Electromagnetics:
Electromagnetic Field Theory

Skin Depth & Power Flow

Lecture Outline

• Skin Depth
• Power Flow
Skin Depth

Waves in good conductors attenuate very quickly. The distance over which they decay by a factor of $1/e$ is called the skin depth $\delta$.

**Definition of Skin Depth**

$$\delta = \frac{1}{\alpha} \text{ (m)}$$

**Relation to Impedance**

$$\eta = \frac{1 + j}{\sigma \delta} = \frac{\sqrt{2}}{\sigma \delta} \angle 45^\circ$$

**In Terms of Fundamental Parameters**

$$\delta = \sqrt{\frac{2}{\omega \mu \sigma}} = \frac{1}{\sqrt{\pi f \mu \sigma}}$$
Skin Depth for Various Materials

Skin depth vs. frequency for some materials at room temperature. Red vertical line denotes 50 Hz frequency:
- Mn-Zn = magnetostrictive ferrite
- Al = metallic aluminum
- Cu = metallic copper
- steel 410 = magnetic stainless steel
- Fe-Si = grain-oriented electrical steel
- Fe-Ni = high-permeability permalloy (80%Ni-20%Fe)

https://en.wikipedia.org/wiki/Skin_effect

DC Resistance, $R_{DC}$

Current density is uniform throughout the conductor so the entire conductor contributes to current flow.

Conductor Area $= A_{DC} = \pi r^2$

$$R_{DC} = \frac{\ell}{\sigma A_{DC}} = \frac{\ell}{\sigma \pi r^2}$$
AC Resistance, $R_{AC}$

Current density is NOT uniform throughout the conductor so only part of the conductor contributes to current flow.

Conductor Area = $A_{AC} \approx 2\pi r\delta$

$R_{AC} = \frac{\ell}{\sigma A_{AC}} = \frac{\ell}{2\pi r\sigma\delta}$ for $\delta \ll r$

Conductor Area for AC Resistance

Area Described by $\delta$

$A_{AC} = \pi r^2 - \pi (r - \delta)^2$

$= \pi r^2 - \pi (r^2 - 2r\delta + \delta^2)$

$= \pi r^2 - \pi r^2 + 2\pi r\delta - \pi\delta^2$

$= 2\pi r\delta - \pi\delta^2$ for $\delta \ll r$

$\approx 2\pi r\delta$

Effective Conductor Area

$A_{AC} \approx 2\pi r\delta$ for $\delta \ll r$
Notes on Skin Depth and AC Resistance

- High frequencies experience so much loss that they do not penetrate very far into a conductor.
- The depth of penetration is called skin depth $\delta$.
- Due to the skin depth at high frequencies, only part of the conductor contributes to current flow. This makes resistance increase as a function of frequency.
- Drawbacks
  - High frequencies experience more loss.
  - Signals get distorted
- Benefits
  - Conductors can be made hollow – cheaper, lighter, etc.
  - Can make inner part of conductor out of a different material.

Power Flow
Poynting’s Theorem

Poynting’s theorem is a conservation of power equation.

The total power leaving a volume must be equal to the rate of decrease of the total energy stored in the field plus the energy lost due to heat (or something else).

\[ \oint (\vec{E} \times \vec{H}) \cdot d\vec{s} = -\frac{1}{2} \frac{\partial}{\partial t} \iint (\mu |\vec{H}|^2 + \varepsilon |\vec{E}|^2) \, dv - \iint (\sigma |\vec{E}|^2) \, dv \]

Poynting Vector,

We read off of Poynting’s theorem the term responsible for power leaving the volume.

\[ \oint (\vec{E} \times \vec{H}) \cdot d\vec{s} = -\frac{1}{2} \frac{\partial}{\partial t} \iint (\mu |\vec{H}|^2 + \varepsilon |\vec{E}|^2) \, dv - \iint (\sigma |\vec{E}|^2) \, dv \]

Here we are integrating a flux to get total power. The argument must be power density (W/m²). We call this argument the instantaneous Poynting vector.

\[ \vec{\phi}(t) = \vec{E}(t) \times \vec{H}(t) \]

Due to the cross product, the Poynting vector is perpendicular to both \( \vec{E} \) and \( \vec{H} \). For LHI materials, the Poynting vector is in the same direction as the wave vector.

\[ \vec{E} \perp \vec{k} \quad \vec{H} \perp \vec{k} \quad \vec{\phi} \parallel \vec{k} \]
Instantaneous Poynting Vector

Recall the electric and magnetic field components of a plane wave travelling in the +z direction can be written in the time-domain as:

\[ E(z,t) = E_0 e^{-az} \cos(\omega t - \beta z) \hat{a}_z \]
\[ H(z,t) = \frac{E_0}{\mu} e^{-az} \cos(\omega t - \beta z - \eta) \hat{a}_z \]

Substituting these expressions into the definition of the instantaneous Poynting vector gives:

\[ \tilde{\phi}(t) = \tilde{E}(t) \times \tilde{H}(t) \]
\[ = \left[ E_0 e^{-az} \cos(\omega t - \beta z) \hat{a}_z \right] \times \left[ \frac{E_0}{\mu} e^{-az} \cos(\omega t - \beta z - \eta) \hat{a}_z \right] \]
\[ = \frac{E_0^2}{2\mu} e^{-2az} \cos(\eta z) \hat{a}_z + \frac{E_0^2}{2\mu} e^{-2az} \cos(2\omega t - 2\beta z - \eta) \hat{a}_z \]

Constant power flow
Rapidly oscillating fluctuation in power flow

Animation of Instantaneous Power Flow

\[ \tilde{\phi}(t) = \frac{E_0^2}{2\mu} e^{-2az} \cos(\eta z) \hat{a}_z + \frac{E_0^2}{2\mu} e^{-2az} \cos(2\omega t - 2\beta z - \eta) \hat{a}_z \]

Constant power flow
Rapidly oscillating fluctuation in power flow
**Average Poynting Vector**

The instantaneous power flow is rarely of interest because the rapidly fluctuating term does not transport any net power. The more practical and useful quantity is the time-average Poynting Vector.

To obtain the time-average Poynting vector, we integrate over one wave cycle.

\[
\mathbf{\dot{\omega}}_{\text{avg}} = \frac{1}{\tau} \int_0^\tau \mathbf{\dot{\omega}}(t) \, dt
\]

\[
= \frac{1}{\tau} \left[ \frac{E_0^2}{2|\eta|} e^{-2\alpha z} \cos(\omega \tau) \hat{a}_z \right] \, \tau + \frac{1}{\tau} \left[ \frac{E_0^2}{2|\eta|} e^{-2\alpha z} \cos(2\omega \tau - 2\beta z - \omega \tau) \hat{a}_z \right] \, dt
\]

\[
\mathbf{\Phi}_{\text{avg}} = \frac{E_0^2}{2|\eta|} e^{-2\alpha z} \cos(\omega \tau) \hat{a}_z
\]

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**Complex Poynting Vector**

For time-harmonic signals, the frequency-domain Poynting vector is complex.

\[
\mathbf{\dot{\omega}} = \mathbf{E} \times \mathbf{H}^* \\
\mathbf{E}(z) = E_0 e^{-j\gamma z} \hat{a}_z, \\
\mathbf{H}(z) = \frac{E_0}{|\eta|} e^{-j\beta z} e^{-j\omega \tau} \hat{a}_y
\]

The field expressions for our plane wave are

\[
\mathbf{E}(z) = E_0 e^{-j\gamma z} \hat{a}_z, \\
\mathbf{H}(z) = \frac{E_0}{|\eta|} e^{-j\beta z} e^{-j\omega \tau} \hat{a}_y
\]

Substituting these into the definition of complex Poynting vector gives

\[
\mathbf{\Phi} = \mathbf{E}_0 \mathbf{E}^* \times \mathbf{H}^* \\
\mathbf{E}_0 = E_0 e^{-j\gamma z} \hat{a}_z, \\
\mathbf{H} = \frac{E_0}{|\eta|} e^{-j\beta z} e^{-j\omega \tau} \hat{a}_y
\]

\[
\mathbf{\Phi} = \frac{E_0^2}{|\eta|} e^{2\alpha z} e^{j\omega \tau} \hat{a}_z
\]
RMS Poynting Vector

The complex Poynting vector is like the instantaneous Poynting vector and contains the rapidly varying fluctuations in power flow.

A more meaningful quantity is the root-mean-square (RMS) power flow that is easily calculated from the complex Poynting vector.

\[
\tilde{\phi}_{\text{avg}} = \frac{1}{2} \text{Re} \left( \tilde{\phi} \right) = \frac{1}{2} \text{Re} \left[ \vec{E} \times \vec{H}^* \right]
\]

\[
= \frac{1}{2} \text{Re} \left[ \frac{E_0^2 e^{-2az} e^{j\alpha t}}{|\eta|} \hat{z} \right]
\]

\[
= \frac{E_0^2}{2|\eta|} e^{-2az} \cos (\angle \eta) \hat{a}_z
\]

Total Power

The Poynting vector is a power density with units of W/m².

To calculate total power flow through some area, we must integrate the Poynting vector over that area.

\[
P(t) = \iiint_S \vec{\phi}(t) \cdot d\vec{s}
\]

The average power flow is simply calculated from the average Poynting vector.

\[
P_{\text{avg}} = \iint_S \tilde{\phi}_{\text{avg}} \cdot d\vec{s}
\]
Phase propagates in the direction of \( \mathbf{k} \). Therefore, the refractive index derived from \(|\mathbf{k}|\) is best described as the phase refractive index. Velocity here is the phase velocity.

Power propagates in the direction of the Poynting vector which is always normal to the surface of the index ellipsoid. From this, we can define a group velocity and a group refractive index.