sin(k_c d)\right] = B\frac{j\omega\mu}{k_c}\sin(k_c d)$$

$B = 0$ cannot be chosen because that would lead to a trivial solution. Instead, it must be the $\sin(k_c d)$ term that is zero at $y = d$.

$$\sin(k_c d) = 0 \quad \rightarrow \quad k_c d = m\pi \qquad m = 1,2,3,\ldots$$

Note that $m = 0$ would force the entire field to be zero so this is not a valid solution.

The cutoff wave number k_c is then

$$k_c = \frac{m\pi}{d} \qquad m = 1,2,3,\ldots$$

Remember this equation k_c for the next slide.

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Phase Constant β (1 of 2)

Recall the original definition of the cutoff wave number k_c. Solve this equation for β.

$$k_c^2 = k^2 - \beta^2 \quad \rightarrow \quad \beta = \sqrt{k^2 - k_c^2}$$

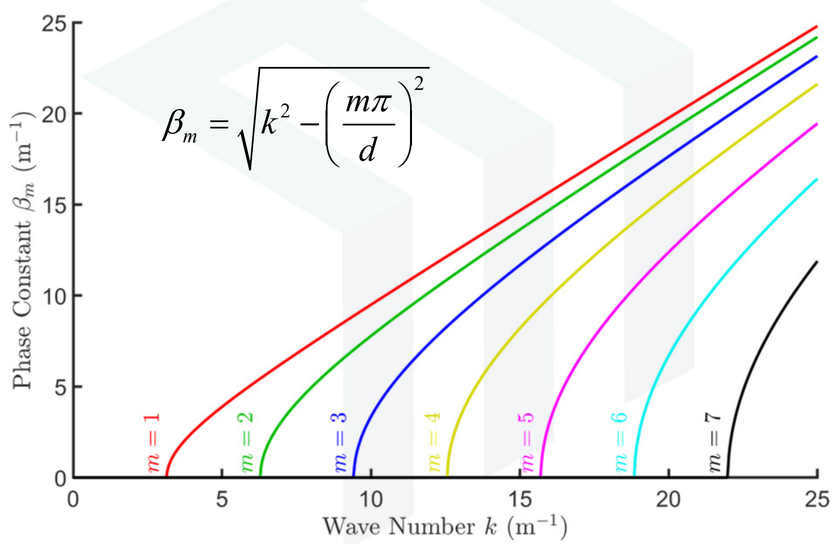
An expression for k_c was derived on the previous slide that arose from the boundary conditions.

$$\beta = \sqrt{k^2 - \left(\frac{m\pi}{d}\right)^2} \qquad m = 1, 2, 3, \ldots$$

Since m is only integer values, it is concluded that there are an infinite number of discrete solutions and the order of the mode is m.

$$\boxed{\beta_m = \sqrt{k^2 - \left(\frac{m\pi}{d}\right)^2} \qquad m = 1, 2, 3, \ldots}$$

Phase Constant β (2 of 2)



Final Solution

Recall the general solution was

$$\frac{d^2 H_{0,z}}{dy^2} - k_c^2 H_{0,z} = 0 \quad \rightarrow \quad H_{0,z} = A\sin(k_c y) + B\cos(k_c y)$$

But now it is known that $A = 0$ and $k_c = m\pi/d$. The final solution is

$$H_{0,z}(x,y) = B_m \cos\left(\frac{m\pi y}{d}\right) \quad \rightarrow \quad \boxed{H_z(x,y,z) = B_m \cos\left(\frac{m\pi y}{d}\right) e^{-j\beta_m z}}$$

The remaining field components are calculated from this result.

$$H_x(x,y,z) = -\frac{j\beta_m}{k_c^2}\frac{\partial H_z}{\partial x} = -\frac{j\beta_m}{k_c^2}\frac{\partial}{\partial x}\left[B_m \cos\left(\frac{m\pi y}{d}\right) e^{-j\beta_m z}\right] = 0$$

$$H_y(x,y,z) = -\frac{j\beta_m}{k_c^2}\frac{\partial H_z}{\partial y} = -\frac{j\beta_m}{k_c^2}\frac{\partial}{\partial y}\left[B_m \cos\left(\frac{m\pi y}{d}\right) e^{-j\beta_m z}\right] = \frac{j\beta_m}{k_c} B_m \sin\left(\frac{m\pi y}{d}\right) e^{-j\beta_m z}$$

$$E_x(x,y,z) = -\frac{j\omega\mu}{k_c^2}\frac{\partial H_z}{\partial y} = -\frac{j\omega\mu}{k_c^2}\frac{\partial}{\partial y}\left[B_m \cos\left(\frac{m\pi y}{d}\right) e^{-j\beta_m z}\right] = \frac{j\omega\mu}{k_c} B_m \sin\left(\frac{m\pi y}{d}\right) e^{-j\beta_m z}$$

$$E_y(x,y,z) = \frac{j\omega\mu}{k_c^2}\frac{\partial H_z}{\partial x} = \frac{j\omega\mu}{k_c^2}\frac{\partial}{\partial x}\left[B_m \cos\left(\frac{m\pi y}{d}\right) e^{-j\beta_n z}\right] = 0$$

$$E_z(x,y,z) = 0$$

Why No TE_0 Mode?

For $m = 0$, the field components would be

$$H_x = 0$$

$$H_y = \frac{j\beta_m}{k_c} B_m \sin(0) e^{-j\beta_m z} = 0$$

$$H_z = B_m \cos(0) e^{-j\beta_m z} = B_m e^{-j\beta_m z}$$

$$E_x = \frac{j\omega\mu}{k_c} B_m \sin(0) e^{-j\beta_m z} = 0$$

$$E_y = 0$$

$$E_z = 0$$

This is not a physical solution because the electric field is entirely zero.

Cutoff Condition

Recall the phase constant β_m is calculated as

$$\beta_m = \sqrt{k^2 - k_c^2} = \sqrt{k^2 - \left(\frac{m\pi}{d}\right)^2} \qquad m = 1, 2, 3, \ldots$$

This becomes imaginary when $k_c > k$. Values of m that cause this condition correspond to modes that are "cutoff." These are still modes, but they decay very quickly so they are almost never used and are not considered *guided* modes.

$$k_c = \omega_c \sqrt{\mu\varepsilon} = \frac{m\pi}{d}$$

$$2\pi f_c \sqrt{\mu\varepsilon} = \frac{m\pi}{d}$$

$$f_c = \frac{m}{2d\sqrt{\mu\varepsilon}} = \frac{k_c}{2\pi\sqrt{\mu\varepsilon}}$$

→ This is the cutoff frequency for the TE_m mode.

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Characteristic Impedance Z_{TE}

The characteristic impedance of the TE mode is defined as

$$Z_{TE} = \frac{E_{0,x}}{H_{0,y}} = -\frac{E_{0,y}}{H_{0,x}}$$

It is derived by substituting in the expressions for the field components.

$$Z_{TE} = \frac{E_{0,x}}{H_{0,y}} = \frac{\frac{j\omega\mu}{k_c} B_m \sin\left(\frac{n\pi y}{d}\right)}{\frac{j\beta_m}{k_c} B_m \sin\left(\frac{n\pi y}{d}\right)} = \frac{\omega\mu}{\beta_m} = \eta \frac{k}{\beta_m}$$

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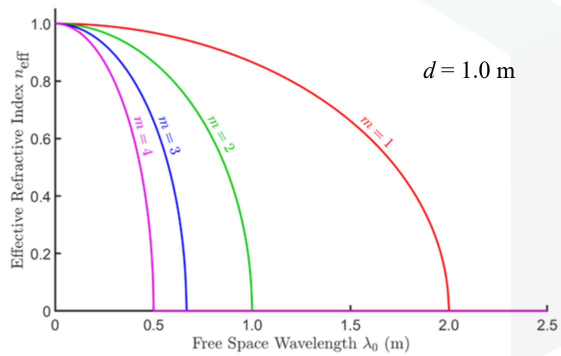
Effective Refractive Index n_{eff}

A wave propagates in a waveguide at a speed quantified by the effective refractive index n_{eff}.

$$\beta_m = k_0 n_{\text{eff}} = \sqrt{k^2 - \left(\frac{m\pi}{d}\right)^2} \quad \rightarrow \quad n_{\text{eff}} = n\sqrt{1 - \left(\frac{m\lambda_0}{2nd}\right)^2}$$



This term acts to make $n_{\text{eff}} < n$.

n is the refractive index of the dielectric in the waveguide.

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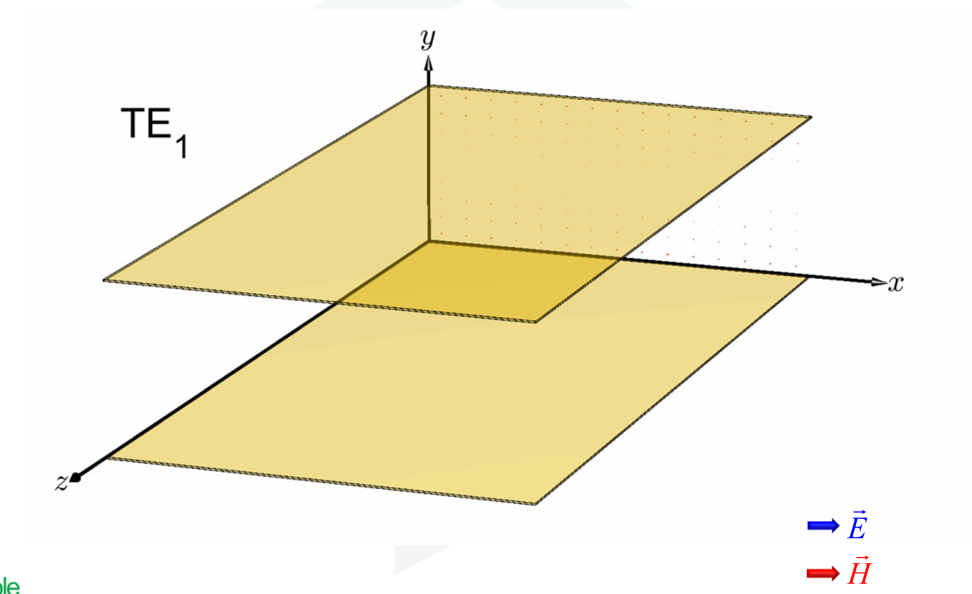
Visualization of TE_1 Mode



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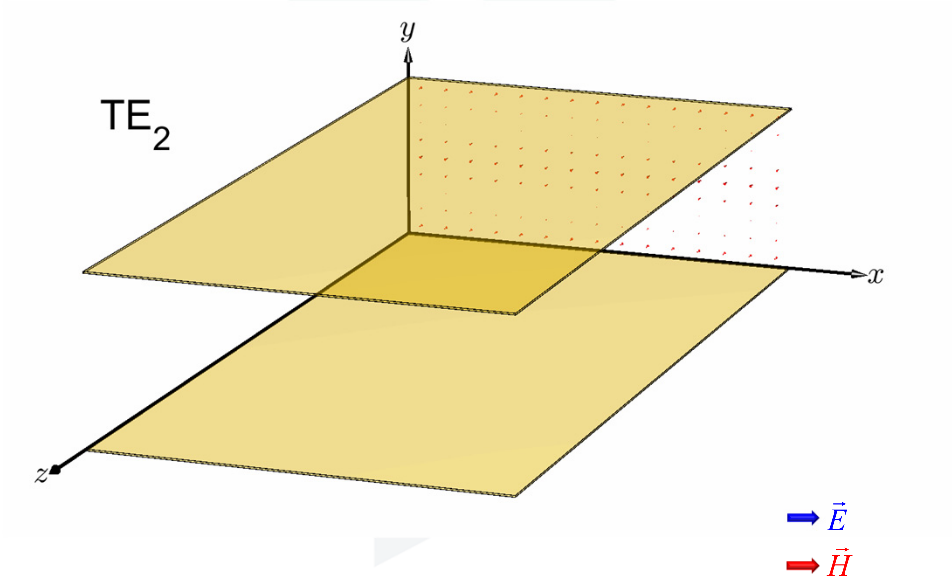
Visualization of TE_2 Mode



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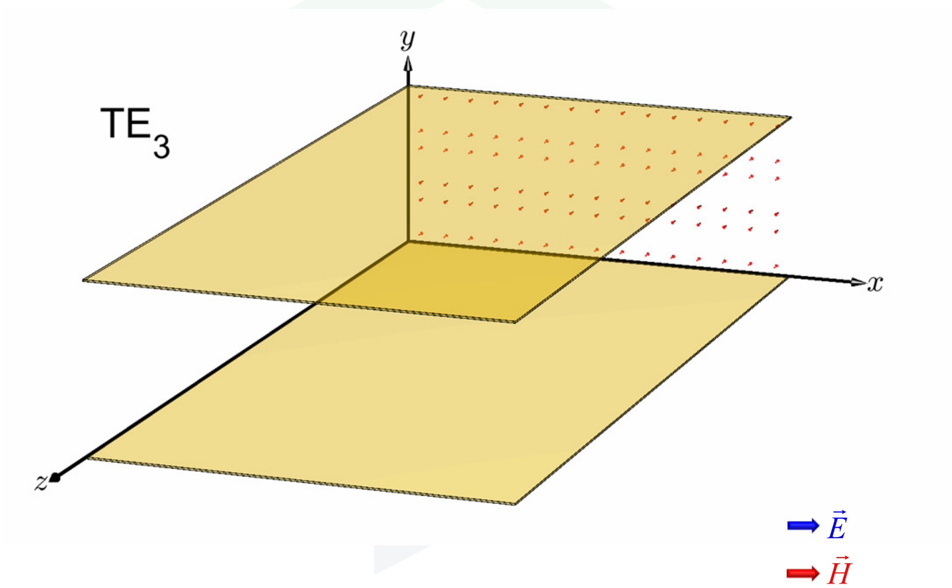
Visualization of TE_3 Mode



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Summary of TE Analysis

Field Solution

$$E_x(x,y,z) = \frac{j\omega\mu}{k_c} B_m \sin\left(\frac{m\pi y}{d}\right) e^{-j\beta_m z}$$

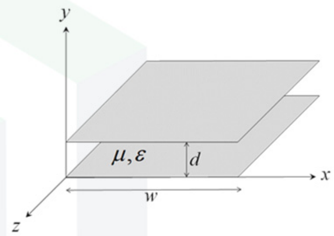
$$E_y(x,y,z) = 0$$

$$E_z(x,y,z) = 0$$

$$H_x(x,y,z) = 0$$

$$H_y(x,y,z) = \frac{j\beta_m}{k_c} B_m \sin\left(\frac{m\pi y}{d}\right) e^{-j\beta_m z}$$

$$H_z(x,y,z) = B_m \cos\left(\frac{m\pi y}{d}\right) e^{-j\beta_m z}$$



- TE_0 mode does not exist
- TE_1 is the lowest order TE mode

Phase Constant

$$\beta_m = \sqrt{k^2 - \left(\frac{m\pi}{d}\right)^2}$$

$$m = 1, 2, 3, \ldots$$

Same equation as for TM

Cutoff Frequency

$$f_{c,m} = \frac{m}{2d\sqrt{\mu\varepsilon}} = \frac{mc_0}{2nd}$$

Same equation as for TM

Characteristic Impedance

$$Z_{TE,m} = \frac{k\eta}{\beta_m}$$

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TM Analysis

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Recall the Starting Point

The governing equation for TM analysis (i.e. $H_{0,z} = 0$) is

$$\frac{\partial^2 E_{0,z}}{\partial x^2} + \frac{\partial^2 E_{0,z}}{\partial y^2} - k_c^2 E_{0,z} = 0 \qquad k_c^2 = k^2 - \beta^2$$

After a solution is obtained, the remaining field components are calculated according to

$$H_{0,x} = \frac{j\omega\varepsilon}{k_c^2}\frac{\partial E_{0,z}}{\partial y} \qquad E_{0,x} = -\frac{j\beta}{k_c^2}\frac{\partial E_{0,z}}{\partial x}$$

$$H_{0,y} = -\frac{j\omega\varepsilon}{k_c^2}\frac{\partial E_{0,z}}{\partial x} \qquad E_{0,y} = -\frac{j\beta}{k_c^2}\frac{\partial E_{0,z}}{\partial y}$$

$$H_{0,z} = 0$$

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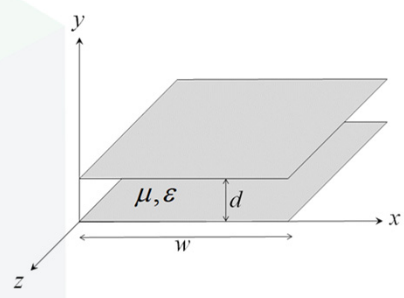
Simplify Governing Equation

Assuming the waveguide is uniform in the direction of x

$$\frac{\partial}{\partial x} = 0 \quad \text{and} \quad \frac{\partial^2}{\partial x^2} = 0$$



The governing equation reduces to

$$\cancel{\frac{\partial^2 E_{0,z}}{\partial x^2}} + \frac{\partial^2 E_{0,z}}{\partial y^2} - k_c^2 E_{0,z} = 0 \quad \rightarrow \quad \frac{d^2 E_{0,z}}{dy^2} - k_c^2 E_{0,z} = 0$$

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General Solution

The general solution to the governing equation is

$$\frac{d^2 E_{0,z}}{dy^2} - k_c^2 E_{0,z} = 0 \quad \rightarrow \quad E_{0,z} = A\sin(k_c y) + B\cos(k_c y)$$

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Boundary Conditions

The electric field component $E_{0,z}$ is tangential to the interfaces. So, the boundary conditions are applied to this directly.

The first boundary condition is

$$E_{0,z}(x,0) = A\sin(0) + B\cos(0) = B = 0 \quad \rightarrow \quad B = 0$$

The second boundary condition is

$$E_{0,z}(x,d) = A\sin(k_c d) = 0$$

$A = 0$ cannot be chosen because that would lead to a trivial solution. Instead, it must be the $\sin(k_c d)$ term that is zero at $y = d$.

$$\sin(k_c d) = 0 \quad \rightarrow \quad k_c d = m\pi \quad m = 0,1,2,3,\ldots$$

The cutoff wave number is then

$$k_c = \frac{m\pi}{d} \quad m = 0,1,2,3,\ldots$$

Note that $m = 0$ is allowed in this case because it does not force the field to be entirely zero. It does, however, force the field to be perfectly uniform. Thus, TM_0 is the TEM mode.

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Phase Constant β

Recall the original definition of the cutoff wave number. Solve equation this for β.

$$k_c^2 = k^2 - \beta^2 \quad \rightarrow \quad \beta = \sqrt{k^2 - k_c^2}$$

Substitute in $k_c = m\pi/d$ from the previous slide to get

$$\beta = \sqrt{k^2 - \left(\frac{m\pi}{d}\right)^2} \qquad m = 0,1,2,3,\ldots$$

It is observed that there is an infinite number of discrete solutions where the order of the mode is m.

$$\boxed{\beta_m = \sqrt{k^2 - \left(\frac{m\pi}{d}\right)^2} \qquad m = 0,1,2,3,\ldots}$$

TM_0 is the TEM mode

Final Solution

Recall that the general solution was

$$\frac{d^2 E_{0,z}}{dy^2} - k_c^2 E_{0,z} = 0 \quad \rightarrow \quad E_{0,z} = A\sin(k_c y) + B\cos(k_c y)$$

But now it is known that $B = 0$ and $k_c = m\pi/d$. The final solution is

$$E_{0,z}(x,y) = A_m \sin\left(\frac{m\pi y}{d}\right) \quad \rightarrow \quad \boxed{E_z(x,y,z) = A_m \sin\left(\frac{m\pi y}{d}\right) e^{-j\beta_m z}}$$

The remaining field components are calculated from this result.

$$H_x(x,y,z) = \frac{j\omega\varepsilon}{k_c^2}\frac{\partial E_z}{\partial y} = \frac{j\omega\varepsilon}{k_c^2}\frac{\partial}{\partial y}\left[A_m \sin\left(\frac{m\pi y}{d}\right)e^{-j\beta_m z}\right] = \frac{j\omega\varepsilon}{k_c} A_m \cos\left(\frac{m\pi y}{d}\right)e^{-j\beta_m z}$$

$$H_y(x,y,z) = -\frac{j\omega\varepsilon}{k_c^2}\frac{\partial E_z}{\partial x} = -\frac{j\omega\varepsilon}{k_c^2}\frac{\partial}{\partial x}\left[A_m \sin\left(\frac{m\pi y}{d}\right)e^{-j\beta_m z}\right] = 0$$

$$H_z(x,y,z) = 0$$

$$E_x(x,y,z) = -\frac{j\beta_m}{k_c^2}\frac{\partial E_z}{\partial x} = -\frac{j\beta_m}{k_c^2}\frac{\partial}{\partial x}\left[A_m \sin\left(\frac{m\pi y}{d}\right)e^{-j\beta_m z}\right] = 0$$

$$E_y(x,y,z) = -\frac{j\beta_m}{k_c^2}\frac{\partial E_z}{\partial y} = -\frac{j\beta_m}{k_c^2}\frac{\partial}{\partial y}\left[A_m \sin\left(\frac{m\pi y}{d}\right)e^{-j\beta_m z}\right] = -\frac{j\beta_m}{k_c} A_m \cos\left(\frac{m\pi y}{d}\right)e^{-j\beta_m z}$$

Why Does TM_0 Mode Exist?

For $m = 0$, the field components are

$$H_x = \frac{j\omega\varepsilon}{k_c} A_m \cos(0) e^{-j\beta_m z} = \frac{j\omega\varepsilon}{k_c} A_m e^{-j\beta_m z}$$

$$H_y = 0$$

$$H_z = 0$$

$$E_x = 0$$

$$E_y = -\frac{j\beta_m}{k_c} A_m \cos(0) e^{-j\beta_m z} = -\frac{j\beta_m}{k_c} A_m e^{-j\beta_m z}$$

$$E_z(x,y,z) = A_m \sin(0) e^{-j\beta_m z} = 0$$

This is a valid solution.

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Cutoff Condition

Recall that the phase constant β_m is calculated as

$$\beta_m = \sqrt{k^2 - k_c^2} = \sqrt{k^2 - \left(\frac{m\pi}{d}\right)^2} \qquad m = 0, 1, 2, 3, \ldots$$

This becomes imaginary when $k_c > k$. Values of m that cause this condition correspond to modes that are "cutoff." These are still modes, but they decay very quickly so they are almost never used and are not considered *guided modes*.

$$k_c = \omega_c \sqrt{\mu\varepsilon} = \frac{m\pi}{d}$$

$$2\pi f_c \sqrt{\mu\varepsilon} = \frac{m\pi}{d}$$

$$f_c = \frac{m}{2d\sqrt{\mu\varepsilon}} = \frac{k_c}{2\pi\sqrt{\mu\varepsilon}}$$

→ This is the cutoff frequency for the TM_m mode.

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Characteristic Impedance Z_{TM}

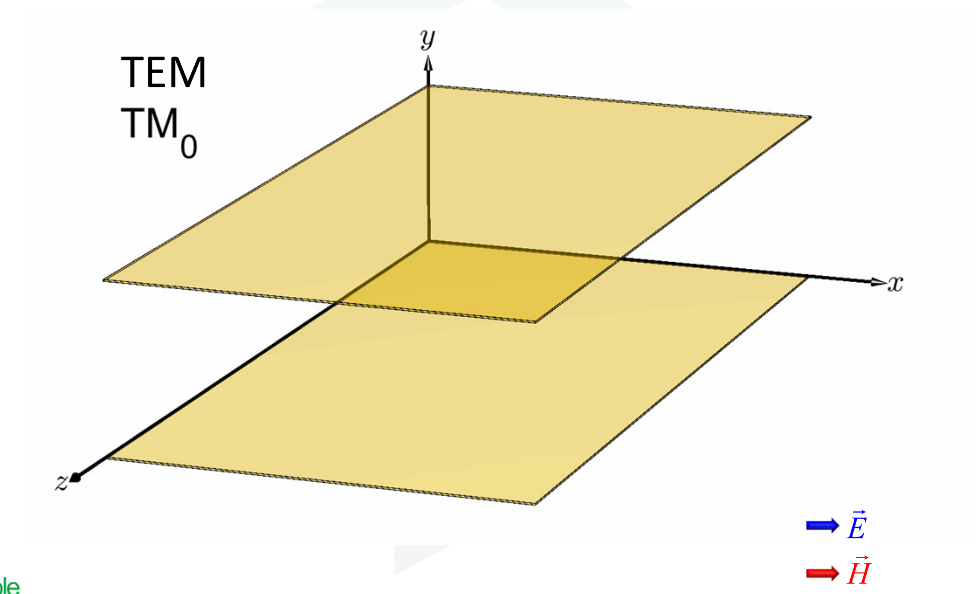
The characteristic impedance Z_{TM} of the TM mode is defined as

$$Z_{\text{TM}} = \frac{E_{0,x}}{H_{0,y}} = -\frac{E_{0,y}}{H_{0,x}}$$

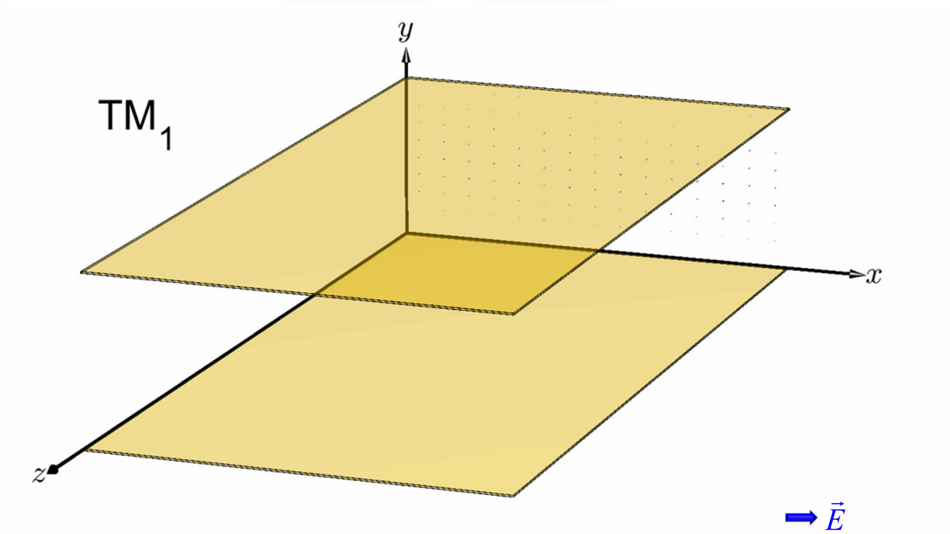
An expression for Z_{TM} is derived by substituting in the expressions for field components.

$$Z_{\text{TM}} = -\frac{E_{0,y}}{H_{0,x}} = -\frac{-\dfrac{j\beta_m}{k_c} A_m \cos\left(\dfrac{m\pi y}{d}\right)}{\dfrac{j\omega\varepsilon}{k_c} A_m \cos\left(\dfrac{m\pi y}{d}\right)} = \frac{\beta_m}{\omega\varepsilon} = \eta\frac{\beta_m}{k}$$

Visualization of TM_0 Mode



Visualization of TM_1 Mode



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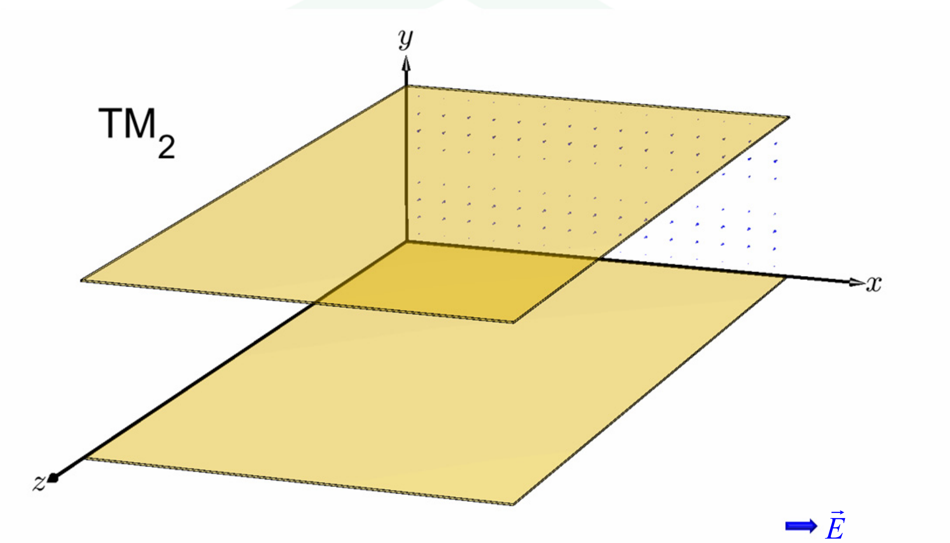
Visualization of TM_2 Mode



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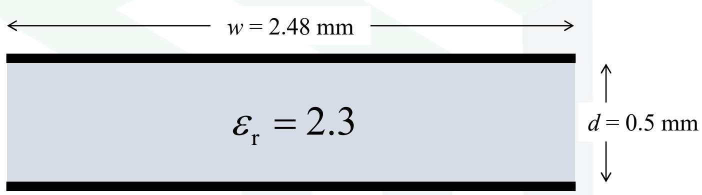
Example #2 (1 of 2)

Given the following parallel plate waveguide...



What is the bandwidth of this waveguide when used as a transmission line?

Example #2 (2 of 2)

When used as a transmission line, it is only the TEM mode that is of interest. The bandwidth is the range of frequencies for which the waveguide supports only the TEM mode.

The cutoff frequencies are the same for the TE and TM modes, so they are essentially checked at the same time.

The second order modes are TE_1 and TM_1. The bandwidth is simply the cutoff frequency of these modes.

$$f_c(m=1) = \frac{k_c}{2\pi\sqrt{\mu\varepsilon}} = \frac{mc_0}{2d\sqrt{\mu_r\varepsilon_r}} = \frac{(1)(299792458\ \text{m/s})}{2(0.5\ \text{mm})\sqrt{(1.0)(2.3)}} = \boxed{197.6\ \text{GHz}}$$

Summary of TM Analysis

Field Solution

$$E_x(x,y,z)=0$$

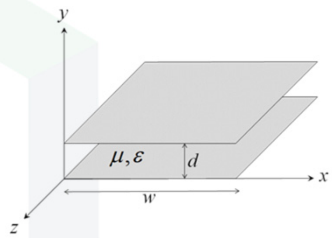
$$E_y(x,y,z)=-\frac{j\beta_m}{k_c}A_m\cos\left(\frac{m\pi y}{d}\right)e^{-j\beta_m z}$$

$$E_z(x,y,z)=A_m\sin\left(\frac{m\pi y}{d}\right)e^{-j\beta_m z}$$

$$H_x(x,y,z)=\frac{j\omega\varepsilon}{k_c}A_m\cos\left(\frac{m\pi y}{d}\right)e^{-j\beta_m z}$$

$$H_y(x,y,z)=0$$

$$H_z(x,y,z)=0$$



- TM_0 mode is the TEM mode

Phase Constant

$$\beta_m=\sqrt{k^2-\left(\frac{m\pi}{d}\right)^2}$$

$$m=0,1,2,3,\ldots$$

Same equation as for TE

Cutoff Frequency

$$f_{c,m}=\frac{m}{2d\sqrt{\mu\varepsilon}}$$

Same equation as for TE

Characteristic Impedance

$$Z_{TM,m}=\eta\frac{\beta_m}{k}$$

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Conclusion

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Summary of Parallel Plate Waveguide

Parameter	TEM	TM_m $m = 0,1,2,3\ldots$	TE_m $m = 1,2,3\ldots$
k	$\omega\sqrt{\mu\varepsilon}$	$\omega\sqrt{\mu\varepsilon}$	$\omega\sqrt{\mu\varepsilon}$
k_c	0	$m\pi/d$	$m\pi/d$
β	$k = \omega\sqrt{\mu\varepsilon}$	$\sqrt{k^2 - k_c^2}$	$\sqrt{k^2 - k_c^2}$
λ_c	∞	$2\pi/k_c = 2d/n$	$2\pi/k_c = 2d/n$
λ_g	$2\pi/k$	$2\pi/\beta_m$	$2\pi/\beta_m$
v_p	$\omega/k = 1/\sqrt{\mu\varepsilon}$	ω/β_m	ω/β_m
α_d	$k\tan\delta/2$	$k^2\tan\delta/2\beta_m$	$k^2\tan\delta/2\beta_m$
α_c	$R_s/\eta d$	$2kR_s/\beta_m\eta d$	$2k_c^2 R_s/k\beta_m\eta d$
E_x	0	0	$(j\omega\mu/k_c)B_m\sin(m\pi y/d)e^{-j\beta_m z}$
E_y	$(-V_0/d)e^{-j\beta z}$	$(-j\beta_m/k_c)A_m\cos(m\pi y/d)e^{-j\beta_m z}$	0
E_z	0	$A_m\sin(m\pi y/d)e^{-j\beta_m z}$	0
H_x	$(-V_0/\eta d)e^{-j\beta z}$	$(j\omega\varepsilon/k_c)A_m\cos(m\pi y/d)e^{-j\beta_m z}$	0
H_y	0	0	$(j\beta_m/k_c)B_m\sin(m\pi y/d)e^{-j\beta_m z}$
H_z	0	0	$B_m\cos(m\pi y/d)e^{-j\beta_m z}$
Z	$\eta d/w$	$\beta_m\eta/k$	$k\eta/\beta_m$

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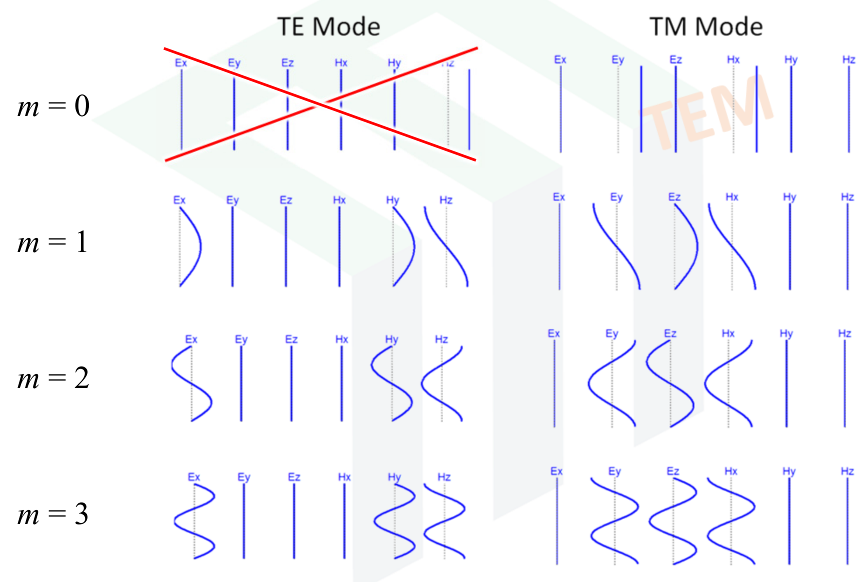
Modes in Parallel Plate Waveguide



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Notes

- The parallel plate supports a TEM mode when it has a homogeneous dielectric because it has two conductors.
- Supports TE and TM modes when it has a homogeneous dielectric
- The lowest order mode is TM_0 which is the TEM mode.

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