Electromagnetics:
Electromagnetic Field Theory

Phase Matching & Special Angles

Lecture Outline

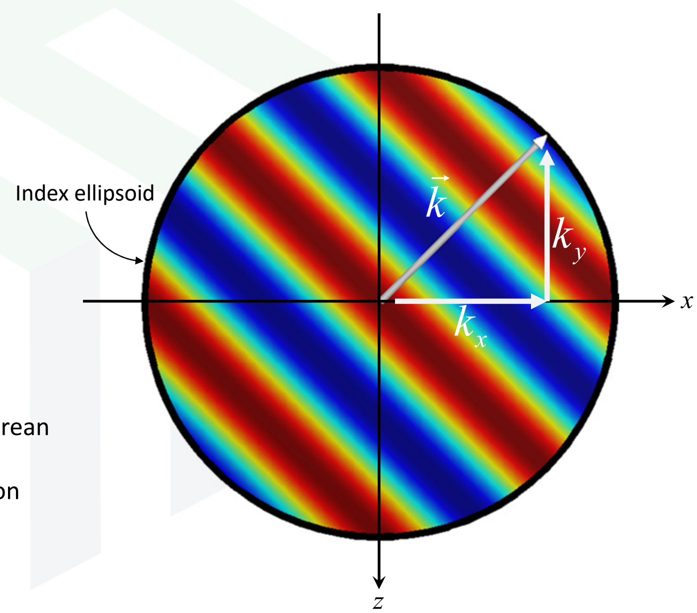
- Phase Matching at an Interface
- The Critical Angle
- Brewster's Angle

Phase Matching at an Interface

Slide 3

Illustration of the Dispersion Relation

$$k_x^2 + k_z^2 = \left|\vec{k}\right|^2 = \left(k_0 n\right)^2$$



The dispersion relation for isotropic materials is essentially just the Pythagorean theorem. It says a wave sees the same refractive index no matter what direction the wave is travelling.

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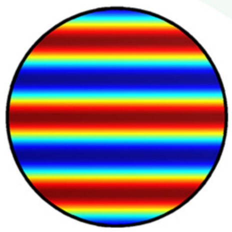
Index Ellipsoid in Two Different Materials

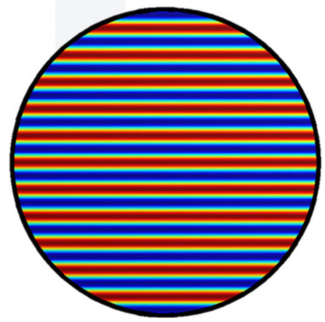
Material 1 (Low n)

$$k_{x,1}^2 + k_{z,1}^2 = \left|\vec{k}_1\right|^2 = \left(k_0 n_1\right)^2$$

Material 2 (High n)

$$k_{x,2}^2 + k_{z,2}^2 = \left|\vec{k}_2\right|^2 = \left(k_0 n_2\right)^2$$





$n_1 < n_2$

Phase Matching When $n_1 < n_2$

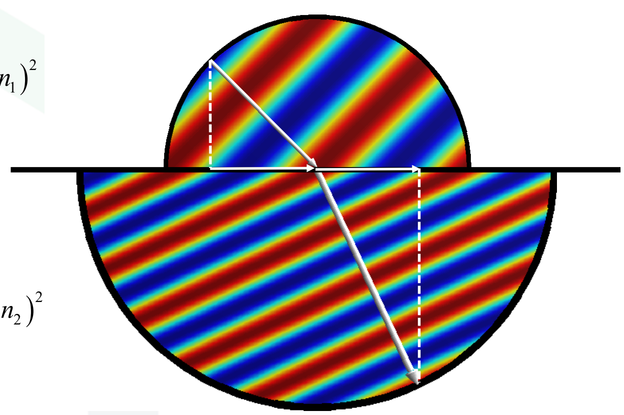
$n_1 < n_2$

Material 1

$$k_{x,1}^2 + k_{z,1}^2 = \left|\vec{k}_1\right|^2 = \left(k_0 n_1\right)^2$$

Material 2

$$k_{x,2}^2 + k_{z,2}^2 = \left|\vec{k}_2\right|^2 = \left(k_0 n_2\right)^2$$



Summary of Phase Matching for $n_1 < n_2$

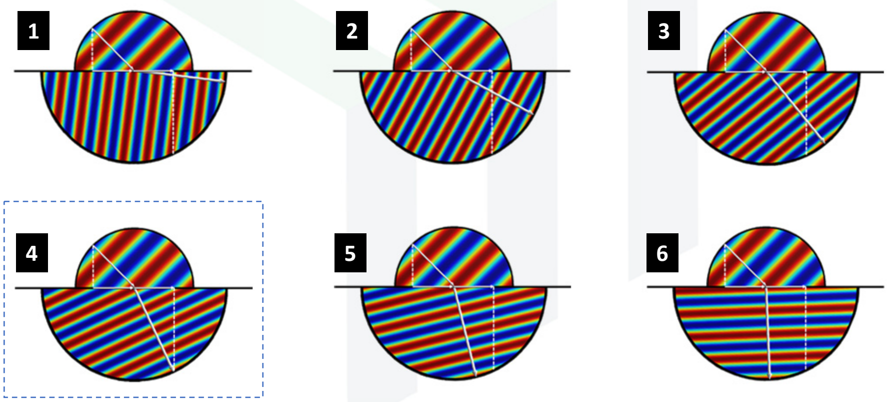
$$n_1 < n_2$$

Material 1

$$k_{x,1}^2 + k_{z,1}^2 = \left|\vec{k}_1\right|^2 = \left(k_0 n_1\right)^2$$

Material 2

$$k_{x,2}^2 + k_{z,2}^2 = \left|\vec{k}_2\right|^2 = \left(k_0 n_2\right)^2$$



Properly phased matched at the interface.

Phase Matching When $n_1 > n_2$

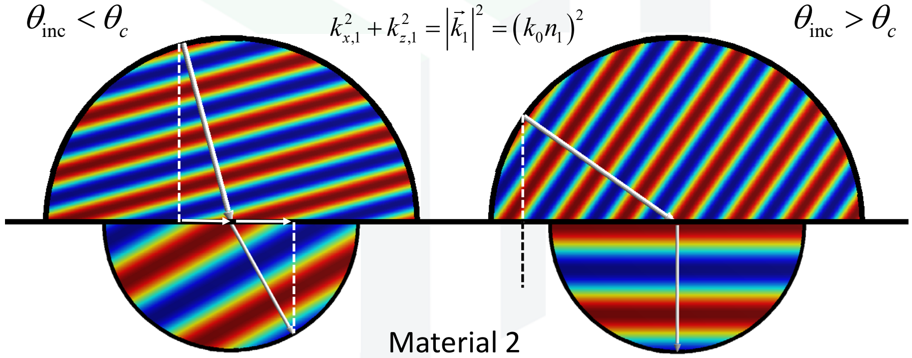
$$n_1 > n_2$$

Material 1

$$k_{x,1}^2 + k_{z,1}^2 = \left|\vec{k}_1\right|^2 = \left(k_0 n_1\right)^2$$

$\theta_{\text{inc}} < \theta_c$

$\theta_{\text{inc}} > \theta_c$



Material 2

$$k_{x,2}^2 + k_{z,2}^2 = \left|\vec{k}_2\right|^2 = \left(k_0 n_2\right)^2$$

Summary of Phase Matching for $n_1 > n_1$

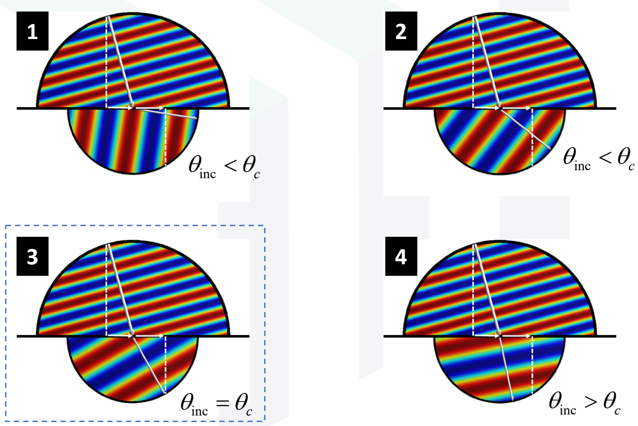
$n_1 > n_2$

Material 1

$$k_{x,1}^2 + k_{z,1}^2 = \left|\vec{k}_1\right|^2 = \left(k_0 n_1\right)^2$$

Material 2

$$k_{x,2}^2 + k_{z,2}^2 = \left|\vec{k}_2\right|^2 = \left(k_0 n_2\right)^2$$



Properly phased matched at the interface.

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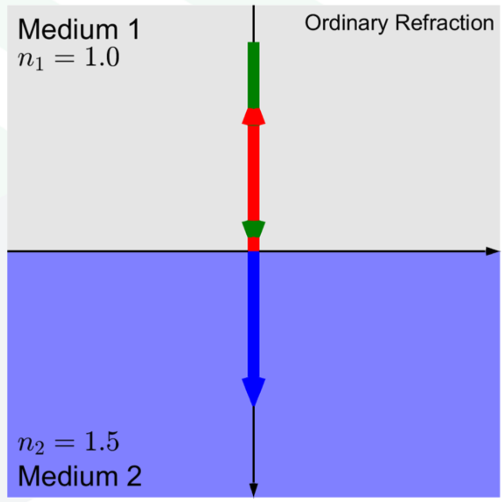
The Critical Angle

Animation of Snell's Law (1 of 2)

Beam propagates from low-index medium to a high-index medium.

$$n_1 \sin\theta_{\mathrm{i}} = n_2 \sin\theta_{\mathrm{t}}$$



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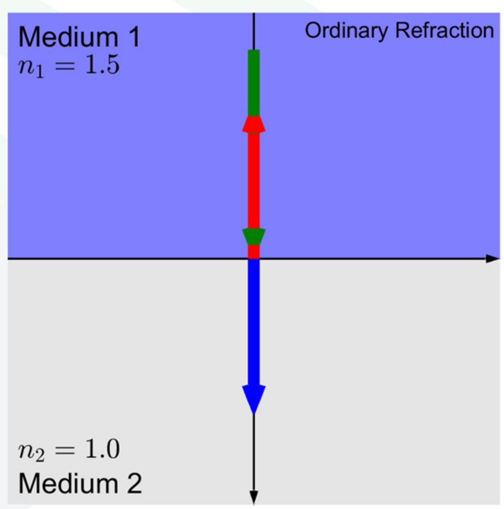
Animation of Snell's Law (2 of 2)

Beam propagates from high-index medium to a low-index medium.

$$n_1 \sin\theta_{\mathrm{i}} = n_2 \sin\theta_{\mathrm{t}}$$



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The Critical Angle θ_c

The critical angle θ_c is the angle of incidence θ_i that produces an angle of transmission θ_t that is exactly 90°.

$$n_1 \sin\theta_i = n_2 \sin\theta_t$$

$$\downarrow$$

$$n_1 \sin\theta_c = n_2 \sin(90°) \quad \rightarrow \quad \theta_c = \sin^{-1}\left(\frac{n_2}{n_1}\sin(90°)\right)$$

$$\downarrow$$

$$\boxed{\theta_c = \sin^{-1}\left(\frac{n_2}{n_1}\right) \qquad n_1 > n_2}$$

In order for there to be a critical angle θ_c, the wave must be incident onto a low-index medium from a high-index medium.

$$n_1 > n_2$$

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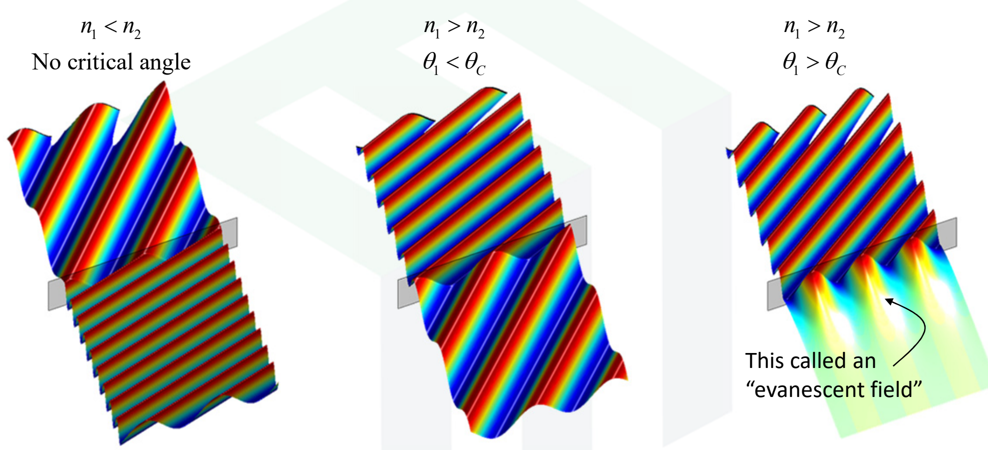
Field at an Interface Above and Below the Critical Angle (Ignoring Reflections)



1. The field always penetrates into material 2, but it may not propagate.
2. Above the critical angle, penetration is greatest near the critical angle.
3. Very high spatial frequencies are supported in material 2 despite the dispersion relation.
4. In material 2, power always flows in the transverse direction, but not in the longitudinal direction.

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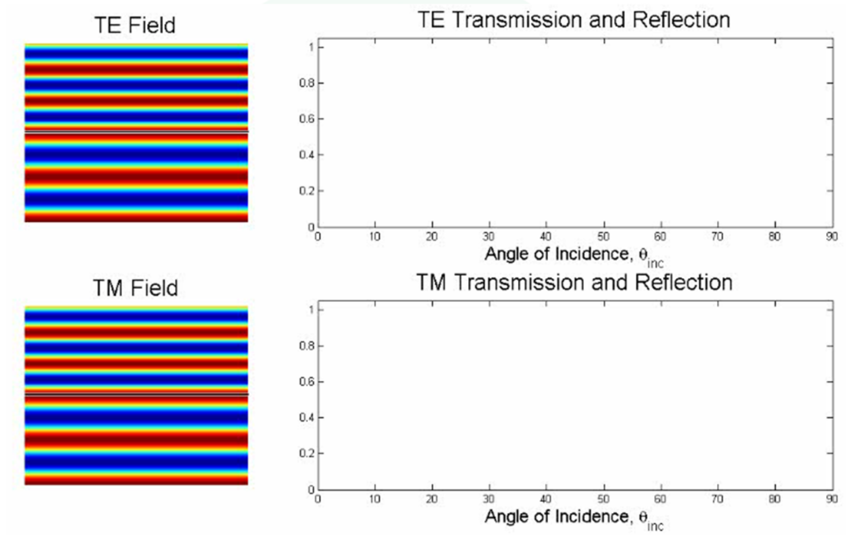
Simulation of Reflection and Transmission at a Single Interface $(n_1 > n_2)$



n_1=1.41, n_2=1.0 → θ_c=45°

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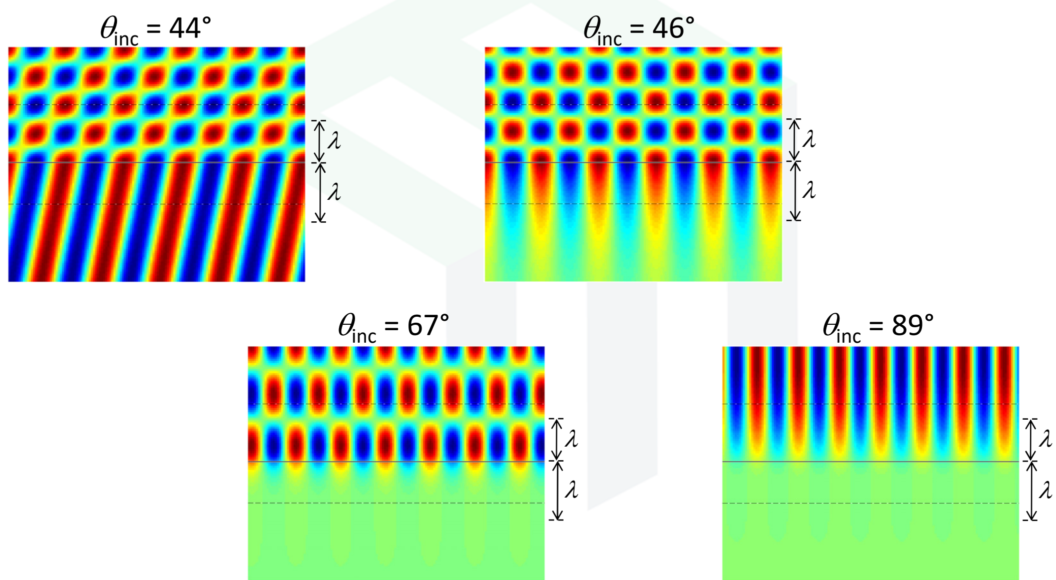
Field Visualization for $\theta_c = 45°$



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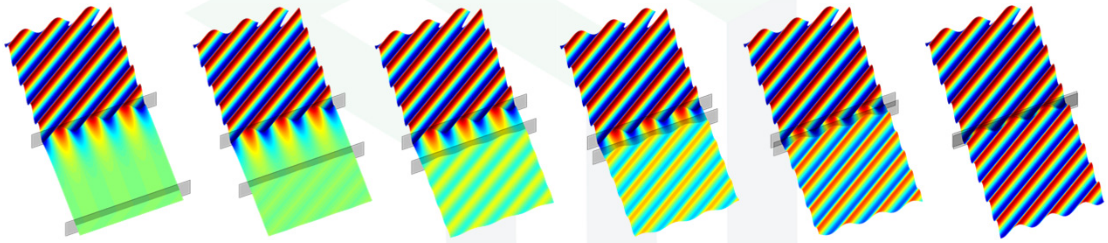
Electromagnetic Tunneling

If an evanescent field touches a medium with higher refractive index, the field may no longer be cutoff and become a propagating wave.



This is a very unusual phenomenon because the evanescent field is contributing to power flow.

This is called electromagnetic tunneling and is analogous to electron tunneling through thin insulators.

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Brewster's Angle

Slide 18

Can Reflection Ever Be Zero?

Observe that in the Fresnel equations the reflection coefficients have a difference in their numerator. This means there must exist special conditions where reflection can be zero. These are the *Brewster's angles*.

TE, s, ⊥ Polarization

$$r_{\mathrm{TE}} = \frac{\eta_2 \cos\theta_{\mathrm{i}} - \eta_1 \cos\theta_{\mathrm{t}}}{\eta_2 \cos\theta_{\mathrm{i}} + \eta_1 \cos\theta_{\mathrm{t}}}$$

$$t_{\mathrm{TE}} = \frac{2\eta_2 \cos\theta_{\mathrm{i}}}{\eta_2 \cos\theta_{\mathrm{i}} + \eta_1 \cos\theta_{\mathrm{t}}}$$

$$1 + r_{\mathrm{TE}} = t_{\mathrm{TE}}$$

TM, p, || Polarization

$$r_{\mathrm{TM}} = \frac{\eta_2 \cos\theta_{\mathrm{t}} - \eta_1 \cos\theta_{\mathrm{i}}}{\eta_2 \cos\theta_{\mathrm{t}} + \eta_1 \cos\theta_{\mathrm{i}}}$$

$$t_{\mathrm{TM}} = \frac{2\eta_2 \cos\theta_{\mathrm{i}}}{\eta_2 \cos\theta_{\mathrm{t}} + \eta_1 \cos\theta_{\mathrm{i}}}$$

$$1 + r_{\mathrm{TM}} = \frac{\cos\theta_{\mathrm{t}}}{\cos\theta_{\mathrm{i}}} t_{\mathrm{TM}}$$

A similar condition is not observed for transmission.

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Brewster's Angle for TE Polarization (1 of 3)

Start with the Fresnel equation for reflection of the TE polarization.

$$r_{\mathrm{TE}} = \frac{\eta_2 \cos\theta_{\mathrm{i}} - \eta_1 \cos\theta_{\mathrm{t}}}{\eta_2 \cos\theta_{\mathrm{i}} + \eta_1 \cos\theta_{\mathrm{t}}}$$

We set the numerator equation to zero.

$$\eta_2 \cos\theta_{\mathrm{i}} - \eta_1 \cos\theta_{\mathrm{t}} = 0$$

The Brewster's angle θ_{B} is the angle of incidence θ_{i} that satisfies this expression and makes reflection go to zero.

$$\eta_2 \cos\theta_{\mathrm{B}} - \eta_1 \cos\theta_{\mathrm{t}} = 0$$

$$\eta_2 \cos\theta_{\mathrm{B}} = \eta_1 \cos\theta_{\mathrm{t}}$$

$$\cos\theta_{\mathrm{B}} = \frac{\eta_1}{\eta_2} \cos\theta_{\mathrm{t}}$$

$$\cos^2\theta_{\mathrm{B}} = \left(\frac{\eta_1}{\eta_2}\right)^2 \cos^2\theta_{\mathrm{t}}$$

$$1 - \sin^2\theta_{\mathrm{B}} = \left(\frac{\eta_1}{\eta_2}\right)^2 - \left(\frac{\eta_1}{\eta_2}\right)^2 \sin^2\theta_{\mathrm{t}}$$

We would like to eliminate this term.

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Slide 20

Brewster's Angle for TE Polarization (2 of 3)

Solve Snell's law for $\sin^2\theta_t$.

$$n_1 \sin\theta_B = n_2 \sin\theta_t \quad \rightarrow \quad \sin^2\theta_t = \left(\frac{n_1}{n_2}\right)^2 \sin^2\theta_B$$

Substitute this result into the previous expression for the Brewster's angle.

$$1-\sin^2\theta_B = \left(\frac{\eta_1}{\eta_2}\right)^2 - \left(\frac{\eta_1}{\eta_2}\right)^2 \sin^2\theta_t$$

$$\downarrow$$

$$1-\sin^2\theta_B = \left(\frac{\eta_1}{\eta_2}\right)^2 - \left(\frac{\eta_1}{\eta_2}\right)^2 \left(\frac{n_1}{n_2}\right)^2 \sin^2\theta_B \quad \rightarrow \quad \sin^2\theta_{B,TE} = \frac{1-(\eta_1/\eta_2)^2}{1-(n_1\eta_1/n_2\eta_2)^2}$$

Brewster's Angle for TE Polarization (3 of 3)

Now write η and n and in terms of μ and ε.

$$\eta_1 = \eta_0\sqrt{\frac{\mu_{r1}}{\varepsilon_{r1}}} \qquad \eta_2 = \eta_0\sqrt{\frac{\mu_{r2}}{\varepsilon_{r2}}} \qquad n_1 = \sqrt{\mu_{r1}\varepsilon_{r1}} \qquad n_2 = \sqrt{\mu_{r2}\varepsilon_{r2}}$$

The expression for the Brewster's angle becomes

$$\boxed{\sin^2\theta_{B,TE} = \frac{1-\frac{\mu_{r1}}{\mu_{r2}}\frac{\varepsilon_{r2}}{\varepsilon_{r1}}}{1-\left(\frac{\mu_{r1}}{\mu_{r2}}\right)^2}}$$

Inspecting this equation, observe that there is no Brewster's angle for the TE polarization unless the permeability is different in each medium.

Brewster's Angle for TM Polarization (1 of 3)

Start with the Fresnel equation for reflection of the TM polarization.

$$r_{\mathrm{TM}} = \frac{\eta_2 \cos\theta_t - \eta_1 \cos\theta_i}{\eta_2 \cos\theta_t + \eta_1 \cos\theta_i}$$

Set the numerator equation to zero.

$$\eta_2 \cos\theta_t - \eta_1 \cos\theta_i = 0$$

The Brewster's angle θ_B is the angle of incidence θ_i that satisfies this expression and makes reflection go to zero.

$$\eta_2 \cos\theta_t - \eta_1 \cos\theta_B = 0$$

$$\downarrow$$

$$1 - \sin^2\theta_B = \left(\frac{\eta_2}{\eta_1}\right)^2 - \left(\frac{\eta_2}{\eta_1}\right)^2 \sin^2\theta_t$$

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Brewster's Angle for TM Polarization (2 of 3)

Solve Snell's law for $\sin^2\theta_t$.

$$n_1 \sin\theta_B = n_2 \sin\theta_t \quad \rightarrow \quad \sin^2\theta_t = \left(\frac{n_1}{n_2}\right)^2 \sin^2\theta_B$$

Substitute this result into our previous expression for the Brewster's angle.

$$1 - \sin^2\theta_B = \left(\frac{\eta_2}{\eta_1}\right)^2 - \left(\frac{\eta_2}{\eta_1}\right)^2 \sin^2\theta_t$$

$$\downarrow$$

$$1 - \sin^2\theta_B = \left(\frac{\eta_2}{\eta_1}\right)^2 - \left(\frac{\eta_2}{\eta_1}\right)^2 \left(\frac{n_1}{n_2}\right)^2 \sin^2\theta_B \quad \rightarrow \quad \sin^2\theta_{B,\mathrm{TM}} = \frac{1 - (\eta_2/\eta_1)^2}{1 - (n_1\eta_2/n_2\eta_1)^2}$$

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Slide 24

Brewster's Angle for TM Polarization (3 of 3)

We now write the impedances and refractive indices in terms of

$$\eta_1 = \eta_0\sqrt{\frac{\mu_{r1}}{\varepsilon_{r1}}} \qquad \eta_2 = \eta_0\sqrt{\frac{\mu_{r2}}{\varepsilon_{r2}}} \qquad n_1 = \sqrt{\mu_{r1}\varepsilon_{r1}} \qquad n_2 = \sqrt{\mu_{r2}\varepsilon_{r2}}$$

Our expression for the Brewster's angle becomes

$$\sin^2\theta_{B,TM} = \frac{1 - \frac{\mu_{r2}}{\mu_{r1}}\frac{\varepsilon_{r1}}{\varepsilon_{r2}}}{1 - \left(\frac{\varepsilon_{r1}}{\varepsilon_{r2}}\right)^2}$$

Inspecting this equation, we see that we still have a Brewster's angle even when the materials do not have a magnetic response.

$$\tan\theta_{B,TM} = \sqrt{\frac{\varepsilon_{r2}}{\varepsilon_{r1}}} = \frac{n_2}{n_1}$$

$$\mu_{r1} = \mu_{r2} = 1$$

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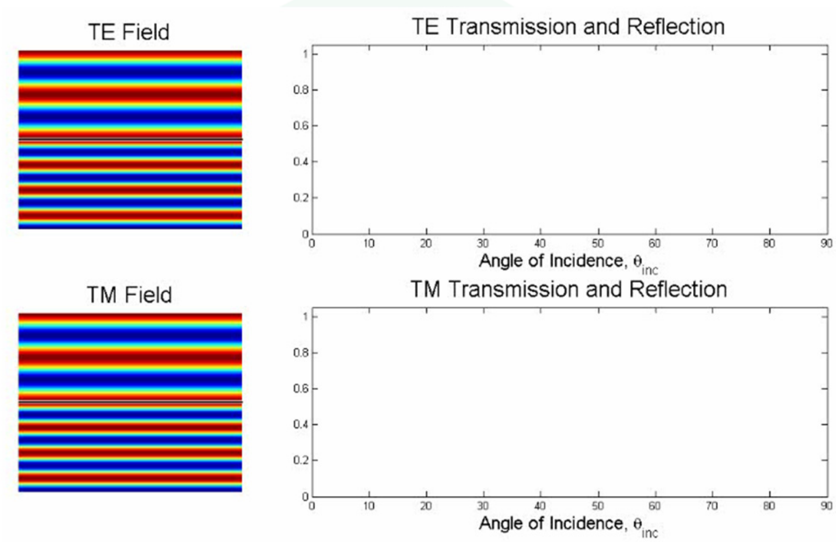
Simulation of Reflection and Transmission at a Single Interface ($n_1<n_2$)



n_1=1.0, n_2=1.73 → θ_B=60°

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Slide 26

Example – Plot of Fresnel Equations

Slide 27

Plots of the Fresnel Equations

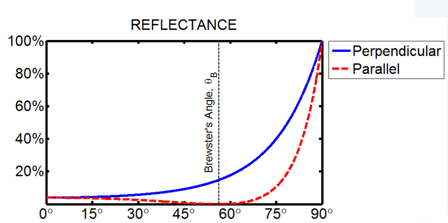
Low to High Index
($n_1 = 1.0$ and $n_2 = 1.5$)

No critical angle

$$\theta_c = \sin^{-1}\left(\frac{n_2}{n_1}\right) = \sin^{-1}(1.5)$$

Brewster's angle

$$\theta_{B,TM} = \tan^{-1}\left(\frac{n_2}{n_1}\right) = \tan^{-1}(1.5) = 56.31°$$



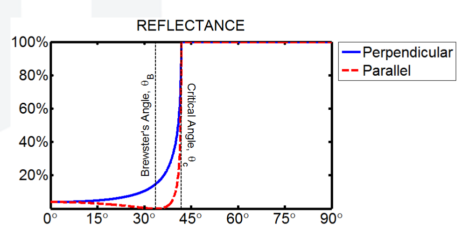
High to Low Index
($n_1 = 1.5$ and $n_2 = 1.0$)

Critical angle

$$\theta_c = \sin^{-1}\left(\frac{n_2}{n_1}\right) = \sin^{-1}(0.667) = 41.81°$$

Brewster's angle

$$\theta_{B,TM} = \tan^{-1}\left(\frac{n_2}{n_1}\right) = \tan^{-1}(0.667) = 33.69°$$



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