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Electromagnetics:
Electromagnetic Field Theory

Skin Depth & Power Flow

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Lecture Outline

- Skin Depth δ
- Power Flow

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Skin Depth δ

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Skin Depth δ

Waves in good conductors attenuate very quickly. The distance over which they decay by a factor of $1/e$ is called the skin depth δ.

Definition of Skin Depth

$$\delta = 1/\alpha \quad (\mathrm{m})$$

Relation to Impedance

$$\eta = \frac{1+j}{\sigma\delta} = \frac{\sqrt{2}}{\sigma\delta}\measuredangle 45^\circ$$

In Terms of Fundamental Parameters

$$\delta = \sqrt{\frac{2}{\omega\mu\sigma}} = \frac{1}{\sqrt{\pi f \mu\sigma}}$$

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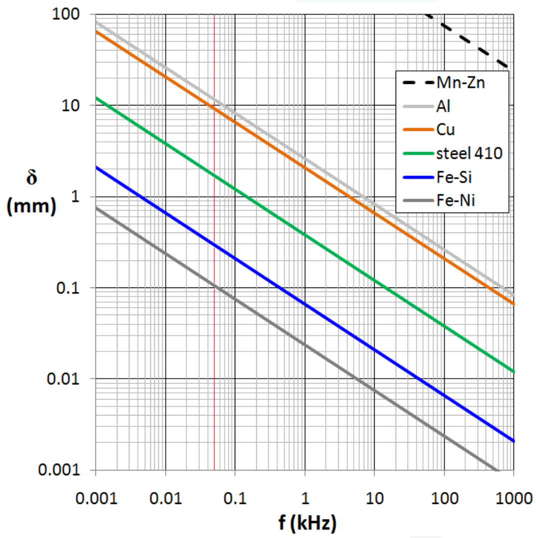
Skin Depth for Various Materials



Skin depth vs. frequency for some materials at room temperature, red vertical line denotes 50 Hz frequency:
Mn-Zn – magnetically soft ferrite
Al – metallic aluminum
Cu – metallic copper
steel 410 – magnetic stainless steel
Fe-Si – grain-oriented electrical steel
Fe-Ni – high-permeability permalloy (80%Ni-20%Fe)

https://en.wikipedia.org/wiki/Skin_effect

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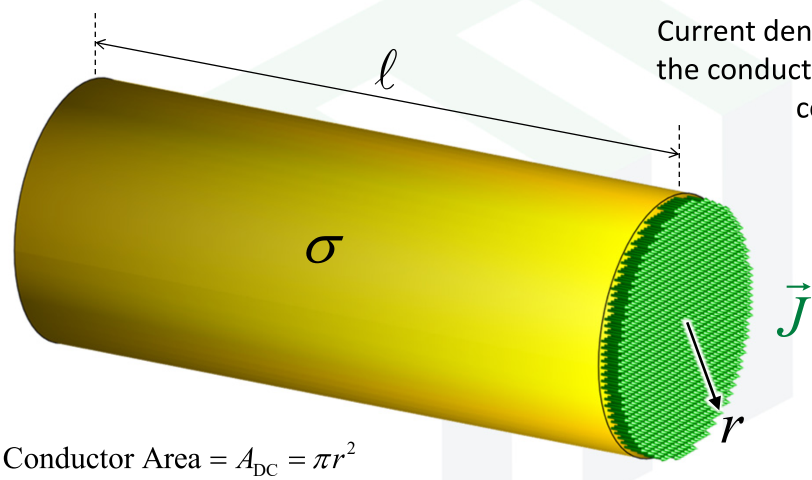
DC Resistance, R_{DC}



Current density is uniform throughout the conductor so the entire conductor contributes to current flow.

$$\text{Conductor Area} = A_{\mathrm{DC}} = \pi r^2$$

$$R_{\mathrm{DC}} = \frac{\ell}{\sigma A_{\mathrm{DC}}} = \frac{\ell}{\sigma \pi r^2}$$

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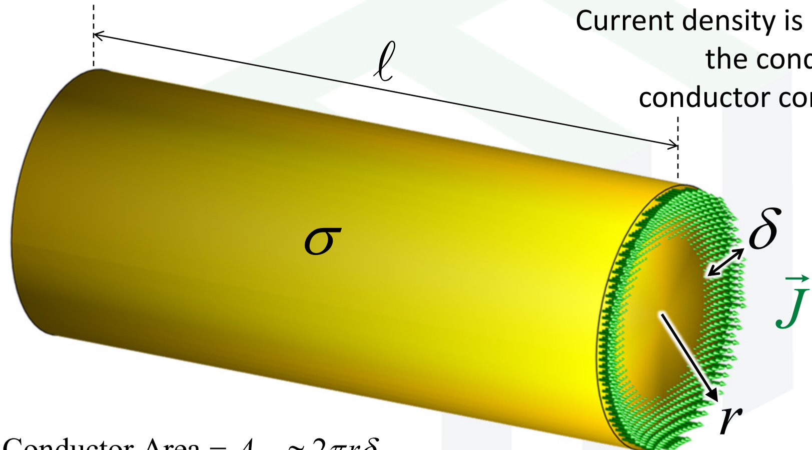
AC Resistance, R_{AC}

Current density is NOT uniform throughout the conductor so only part of the conductor contributes to current flow.



$$\text{Conductor Area} = A_{\text{AC}} \approx 2\pi r\delta$$

$$R_{\text{AC}} = \frac{\ell}{\sigma A_{\text{AC}}} = \frac{\ell}{2\pi r\sigma\delta} \qquad \text{for } \delta \ll r$$

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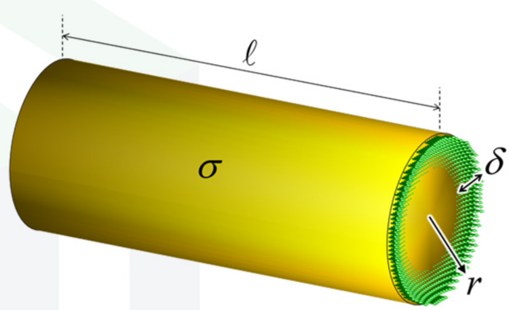
Conductor Area for AC Resistance

Area Described by δ

$$\begin{aligned} A_{\text{AC}} &= \pi r^2 - \pi(r-\delta)^2 \\ &= \pi r^2 - \pi\left(r^2 - 2r\delta + \delta^2\right) \\ &= \pi r^2 - \pi r^2 + 2\pi r\delta - \pi\delta^2 \\ &= 2\pi r\delta - \cancel{\pi\delta^2} \quad \text{for } \delta \ll r \\ &\approx 2\pi r\delta \end{aligned}$$



Effective Conductor Area

$A_{\text{AC}} \approx 2\pi r\delta \qquad \text{for } \delta \ll r$

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Notes on Skin Depth and AC Resistance

- High frequencies experience so much loss that they do not penetrate very far into a conductor.
- The depth of penetration is called skin depth δ.
- Due to the skin depth at high frequencies, only part of the conductor contributes to current flow. This makes resistance increase as a function of frequency.
- Drawbacks
  - High frequencies experience more loss.
  - Signals get distorted
- Benefits
  - Conductors can be made hollow – cheaper, lighter, etc.
  - Can make inner part of conductor out of a different material.

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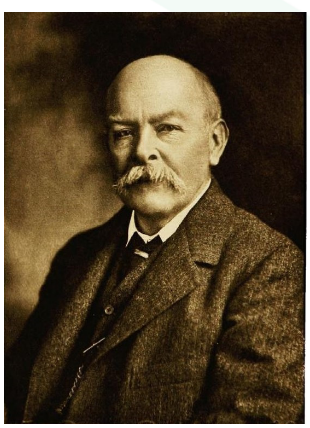
Power Flow



John Henry Poynting
1852 – 1914
Academic Advisor: James Clerk Maxwell

https://en.wikipedia.org/wiki/John_Henry_Poynting

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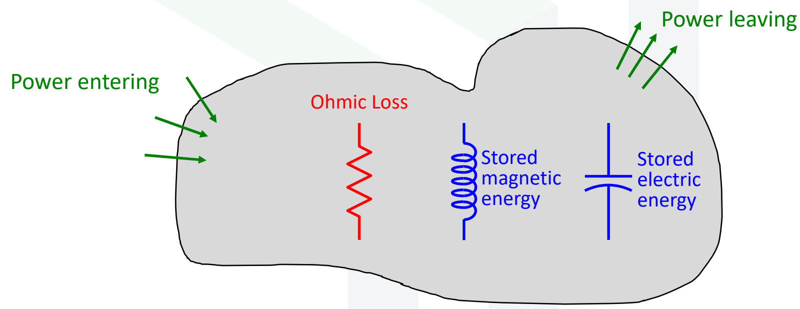
Poynting's Theorem

Poynting's theorem is a conservation of power equation.

The total power leaving a volume must be equal to the rate of decrease of the total energy stored in the field plus the energy lost due to heat (or something else).



$$\underbrace{\oint\!\!\!\oint_S \left(\vec{E}\times\vec{H}\right)\bullet d\vec{s}}_{\text{Total power leaving volume}} = \underbrace{-\frac{1}{2}\frac{\partial}{\partial t}\iiint_V \left(\mu\left|\vec{H}\right|^2 + \varepsilon\left|\vec{E}\right|^2\right)dv}_{\text{Rate of decrease of stored electric and magnetic energy}} \underbrace{-\iiint_V \left(\sigma\left|\vec{E}\right|^2\right)dv}_{\text{Ohmic power dissipated}}$$

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Poynting Vector

From Poynting's theorem, the term responsible for power leaving the volume is identified.

$$\oint\!\!\!\oint_S \left(\vec{E}\times\vec{H}\right)\bullet d\vec{s} = -\frac{1}{2}\frac{\partial}{\partial t}\iiint_V \left(\mu\left|\vec{H}\right|^2 + \varepsilon\left|\vec{E}\right|^2\right)dv - \iiint_V \left(\sigma\left|\vec{E}\right|^2\right)dv$$

Here flux is being integrated to get total power. The argument must be power density (W/m²). This term is called the *instantaneous Poynting vector*.

$$\boxed{\vec{\wp}(t) = \vec{E}(t)\times\vec{H}(t)}$$

Due to the cross product, the Poynting vector $\vec{\wp}$ is perpendicular to both $\vec{E}$ and $\vec{H}$. For LHI materials, the Poynting vector is in the same direction as the wave vector $\vec{k}$.

$\vec{E}\perp\vec{k}\perp\vec{H}$ $\qquad$ $\vec{E}\perp\vec{\wp}\perp\vec{H}$ $\qquad$ $\vec{\wp}\parallel\vec{k}$

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Instantaneous Poynting Vector $\vec{\wp}(t)$

Recall the electric and magnetic field components of a plane wave travelling in the $+z$ direction can be written in the time-domain as

$$\vec{E}(z,t) = E_0 e^{-\alpha z}\cos(\omega t - \beta z)\hat{a}_x \qquad \vec{H}(z,t) = \frac{E_0}{|\eta|}e^{-\alpha z}\cos(\omega t - \beta z - \measuredangle\eta)\hat{a}_y$$

Substituting these expressions into the definition of the instantaneous Poynting vector gives

$$\vec{\wp}(t) = \vec{E}(t)\times\vec{H}(t)$$

$$= \left[E_0 e^{-\alpha z}\cos(\omega t - \beta z)\hat{a}_x\right]\times\left[\frac{E_0}{|\eta|}e^{-\alpha z}\cos(\omega t - \beta z - \measuredangle\eta)\hat{a}_y\right]$$

$$\vec{\wp}(t) = \underbrace{\frac{E_0^2}{2|\eta|}e^{-2\alpha z}\cos(\measuredangle\eta)\hat{a}_z}_{\text{Constant power flow}} + \underbrace{\frac{E_0^2}{2|\eta|}e^{-2\alpha z}\cos(2\omega t - 2\beta z - \measuredangle\eta)\hat{a}_z}_{\text{Rapidly oscillating fluctuation in power flow}}$$

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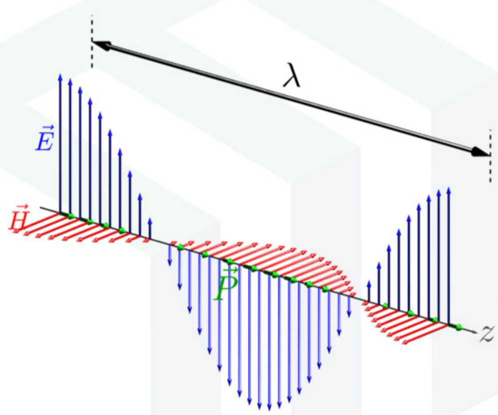
Animation of Instantaneous Power Flow



$$\vec{\wp}(t) = \underbrace{\frac{E_0^2}{2|\eta|}e^{-2\alpha z}\cos(\measuredangle\eta)\hat{a}_z}_{\text{Constant power flow}} + \underbrace{\frac{E_0^2}{2|\eta|}e^{-2\alpha z}\cos(2\omega t - 2\beta z - \measuredangle\eta)\hat{a}_z}_{\text{Rapidly oscillating fluctuation in power flow}}$$

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Average Poynting Vector $\vec{\wp}_{\text{avg}}$

The instantaneous power flow is rarely of interest because the rapidly fluctuating term does not transport any net power. The more practical and useful quantity is the time-average Poynting Vector $\vec{\wp}_{\text{avg}}$.

To obtain the time-average Poynting vector $\vec{\wp}_{\text{avg}}$, integrate over one wave cycle τ.

$$\vec{\wp}_{\text{avg}} = \frac{1}{\tau}\int_0^\tau \vec{\wp}(t)\,dt$$

Integrating cosine over one wave cycle equals zero because cosine is both negative and positive equally.

$$= \frac{1}{\tau}\int_0^\tau \left[\frac{E_0^2}{2|\eta|}e^{-2\alpha z}\cos(\measuredangle\eta)\hat{a}_z\right]dt + \frac{1}{\tau}\int_0^\tau \left[\frac{E_0^2}{2|\eta|}e^{-2\alpha z}\cos(2\omega t - 2\beta z - \measuredangle\eta)\hat{a}_z\right]dt$$

$$\boxed{\vec{\wp}_{\text{avg}} = \frac{E_0^2}{2|\eta|}e^{-2\alpha z}\cos(\measuredangle\eta)\hat{a}_z}$$

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Complex Poynting Vector $\vec{\wp}$

For time-harmonic signals, the frequency-domain Poynting vector is complex.

$$\boxed{\vec{\wp} = \vec{E}\times\vec{H}^*}$$

The field expressions for the plane wave are

$$\vec{E}(z) = E_0 e^{-\gamma z}\hat{a}_x \qquad \vec{H}(z) = \frac{E_0}{|\eta|}e^{-\gamma z}e^{-j\measuredangle\eta}\hat{a}_y$$

Substituting these into the definition of complex Poynting vector gives

$$\vec{\wp} = \left[E_0 e^{-\gamma z}\hat{a}_x\right]\times\left[\frac{E_0}{|\eta|}e^{-\gamma z}e^{-j\measuredangle\eta}\hat{a}_y\right]^*$$

$$= \left[E_0 e^{-\alpha z}e^{-j\beta z}\hat{a}_x\right]\times\left[\frac{E_0^*}{|\eta|}e^{-\alpha z}e^{j\beta z}e^{j\measuredangle\eta}\hat{a}_y\right]$$

$$= \frac{|E_0|^2}{|\eta|}e^{-2\alpha z}e^{j\measuredangle\eta}\left(\hat{a}_x\times\hat{a}_y\right) \quad\rightarrow\quad \boxed{\vec{\wp} = \frac{E_0^2}{|\eta|}e^{-2\alpha z}e^{j\measuredangle\eta}\hat{a}_z}$$

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RMS Poynting Vector $\vec{\wp}_{\text{avg}}$

The complex Poynting vector $\vec{\wp}$ is like the instantaneous Poynting vector $\vec{\wp}(t)$ and contains the rapidly varying fluctuations in power flow.

A more meaningful quantity is the root-mean-square (RMS) power flow that is easily calculated from the complex Poynting vector.

$$\vec{\wp}_{\text{avg}} = \frac{1}{2}\text{Re}\left[\vec{\wp}\right] = \frac{1}{2}\text{Re}\left[\vec{E}\times\vec{H}^*\right]$$

$$\begin{aligned}\vec{\wp}_{\text{avg}} &= \frac{1}{2}\text{Re}\left[\vec{\wp}\right] \\ &= \frac{1}{2}\text{Re}\left[\frac{E_0^2}{|\eta|}e^{-2\alpha z}e^{j\measuredangle\eta}\hat{a}_z\right] \\ &= \frac{E_0^2}{2|\eta|}e^{-2\alpha z}\cos\left(\measuredangle\eta\right)\hat{a}_z\end{aligned}$$

$$\vec{\wp}_{\text{avg}} = \frac{E_0^2}{2|\eta|}e^{-2\alpha z}\cos\left(\measuredangle\eta\right)\hat{a}_z$$

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Four Common Poynting Vector Quantities

	Time-Domain	Frequency-Domain
Instantaneous	$\vec{\wp}(t) = \vec{E}(t) \times \vec{H}(t)$	$\vec{\wp}_{\text{avg}} = \frac{E_0^2}{2\|\eta\|}e^{-2\alpha z}\cos(\angle\eta)\,\hat{a}_z$
Average	$\vec{\wp} = \vec{E} \times \vec{H}^* = \frac{E_0^2}{\|\eta\|}e^{-2\alpha z}\cos(\angle\eta)\,\hat{a}_z$	$\vec{\wp}_{\text{avg}} = \frac{1}{2}\text{Re}\left[\vec{E} \times \vec{H}^*\right] = \frac{E_0^2}{2\|\eta\|}e^{-2\alpha z}\cos(\angle\eta)\,\hat{a}_z$

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Total Power

The Poynting vector is a power density with units of W/m^2.

To calculate total power flow through some area, integrate the Poynting vector over that area.

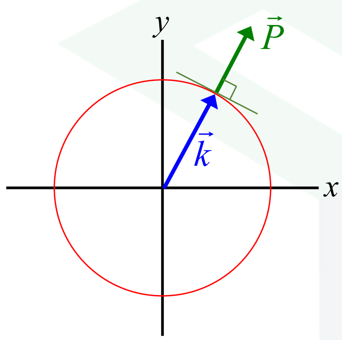
$$P(t) = \iint_S \vec{\wp}(t) \bullet d\vec{s}$$

The average power flow is simply calculated from the average Poynting vector.

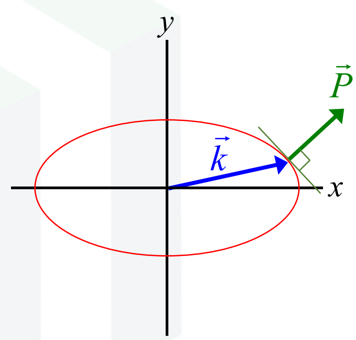
$$P_{\text{avg}} = \iint_S \vec{\wp}_{\text{avg}} \bullet d\vec{s}$$

Index Ellipsoids and Power Flow

Isotropic Materials



Anisotropic Materials



Phase propagates in the direction of $\vec{k}$. Therefore, the refractive index n derived from $|\vec{k}|$ is best described as the phase refractive index n_{p}. Velocity here is the phase velocity $\vec{v}_{\text{p}}$.

Power propagates in the direction of the Poynting vector $\vec{\wp}$ which is always normal to the surface of the index ellipsoid. From this, we can define a group refractive index n_{g} and group velocity $\vec{v}_{\text{g}}$.