



Electromagnetics:
Electromagnetic Field Theory

Standing Waves

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Lecture Outline

- Standing Waves
- Standing Wave Ratio (SWR)

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Standing Waves

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Two Counter-Propagating Waves (1 of 2)

Suppose we have two counter-propagating waves of equal amplitude travelling in opposite directions.



Observations:

1. Things are boring until the waves overlap.
2. Large fluctuations in amplitude are observed.
3. Locations of the fluctuations are stationary.
4. Total field is zero at some points.

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Two Counter-Propagating Waves (2 of 2)

Suppose we have two counter-propagating waves that do not have equal amplitude travelling in opposite directions.



New Observations:

1. Fluctuations are smaller.
2. Fluctuations do not go to zero.

General Expressions for Forward and Backward Waves

An incident wave will be reflected from an interface.

On the reflection side, there will exist two counterpropagating waves.

Incident Wave

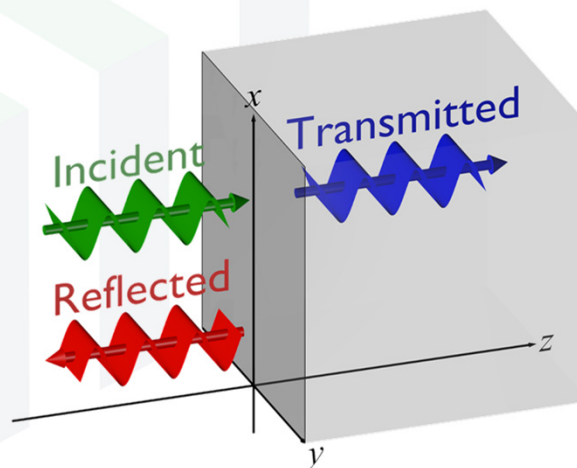
$$\vec{E}_i(z) = E_{0,i} e^{-\gamma_1 z} \hat{a}_x$$

$$\vec{H}_i(z) = \frac{E_{0,i}}{\eta_1} e^{-\gamma_1 z} \hat{a}_y$$

Reflected Wave

$$\vec{E}_r(z) = E_{0,r} e^{+\gamma_1 z} \hat{a}_x$$

$$\vec{H}_r(z) = -\frac{E_{0,r}}{\eta_1} e^{+\gamma_1 z} \hat{a}_y$$



Wave Incident on Metal (1 of 2)

To more easily understand what happens on the reflection side, let the wave be incident from a lossless dielectric (i.e. $\sigma = 0$) onto metal (i.e. $\sigma = \infty$).

In this case, the material parameters are

$$\eta_1 = \sqrt{\frac{\mu}{\epsilon}} \quad \eta_2 = 0$$

The reflection and transmission coefficients are

$$r = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} = \frac{0 - \sqrt{\mu/\epsilon}}{0 + \sqrt{\mu/\epsilon}} = -1$$

$$t = \frac{2\eta_2}{\eta_2 + \eta_1} = \frac{2 \cdot 0}{0 + \sqrt{\frac{\mu}{\epsilon}}} = 0$$

In this case, we get zero transmission and 100% reflection with a 180° phase shift.

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Wave Incident on Metal (2 of 2)

The propagation constant γ_1 is

$$\gamma_1 = \alpha_1 + j\beta_1$$

$$\alpha_1 = \omega \sqrt{\frac{\mu_1 \epsilon_1}{2} \left[\sqrt{1 + \left(\frac{\sigma_1}{\omega \epsilon_1} \right)^2} - 1 \right]} = \omega \sqrt{\frac{\mu_1 \epsilon_1}{2} \left[\sqrt{1 + \left(\frac{0}{\omega \epsilon_1} \right)^2} - 1 \right]} = 0$$

$$\beta_1 = \omega \sqrt{\frac{\mu_1 \epsilon_1}{2} \left[\sqrt{1 + \left(\frac{\sigma_1}{\omega \epsilon_1} \right)^2} + 1 \right]} = \omega \sqrt{\frac{\mu_1 \epsilon_1}{2} \left[\sqrt{1 + \left(\frac{0}{\omega \epsilon_1} \right)^2} + 1 \right]} = \omega \sqrt{\mu_1 \epsilon_1}$$

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Revised Expressions for Our Waves

Given the reflection coefficient r and phase constant β , we can rewrite our wave expressions as

Incident Wave

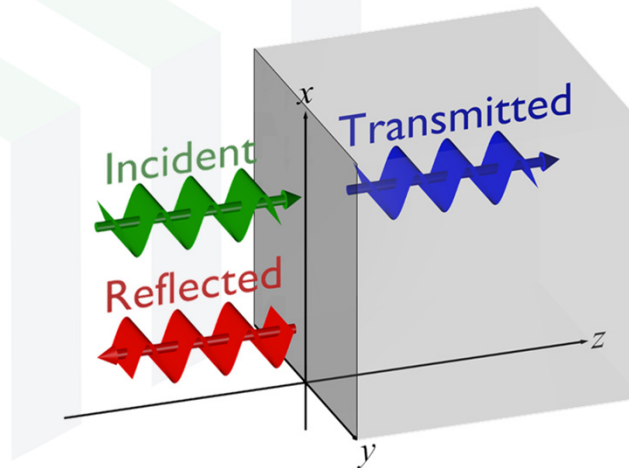
$$\vec{E}_i(z) = E_{0,i} e^{-j\beta_1 z} \hat{a}_x$$

$$\vec{H}_i(z) = \frac{E_{0,i}}{\eta_1} e^{-j\beta_1 z} \hat{a}_y$$

Reflected Wave

$$\vec{E}_r(z) = r E_{0,i} e^{+j\beta_1 z} \hat{a}_x$$

$$\vec{H}_r(z) = -\frac{r E_{0,i}}{\eta_1} e^{+j\beta_1 z} \hat{a}_y$$



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Frequency-domain Standing Waves ($r = -1$)

On the reflection side, the total electromagnetic field is the sum of both the incident and reflected wave.

$$\vec{E}_1(z) = \vec{E}_i(z) + \vec{E}_r(z) = E_{0,i} e^{-j\beta_1 z} \hat{a}_x - E_{0,i} e^{+j\beta_1 z} \hat{a}_x = E_{0,i} (e^{-j\beta_1 z} - e^{+j\beta_1 z}) \hat{a}_x$$

Recall $r = -1$

$$\vec{H}_1(z) = \vec{H}_i(z) + \vec{H}_r(z) = \frac{E_{0,i}}{\eta_1} e^{-j\beta_1 z} \hat{a}_y + \frac{E_{0,i}}{\eta_1} e^{+j\beta_1 z} \hat{a}_y = \frac{E_{0,i}}{\eta_1} (e^{-j\beta_1 z} + e^{+j\beta_1 z}) \hat{a}_y$$

The expressions in parentheses containing complex exponentials are the sine and cosine functions. The equations for the total field become

$$\vec{E}_1(z) = -j2E_{0,i} \sin(\beta_1 z) \hat{a}_x$$

$$\vec{H}_1(z) = \frac{2E_{0,i}}{\eta_1} \cos(\beta_1 z) \hat{a}_y$$

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Time-Domain Standing Waves ($r = -1$)

Converting our standing wave equations to the time-domain, we get

$$\begin{aligned}\vec{E}_1(z, t) &= \text{Re}[-j2E_{0,i} \sin(\beta_1 z) \hat{a}_x \cdot e^{j\omega t}] \\ &= -2E_{0,i} \sin(\beta_1 z) \hat{a}_x \cdot \text{Re}[je^{j\omega t}] \\ &= -2E_{0,i} \sin(\beta_1 z) \hat{a}_x \cdot \text{Re}[j \cos \omega t - \sin \omega t] \\ &= 2E_{0,i} \sin(\beta_1 z) \sin(\omega t) \hat{a}_x\end{aligned}$$

$$\begin{aligned}\vec{H}_1(z, t) &= \text{Re}\left[2 \frac{E_{0,i}}{\eta_1} \cos(\beta_1 z) \hat{a}_y \cdot e^{j\omega t}\right] \\ &= 2 \frac{E_{0,i}}{\eta_1} \cos(\beta_1 z) \hat{a}_y \cdot \text{Re}[e^{j\omega t}] \\ &= 2 \frac{E_{0,i}}{\eta_1} \cos(\beta_1 z) \hat{a}_y \cdot \text{Re}[\cos \omega t + j \sin \omega t] \\ &= 2 \frac{E_{0,i}}{\eta_1} \cos(\beta_1 z) \cos(\omega t) \hat{a}_y\end{aligned}$$

Standing Waves When $r = +1$

In the frequency-domain, we have

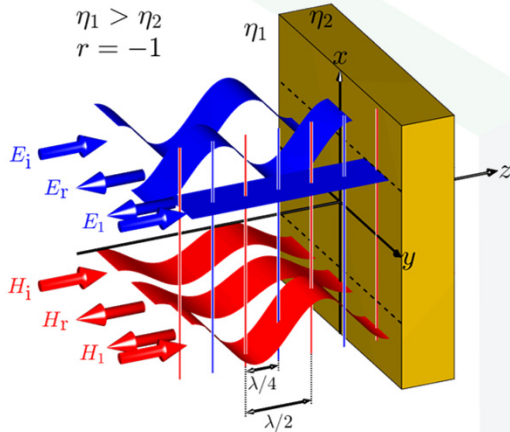
$$\begin{aligned}\vec{E}_1(z) &= \vec{E}_i(z) + \vec{E}_r(z) = E_{0,i} (e^{-j\beta_1 z} + e^{+j\beta_1 z}) \hat{a}_x = 2E_{0,i} \cos(\beta_1 z) \hat{a}_x \\ \vec{H}_1(z) &= \vec{H}_i(z) + \vec{H}_r(z) = \frac{E_{0,i}}{\eta_1} (e^{-j\beta_1 z} - e^{+j\beta_1 z}) \hat{a}_y = -j2 \frac{E_{0,i}}{\eta_1} \sin(\beta_1 z) \hat{a}_y\end{aligned}$$

In the time-domain, we have

$$\begin{aligned}\vec{E}_1(z, t) &= 2E_{0,i} \cos(\beta_1 z) \cos(\omega t) \hat{a}_x \\ \vec{H}_1(z, t) &= 2 \frac{E_{0,i}}{\eta_1} \sin(\beta_1 z) \sin(\omega t) \hat{a}_y\end{aligned}$$

Visualizing the Standing Waves (1 of 2)

We will let $r = -1$ represent the case where $\eta_1 > \eta_2$.



$$\vec{E}_1(z,t) = 2E_{0,i} \sin(\beta_1 z) \sin(\omega t) \hat{a}_x$$

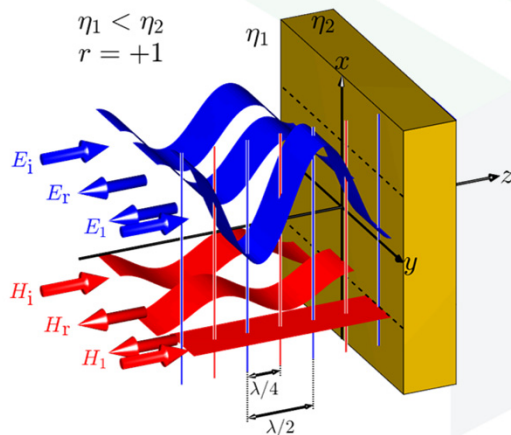
$$\vec{H}_1(z,t) = 2 \frac{E_{0,i}}{\eta_1} \cos(\beta_1 z) \cos(\omega t) \hat{a}_y$$

Observations:

1. 180° phase shift after reflection.
2. Max E and min H occur at the same points.
3. Min E and max H occur at the same points.
4. E is minimum at the interface and H is maximum.
5. Nodes occur a half-wavelength apart.
6. The standing wave is stationary.
7. $\sin(\beta z)$ and $\cos(\beta z)$ terms describe the envelope of the standing wave.

Visualizing the Standing Waves (2 of 2)

We will let $r = +1$ represent the case where $\eta_1 < \eta_2$.



$$\vec{E}_1(z,t) = 2E_{0,i} \cos(\beta_1 z) \cos(\omega t) \hat{a}_x$$

$$\vec{H}_1(z,t) = 2 \frac{E_{0,i}}{\eta_1} \sin(\beta_1 z) \sin(\omega t) \hat{a}_y$$

Observations:

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More Rigorous Visualization (1 of 2)

Electric Field Functions

Standing Wave Envelope $\sin(\beta z)$

Standing Wave $E_i(z) + E_R(z)$

Reflected Wave $E_R(z) = rE_0e^{j\beta z}$

Incident Wave $E_i(z) = E_0e^{-j\beta z}$

Magnetic Field Functions

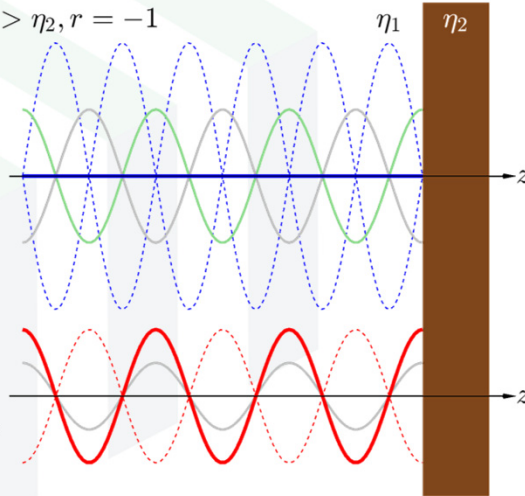
Standing Wave Envelope $\cos(\beta z)$

Standing Wave $H_i(z) + H_R(z)$

Reflected Wave $H_R(z) = -r\frac{E_0}{\eta}e^{j\beta z}$

Incident Wave $H_i(z) = \frac{E_0}{\eta}e^{-j\beta z}$

$\eta_1 > \eta_2, r = -1$



More Rigorous Visualization (2 of 2)

Electric Field Functions

Standing Wave Envelope $\cos(\beta z)$

Standing Wave $E_i(z) + E_R(z)$

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Magnetic Field Functions

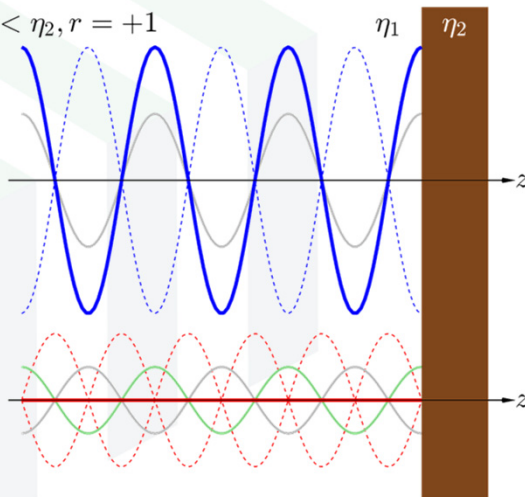
Standing Wave Envelope $\sin(\beta z)$

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Reflected Wave $H_R(z) = -r\frac{E_0}{\eta}e^{j\beta z}$

Incident Wave $H_i(z) = \frac{E_0}{\eta}e^{-j\beta z}$

$\eta_1 < \eta_2, r = +1$



Standing Wave Ratio (SWR)

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Definition of Standing Wave Ratio (SWR)

We wish to have a metric to quantify the severity of the standing wave.

To do this, we define the *standing wave ratio* (SWR) as the maximum electric field observed in the standing wave divided by the minimum electric field observed in the standing wave.

$$\text{SWR} = \frac{|E|_{\max}}{|E|_{\min}}$$

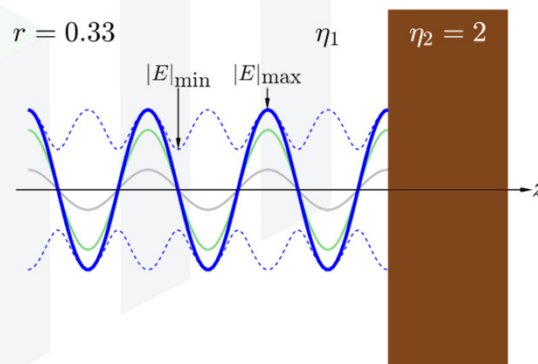
Electric Field Functions

Standing Wave Envelope $\cos(\beta z)$

Standing Wave $E_i(z) + E_R(z)$

Reflected Wave $E_R(z) = rE_0e^{j\beta z}$

Incident Wave $E_i(z) = E_0e^{-j\beta z}$



EMPossible

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Derivation of Standing Wave Ratio (SWR)

Let's examine our expression for the electric field when we have counter propagating waves.

$$\vec{E}_1(z) = \vec{E}_i(z) + \vec{E}_r(z) = E_{0,i} (e^{-\gamma_1 z} + r e^{+\gamma_1 z}) \hat{a}_x$$

This expression has the following maximum and minimum.

$$\max |\vec{E}_1| = E_{0,i} (1 + |r|)$$

$$\min |\vec{E}_1| = E_{0,i} (1 - |r|)$$

Substituting these into our definition of SWR gives

$$\text{SWR} = \frac{\max |\vec{E}_1|}{\min |\vec{E}_1|} = \frac{E_{0,i} (1 + |r|)}{E_{0,i} (1 - |r|)} \rightarrow \boxed{\text{SWR} = \frac{1 + |r|}{1 - |r|}}$$

Derivation in Terms of Magnetic Field

Let's examine our expression for the magnetic field when we have counter propagating waves.

$$\vec{H}_1(z) = \vec{H}_i(z) + \vec{H}_r(z) = \frac{E_{0,i}}{\eta} (e^{-\gamma_1 z} - r e^{+\gamma_1 z}) \hat{a}_y$$

This expression has the following maximum and minimum.

$$\max |\vec{H}_1| = \frac{E_{0,i}}{\eta} (1 + |r|)$$

$$\min |\vec{H}_1| = \frac{E_{0,i}}{\eta} (1 - |r|)$$

Dividing these shows that we get the same expression for SWR

$$\frac{\max |\vec{H}_1|}{\min |\vec{H}_1|} = \frac{\frac{E_{0,i}}{\eta} (1 + |r|)}{\frac{E_{0,i}}{\eta} (1 - |r|)} = \frac{1 + |r|}{1 - |r|} \rightarrow \boxed{\text{SWR} = \frac{|\vec{H}|_{\max}}{|\vec{H}|_{\min}} = \frac{1 + |r|}{1 - |r|}}$$

SWR in Decibel Scale

Very often the SWR is given on a decibel scale.

$$\text{SWR}_{\text{dB}} = 20 \log_{10}(\text{SWR})$$

Given the SWR in dB, we can calculate the SWR on a linear scale.

$$\text{SWR} = 10^{\text{SWR}_{\text{dB}}/20}$$

Usefulness of SWR

The standing wave ratio (SWR) is something that we can directly measure.

Given the SWR, we can calculate the magnitude of the reflection coefficient.

$$|r| = \frac{\text{SWR} - 1}{\text{SWR} + 1}$$

Derivation

$$\text{SWR} = \frac{1 + |r|}{1 - |r|}$$

$$\text{SWR} - |r|\text{SWR} = 1 + |r|$$

$$|r|\text{SWR} + |r| = \text{SWR} - 1$$

$$|r|(\text{SWR} + 1) = \text{SWR} - 1$$

$$|r| = \frac{\text{SWR} - 1}{\text{SWR} + 1}$$

Notes About the SWR

- Since $0 \leq |r| \leq 1$, we conclude that $1 \leq \text{SWR} \leq \infty$.
- SWR is very large when the reflection is very strong.
- $\text{SWR} = 1$ ($\text{SWR}_{\text{dB}} = 0$)
 - Zero standing wave
 - $|r| = 0$
 - No backward wave.
- $\text{SWR} = \infty$ ($\text{SWR}_{\text{dB}} = \infty$)
 - Does NOT imply infinite amplitude standing wave
 - Standing wave has a perfect null (amplitude goes to zero)
 - $|r| = 1$
 - Forward and backward waves have equal amplitude.

Example

Suppose we have a wave inside of a 50Ω medium that is incident onto a second medium with impedance 120Ω . What fraction of power is reflected? What is the standing wave ratio (SWR)?

Solution

The reflection coefficient at the interface is

$$r = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} = \frac{120 \Omega - 50 \Omega}{120 \Omega + 50 \Omega} = \frac{70}{170} = 0.4118$$

The fraction of power reflected is the reflectance.

$$R = |r|^2 = |0.4118|^2 = 0.1696 = \boxed{16.96\%}$$

The SWR is

$$\text{SWR} = \frac{1 + |r|}{1 - |r|} = \frac{1 + |0.4118|}{1 - |0.4118|} = \frac{1.4118}{0.5882} = \boxed{2.4}$$

$$\text{SWR}_{\text{dB}} = 20 \log_{10}(\text{SWR}) = 20 \log_{10}(2.4) = \boxed{7.6 \text{ dB}}$$