Electromagnetics:
Electromagnetic Field Theory

Transmission Line Equations

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Lecture Outline

- Introduction
- Transmission Line Equations
- Transmission Line Wave Equations

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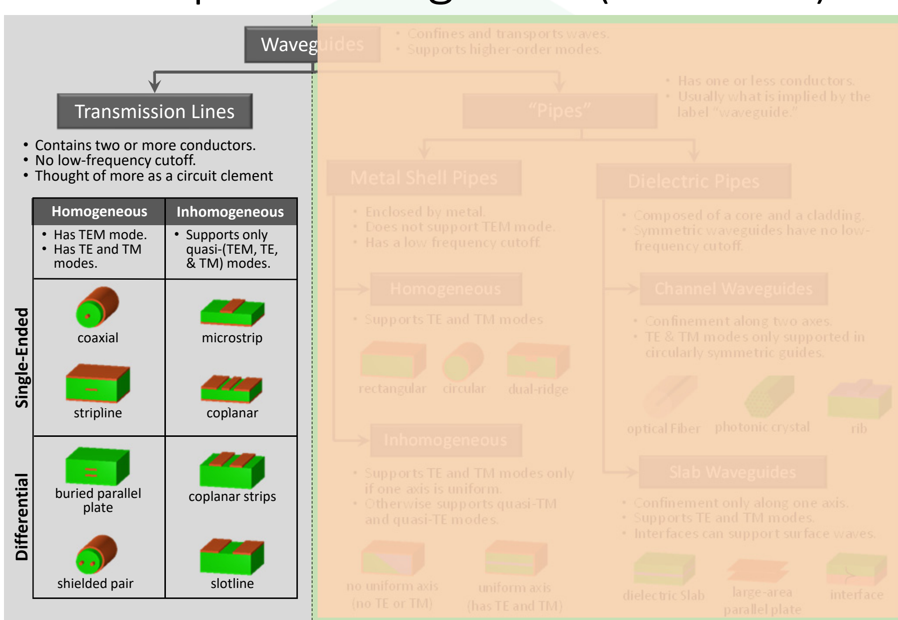
Introduction

Map of Waveguides (LI Media)



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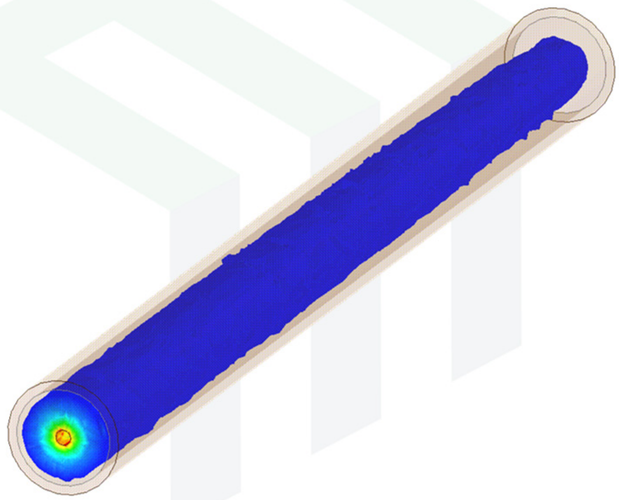
Signals in Transmission Lines: Coax



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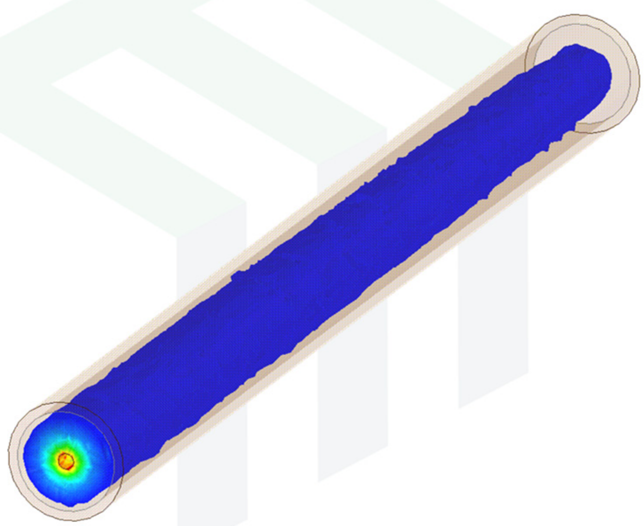
Signals in Transmission Lines: Microstrip



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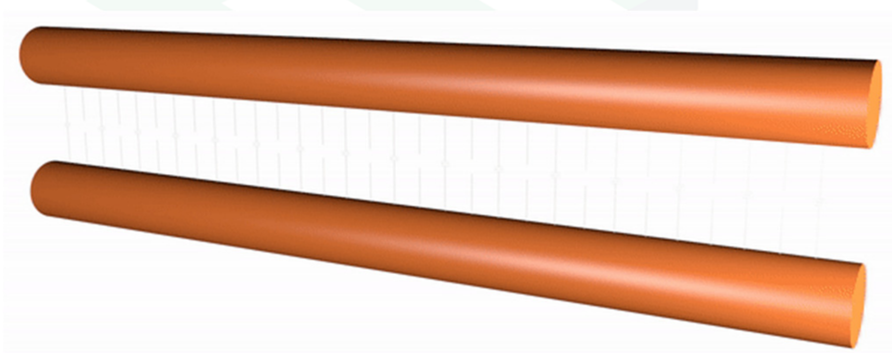
Signals in Transmission Lines: Twisted Pair

Transmission Line Parameters RLGC

It can be useful to think of transmission lines as being composed of millions of tiny little circuit elements that are distributed along the length of the line.

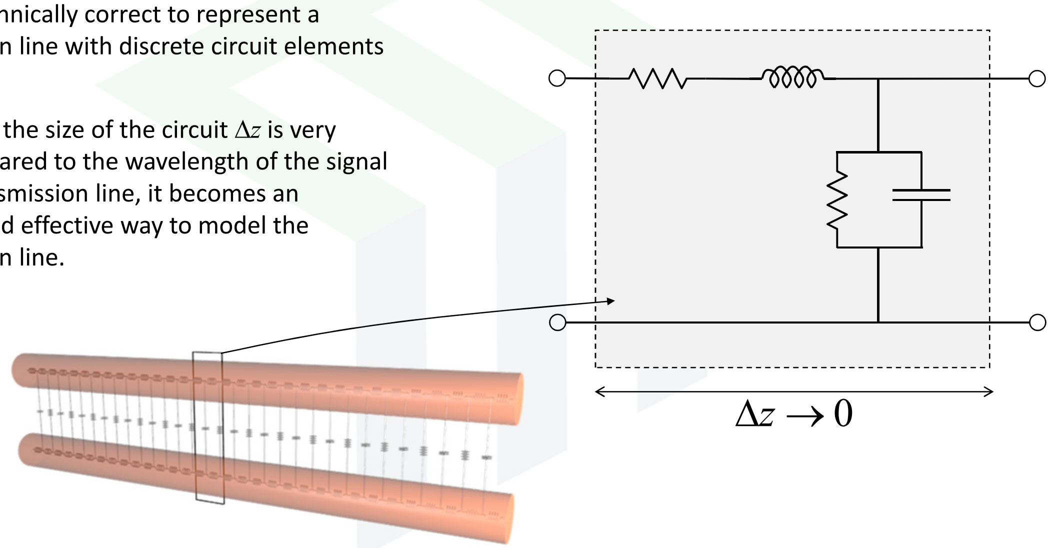


In fact, these circuit element are not discrete, but continuous along the length of the transmission line.

RLGC Circuit Model

It is not technically correct to represent a transmission line with discrete circuit elements like this.

However, if the size of the circuit Δz is very small compared to the wavelength of the signal on the transmission line, it becomes an accurate and effective way to model the transmission line.



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L-Type Equivalent Circuit Model

Distributed Circuit Parameters

R (Ω/m)
Resistance per unit length.
Arises due to resistivity in the conductors.

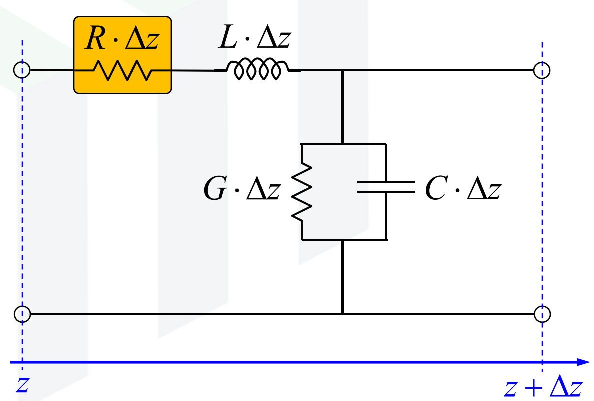
L (H/m)
Inductance per unit length.
Arises due to stored magnetic energy around the line.

G (1/Ω·m)
Conductance per unit length.
Arises due to conductivity in the dielectric separating the conductors.

$$G \neq \frac{1}{R}$$

C (F/m)
Capacitance per unit length.
Arises due to stored electric energy between the conductors.

There are many possible circuit models for transmission lines, but most produce the same equations after analysis.



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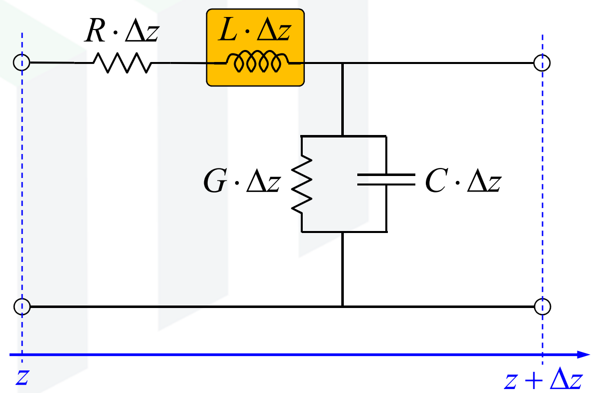

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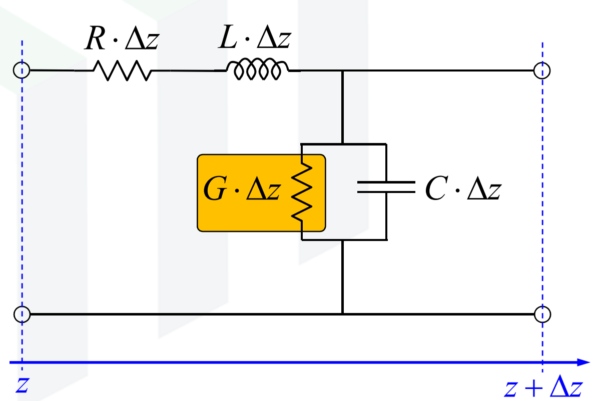

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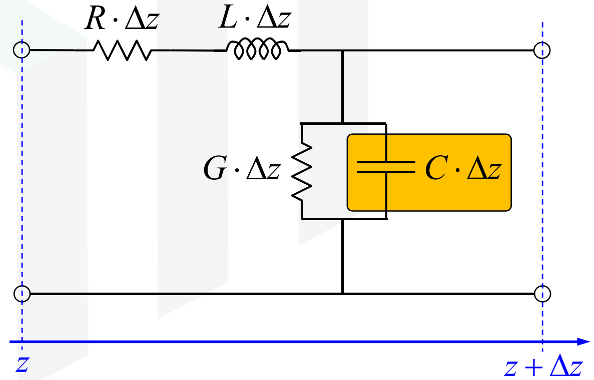

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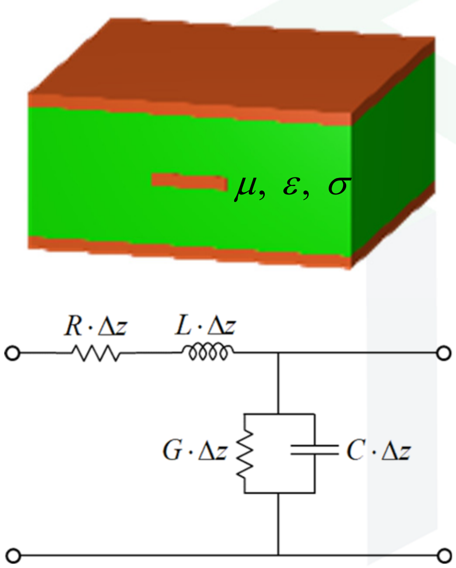
Relation to Electromagnetic Parameters



Every transmission line with a homogeneous fill has:

$$LC = \mu\varepsilon$$

$$\frac{G}{C} = \frac{\sigma}{\varepsilon}$$

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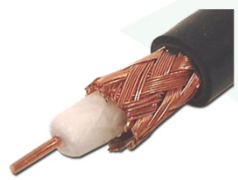
Example RLGC Parameters

RG-59 Coax

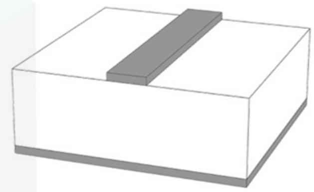


$R = 36 \text{ m}\Omega/\text{m}$
$L = 430 \text{ nH/m}$
$G = 10 \ \mu\mho/\text{m}$
$C = 69 \text{ pF/m}$
$Z_0 = 75 \ \Omega$

CAT5 Twisted Pair

$R = 176 \text{ m}\Omega/\text{m}$
$L = 490 \text{ nH/m}$
$G = 2 \ \mu\mho/\text{m}$
$C = 49 \text{ pF/m}$
$Z_0 = 100 \ \Omega$

Microstrip



$R = 150 \text{ m}\Omega/\text{m}$
$L = 364 \text{ nH/m}$
$G = 3 \ \mu\mho/\text{m}$
$C = 107 \text{ pF/m}$
$Z_0 = 50 \ \Omega$

Surprisingly, almost all transmission lines have parameters very close to these same values.

Transmission Line Equations

E & H → V and I

Fundamentally, all circuit problems are electromagnetic problems and can be solved as such.

All two-conductor transmission lines either support a TEM wave or a wave very closely approximated as TEM.

An important property of TEM waves is that E is uniquely related to V and H and uniquely related to E.

$$V = -\int_L \vec{E} \bullet d\vec{\ell} \qquad I = \oint_L \vec{H} \bullet d\vec{\ell}$$

This reduces analysis of transmission lines to just V and I. This makes analysis much simpler because these are scalar quantities!

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Transmission Line Equations

The transmission line equations do for transmission lines the same thing as Maxwell's curl equations do for waves.

Maxwell's Equations	Transmission Line Equations
$\nabla \times \vec{E} = -\mu \frac{\partial \vec{H}}{\partial t}$	$-\frac{\partial V}{\partial z} = RI + L\frac{\partial I}{\partial t}$
$\nabla \times \vec{H} = \varepsilon \frac{\partial \vec{E}}{\partial t}$	$-\frac{\partial I}{\partial z} = GV + C\frac{\partial V}{\partial t}$

Like Maxwell's equations, the transmission line equations are rarely directly useful. Instead, we will derive all of the useful equations from them.

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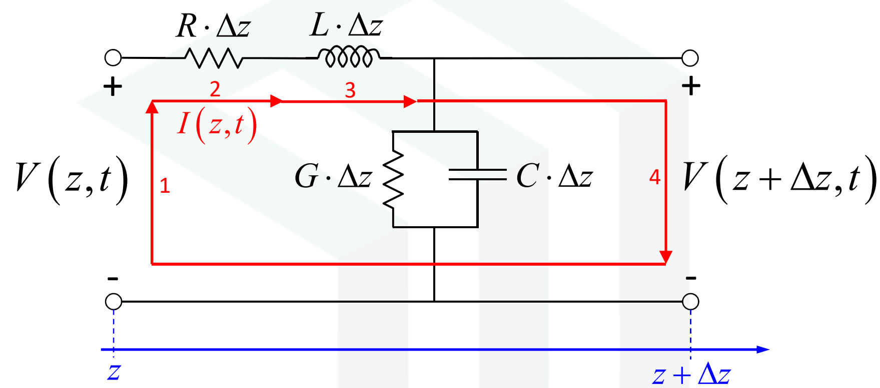
Derivation of First TL Equation (1 of 2)



Apply Kirchoff's voltage law (KVL) to the outer loop of the equivalent circuit:

$$\underbrace{-V(z,t)}_{1}+\underbrace{I(z,t)R\Delta z}_{2}+\underbrace{L\Delta z\frac{\partial I(z,t)}{\partial t}}_{3}+\underbrace{V(z+\Delta z,t)}_{4}=0$$

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Derivation of First TL Equation (2 of 2)

We rearrange the equation by bringing all of the voltage terms to the left-hand side of the equation, bringing all of the current terms to the right-hand side of the equation, and then dividing both sides by Δz.

$$-V(z,t)+I(z,t)R\Delta z+L\Delta z\frac{\partial I(z,t)}{\partial t}+V(z+\Delta z,t)=0$$

$$\downarrow$$

$$-\frac{V(z+\Delta z,t)-V(z,t)}{\Delta z}=RI(z,t)+L\frac{\partial I(z,t)}{\partial t}$$

In the limit as $\Delta z \to 0$, the expression on the left-hand side becomes a derivative with respect to z.

$$\boxed{-\frac{\partial V(z,t)}{\partial z}=RI(z,t)+L\frac{\partial I(z,t)}{\partial t}}$$

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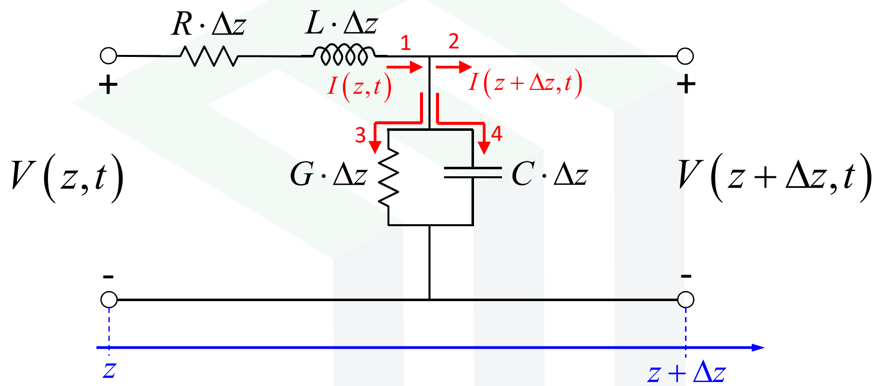
Derivation of Second TL Equation (1 of 2)



Apply Kirchoff's current law (KCL) to the main node the equivalent circuit:

$$\underbrace{I(z,t)}_{1} - \underbrace{I(z+\Delta z,t)}_{2} - \underbrace{G\Delta z V(z+\Delta z,t)}_{3} - \underbrace{C\Delta z \frac{\partial V(z+\Delta z,t)}{\partial t}}_{4} = 0$$

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Derivation of Second TL Equation (2 of 2)

We rearrange the equation by bringing all of the current terms to the left-hand side of the equation, bringing all of the voltage terms to the right-hand side of the equation, and then dividing both sides by Δz.

$$I(z,t) - I(z+\Delta z,t) - G\Delta z V(z+\Delta z,t) - C\Delta z \frac{\partial V(z+\Delta z,t)}{\partial t} = 0$$

$$\downarrow$$

$$-\frac{I(z+\Delta z,t) - I(z,t)}{\Delta z} = GV(z+\Delta z,t) + C\frac{\partial V(z+\Delta z,t)}{\partial t}$$

In the limit as $\Delta z \rightarrow 0$, the expression on the left-hand side becomes a derivative with respect to z.

$$\boxed{-\frac{\partial I(z,t)}{\partial z} = GV(z,t) + C\frac{\partial V(z,t)}{\partial t}}$$

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Transmission Line Wave Equations

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Starting Point – Telegrapher Equations

Start with the transmission line equations derived in the previous section.

$$-\frac{\partial V(z,t)}{\partial z} = RI(z,t) + L\frac{\partial I(z,t)}{\partial t} \qquad -\frac{\partial I(z,t)}{\partial z} = GV(z,t) + C\frac{\partial V(z,t)}{\partial t}$$

time-domain

For time-harmonic (i.e. frequency-domain) analysis, Fourier transform the equations above.

$$\boxed{-\frac{dV(z)}{dz} = (R + j\omega L)I(z)} \qquad \boxed{-\frac{dI(z)}{dz} = (G + j\omega C)V(z)}$$

frequency-domain

Note: The derivative d/dz became an ordinary derivative because z is the only independent variable left.

These last equations are commonly referred to as the *telegrapher equations*.

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Wave Equation in Terms of $V(z)$

$$-\frac{dV(z)}{dz} = (R + j\omega L) I(z) \quad \text{Eq. (1)}$$

$$-\frac{dI(z)}{dz} = (G + j\omega C) V(z) \quad \text{Eq. (2)}$$

To derive a wave equation in terms of $V(z)$, first differentiate Eq. (1) with respect to z.

$$-\frac{d^2V(z)}{dz^2} = (R + j\omega L)\frac{dI(z)}{dz} \quad \text{Eq. (3)}$$

Second, substitute Eq. (2) into the right-hand side of Eq. (3) to eliminate $I(z)$ from the equation.

$$-\frac{d^2V(z)}{dz^2} = -(R + j\omega L)(G + j\omega C) V(z)$$

Last, rearrange the terms to arrive at the final form of the wave equation.

$$\boxed{\frac{d^2V(z)}{dz^2} - (R + j\omega L)(G + j\omega C) V(z) = 0}$$

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Wave Equation in Terms of $I(z)$

$$-\frac{dV(z)}{dz} = (R + j\omega L) I(z) \quad \text{Eq. (1)}$$

$$-\frac{dI(z)}{dz} = (G + j\omega C) V(z) \quad \text{Eq. (2)}$$

To derive a wave equation in terms of $I(z)$, first differentiate Eq. (2) with respect to z.

$$-\frac{d^2I(z)}{dz^2} = (G + j\omega C)\frac{dV(z)}{dz} \quad \text{Eq. (3)}$$

Second, substitute Eq. (1) into the right-hand side of Eq. (3) to eliminate $V(z)$.

$$-\frac{d^2I(z)}{dz^2} = -(G + j\omega C)(R + j\omega L) I(z)$$

Last, rearrange the terms to arrive at the final form of the wave equation.

$$\boxed{\frac{d^2I(z)}{dz^2} - (G + j\omega C)(R + j\omega L) I(z) = 0}$$

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Propagation Constant, γ

In the wave equations, there is the common term $(G + j\omega C)(R + j\omega L)$.

Define the propagation constant γ to be

$$\gamma = \alpha + j\beta = \sqrt{\left(G + j\omega C\right)\left(R + j\omega L\right)}$$

Given this definition, the transmission line equations are written as

$$\frac{d^2V(z)}{dz^2} - \gamma^2 V(z) = 0$$

$$\frac{d^2I(z)}{dz^2} - \gamma^2 I(z) = 0$$

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Solution to the Wave Equations

If the wave equations are handed off to a mathematician, they will return with the following solutions.

$$\frac{d^2V(z)}{dz^2} - \gamma^2 V(z) = 0 \quad \rightarrow \quad V(z) = V_0^+ e^{-\gamma z} + V_0^- e^{\gamma z}$$

$$\frac{d^2I(z)}{dz^2} - \gamma^2 I(z) = 0 \quad \rightarrow \quad I(z) = I_0^+ e^{-\gamma z} + I_0^- e^{\gamma z}$$

Forward wave ($V_0^+ e^{-\gamma z}$, $I_0^+ e^{-\gamma z}$) — Backward wave ($V_0^- e^{\gamma z}$, $I_0^- e^{\gamma z}$)

Both $V(z)$ and $I(z)$ have the same differential equation so it makes sense they have the same solution.

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