Electromagnetics:
Electromagnetic Field Theory

Transmission Line Parameters

Lecture Outline

- Attenuation Coefficient α and Phase Constant β
- Characteristic Impedance, Z_0
- Special Cases of Transmission Lines
  - General transmission lines
  - Lossless lines
  - Weakly absorbing lines
  - Distortionless lines
- Why is 50 Ω a Standard Impedance?

Fundamental Vs. Intuitive Parameters

Fundamental Parameters	Intuitive Parameters
Electromagnetics $\mu,\ \varepsilon,\ \sigma$	Electromagnetics $n,\ \eta,\ \alpha,\ \beta,\ \tan\delta$
Transmission Lines $R,\ L,\ G,\ C$	Transmission Lines $Z_0,\ \alpha,\ \beta,\ \text{VSWR}$

The fundamental parameters are the most basic parameters needed to solve a transmission line problem.

However, it is difficult to be intuitive about how they affect signals on the line.

An electromagnetic analysis is needed to determine R, L, G, and C from the geometry of the transmission line.

The intuitive parameters provide intuitive insight about how signals behave on a transmission line.

They isolate specific information to a single parameter.

The intuitive parameters are calculated from R, L, G, and C .

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Attenuation Coefficient α and Phase Constant β

Derivation α and β (1 of 7)

Step 1 – Start with the expression for γ.

$$\gamma = \alpha + j\beta = \sqrt{(G + j\omega C)(R + j\omega L)}$$

Square this expression to get rid of square-root on right-hand side.

$$(\alpha + j\beta)^2 = (G + j\omega C)(R + j\omega L)$$

Expand this expression.

$$\alpha^2 + j2\alpha\beta - \beta^2 = RG + j\omega RC + j\omega LG - \omega^2 LC$$

Collect real and imaginary parts on the left-hand and right-hand sides.

$$(\alpha^2 - \beta^2) + j2\alpha\beta = (RG - \omega^2 LC) + j\omega(RC + LG)$$

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Derivation α and β (2 of 7)

Step 2 – Generate two equations by equating real and imaginary parts.

$$2\alpha\beta = \omega(RC + LG)$$

$$(\alpha^2 - \beta^2) + j2\alpha\beta = (RG - \omega^2 LC) + j\omega(RC + LG)$$

$$\alpha^2 - \beta^2 = RG - \omega^2 LC$$

There are now two equations and two unknowns.

$$2\alpha\beta = \omega(RC + LG)$$
$$\alpha^2 - \beta^2 = RG - \omega^2 LC$$

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Derivation α and β (3 of 7)

Step 3 – Derive a quadratic equation for α^2.

$$2\alpha\beta = \omega(RC+LG) \qquad \text{Eq. (1a)}$$
$$\alpha^2-\beta^2 = RG-\omega^2 LC \qquad \text{Eq. (1b)}$$

Solve Eq. (1a) for β.

$$\beta = \frac{\omega}{2\alpha}(RC+LG) \qquad \text{Eq. (2)}$$

Substitute Eq. (2) into Eq. (1b) and simplify.

$$\alpha^2-\left[\frac{\omega}{2\alpha}(RC+LG)\right]^2 = RG-\omega^2 LC$$
$$\alpha^2-\frac{\omega^2(RC+LG)^2}{4\alpha^2} = RG-\omega^2 LC$$
$$4\alpha^4-\omega^2(RC+LG)^2 = 4\alpha^2 RG-4\alpha^2\omega^2 LC$$
$$\alpha^4+\alpha^2(\omega^2 LC-RG)-\left[\frac{\omega}{2}(RC+LG)\right]^2 = 0$$

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Derivation α and β (4 of 7)

Step 4 – Solve for α^2 using the quadratic equation.

Recall the quadratic formula: $ax^2+bx+c=0 \rightarrow x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$

The equation for α is in the form of the quadratic equation where

$$\alpha^4+\alpha^2(\omega^2 LC-RG)-\left[\frac{\omega}{2}(RC+LG)\right]^2 = 0 \rightarrow \begin{aligned} a&=1\\ b&=\omega^2 LC-RG\\ c&=-\left[\frac{\omega}{2}(RC+LG)\right]^2\\ x&=\alpha^2\end{aligned}$$

The solution is

$$\alpha^2 = \frac{-(\omega^2 LC-RG)\pm\sqrt{(\omega^2 LC-RG)^2+4\left[\frac{\omega}{2}(RC+LG)\right]^2}}{2}$$
$$= \frac{RG-\omega^2 LC\pm\sqrt{(R^2+\omega^2L^2)(G^2+\omega^2C^2)}}{2}$$

See Appendix for more detail.

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Derivation α and β (5 of 7)

Step 5 – Resolve the sign of the square-root.

$$\alpha^2 = \frac{RG - \omega^2 LC \pm \sqrt{\left(R^2 + \omega^2 L^2\right)\left(G^2 + \omega^2 C^2\right)}}{2}$$

In order for this expression to always give a real value for α, the sign of the square-root must be positive.

The final expression is

$$\alpha^2 = \frac{RG - \omega^2 LC + \sqrt{\left(R^2 + \omega^2 L^2\right)\left(G^2 + \omega^2 C^2\right)}}{2}$$

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Derivation α and β (6 of 7)

Step 6 – Solve for β^2 using the expression for α^2.

Recall Eq. (1b): $\alpha^2 - \beta^2 = RG - \omega^2 LC$

$$\alpha^2 = \frac{RG - \omega^2 LC + \sqrt{\left(R^2 + \omega^2 L^2\right)\left(G^2 + \omega^2 C^2\right)}}{2}$$

Derive an equation for β^2 by substituting the expression for α^2 into Eq. (1b).

$$\frac{RG - \omega^2 LC + \sqrt{\left(R^2 + \omega^2 L^2\right)\left(G^2 + \omega^2 C^2\right)}}{2} - \beta^2 = RG - \omega^2 LC$$

$$\downarrow$$

$$\beta^2 = -\frac{RG - \omega^2 LC - \sqrt{\left(R^2 + \omega^2 L^2\right)\left(G^2 + \omega^2 C^2\right)}}{2}$$

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Derivation α and β (7 of 7)

Step 7 – The final expressions for α and β are derived in terms of the fundamental parameters R, L, G, and C by taking the square-root of the latest expressions for α^2 and β^2.

$$\alpha = \pm\sqrt{\frac{\left(RG-\omega^2 LC\right)+\sqrt{\left(R^2+\omega^2 L^2\right)\left(G^2+\omega^2 C^2\right)}}{2}}$$

$$\beta = \pm\sqrt{\frac{-\left(RG-\omega^2 LC\right)+\sqrt{\left(R^2+\omega^2 L^2\right)\left(G^2+\omega^2 C^2\right)}}{2}}$$

Both α and β must be positive quantities for passive materials.
This means the positive sign is taken for the square-roots.

$$\boxed{\alpha = \sqrt{\frac{\left(RG-\omega^2 LC\right)+\sqrt{\left(R^2+\omega^2 L^2\right)\left(G^2+\omega^2 C^2\right)}}{2}}}$$

$$\boxed{\beta = \sqrt{\frac{-\left(RG-\omega^2 LC\right)+\sqrt{\left(R^2+\omega^2 L^2\right)\left(G^2+\omega^2 C^2\right)}}{2}}}$$

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Characteristic Impedance Z_0

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Characteristic Impedance, Z_0 (Ω)

The characteristic impedance Z_0 of a transmission line is defined as the ratio of the voltage to the current at any point of a forward travelling wave.

$$Z_0 = \frac{V_0^+}{I_0^+} = -\frac{V_0^-}{I_0^-}$$

Definition for a forward travelling wave.

Definition for a backward travelling wave. Notice the negative sign!

Most characteristic impedance values fall in the 50 Ω to 100 Ω range. The specific value of impedance is not usually of importance. What is important is when the impedance changes because this causes reflections, standing waves, and more.

Derivation of Z_0 (1 of 5)

Step 1 – Substitute the solutions back into the transmission line equations.

$$V(z) = V_0^+ e^{-\gamma z} + V_0^- e^{\gamma z}$$

$$I(z) = I_0^+ e^{-\gamma z} + I_0^- e^{\gamma z}$$

$$-\frac{dV(z)}{dz} = (R + j\omega L)I(z) \qquad -\frac{dI(z)}{dz} = (G + j\omega C)V(z)$$

$$\Downarrow \qquad \Downarrow$$

$$-\frac{d}{dz}\left(V_0^+ e^{-\gamma z} + V_0^- e^{\gamma z}\right) = (R + j\omega L)\left(I_0^+ e^{-\gamma z} + I_0^- e^{\gamma z}\right)$$

$$-\frac{d}{dz}\left(I_0^+ e^{-\gamma z} + I_0^- e^{\gamma z}\right) = (G + j\omega C)\left(V_0^+ e^{-\gamma z} + V_0^- e^{\gamma z}\right)$$

Derivation of Z_0 (2 of 5)

Step 2 – Expand the equations and calculate the derivatives.

$$-\frac{d}{dz}\left(V_0^+e^{-\gamma z}+V_0^-e^{\gamma z}\right)=\left(R+j\omega L\right)\left(I_0^+e^{-\gamma z}+I_0^-e^{\gamma z}\right)$$

$$\Downarrow$$

$$\gamma V_0^+e^{-\gamma z}-\gamma V_0^-e^{\gamma z}=\left(R+j\omega L\right)I_0^+e^{-\gamma z}+\left(R+j\omega L\right)I_0^-e^{\gamma z}$$

$$-\frac{d}{dz}\left(I_0^+e^{-\gamma z}+I_0^-e^{\gamma z}\right)=\left(G+j\omega C\right)\left(V_0^+e^{-\gamma z}+V_0^-e^{\gamma z}\right)$$

$$\Downarrow$$

$$\gamma I_0^+e^{-\gamma z}-\gamma I_0^-e^{\gamma z}=\left(G+j\omega C\right)V_0^+e^{-\gamma z}+\left(G+j\omega C\right)V_0^-e^{\gamma z}$$

Derivation of Z_0 (3 of 5)

Step 3 – Equate the expressions multiplying the common exponential terms.

$$\gamma V_0^+=\left(R+j\omega L\right)I_0^+$$

$$\overbrace{\gamma V_0^+}e^{-\gamma z}-\underbrace{\gamma V_0^-}e^{\gamma z}=\overbrace{\left(R+j\omega L\right)I_0^+}e^{-\gamma z}+\underbrace{\left(R+j\omega L\right)I_0^-}e^{\gamma z}$$

$$-\gamma V_0^-=\left(R+j\omega L\right)I_0^-$$

$$\gamma I_0^+=\left(G+j\omega C\right)V_0^+$$

$$\overbrace{\gamma I_0^+}e^{-\gamma z}-\underbrace{\gamma I_0^-}e^{\gamma z}=\overbrace{\left(G+j\omega C\right)V_0^+}e^{-\gamma z}+\underbrace{\left(G+j\omega C\right)V_0^-}e^{\gamma z}$$

$$-\gamma I_0^-=\left(G+j\omega C\right)V_0^-$$

Derivation of Z_0 (4 of 5)

Step 4 – Solve each of our four equations for V_0/I_0 to derive expressions for Z_0.

$$\gamma V_0^+ = (R + j\omega L) I_0^+ \quad \rightarrow \quad \frac{V_0^+}{I_0^+} = \frac{R + j\omega L}{\gamma} = Z_0$$

$$-\gamma V_0^- = (R + j\omega L) I_0^- \quad \rightarrow \quad -\frac{V_0^-}{I_0^-} = \frac{R + j\omega L}{\gamma} = Z_0$$

$$\gamma I_0^+ = (G + j\omega C) V_0^+ \quad \rightarrow \quad \frac{V_0^+}{I_0^+} = \frac{\gamma}{G + j\omega C} = Z_0$$

$$-\gamma I_0^- = (G + j\omega C) V_0^- \quad \rightarrow \quad -\frac{V_0^-}{I_0^-} = \frac{\gamma}{G + j\omega C} = Z_0$$

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Derivation of Z_0 (5 of 5)

Step 5 – Put Z_0 in terms of just R, L, G, and C.

Recall our expression for γ: $\gamma = \alpha + j\beta = \sqrt{(G + j\omega C)(R + j\omega L)}$

We can substitute this into either of our expressions for Z_0.

$$Z_0 = \frac{R + j\omega L}{\gamma} = \frac{\gamma}{G + j\omega C}$$

Proceed with the first expression.

$$Z_0 = \frac{R + j\omega L}{\gamma} = \frac{R + j\omega L}{\sqrt{(G + j\omega C)(R + j\omega L)}}$$

$$= \sqrt{\frac{(R + j\omega L)^2}{(G + j\omega C)(R + j\omega L)}}$$

$$= \sqrt{\frac{R + j\omega L}{G + j\omega C}}$$

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Final Expression for Z_0 (Ω)

We have derived a general expression for the characteristic impedance Z_0 of a transmission line in terms of the fundamental parameters R, L, G, and C.

Definition: $$Z_0 = \frac{V_0^+}{I_0^+} = -\frac{V_0^-}{I_0^-}$$

Expressions: $$Z_0 = \frac{R + j\omega L}{\gamma} = \frac{\gamma}{G + j\omega C} = \sqrt{\frac{R + j\omega L}{G + j\omega C}}$$

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Dissecting the Characteristic Impedance, Z_0

The characteristic impedance describes the amplitude and phase relation between voltage and current along a transmission line. With this picture in mind, the characteristic impedance can be written as

$$Z_0 = |Z_0| \measuredangle \theta_{Z_0}$$

$$V(z) = V_0^+ e^{-\gamma z}$$

$$I(z) = I_0^+ e^{-\gamma z} = \frac{V_0^+}{Z_0} e^{-\gamma z} = \frac{V_0^+}{|Z_0|} e^{-\gamma z} e^{j\measuredangle\theta_{Z_0}}$$

The characteristic impedance can also be written in terms of its real and imaginary parts.

$$Z_0 = R_0 + jX_0$$

Reactive part of Z_0. This is not equal to $j\omega L$ or $1/j\omega C$.

Resistive part of Z_0. This is not equal to R or G.

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Special Cases of Transmission Lines:

General Transmission Line

Parameters for General TLs

Propagation Constant, γ

$$\gamma = \alpha + j\beta = \sqrt{(G + j\omega C)(R + j\omega L)}$$

Attenuation Coefficient, α

$$\alpha = \sqrt{\frac{(RG - \omega^2 LC) + \sqrt{(R^2 + \omega^2 L^2)(G^2 + \omega^2 C^2)}}{2}}$$

Phase Constant, β

$$\beta = \sqrt{\frac{-(RG - \omega^2 LC) + \sqrt{(R^2 + \omega^2 L^2)(G^2 + \omega^2 C^2)}}{2}}$$

Characteristic Impedance, Z_0

$$Z_0 = R_0 + jX_0 = \sqrt{\frac{R + j\omega L}{G + j\omega C}}$$

Special Cases of Transmission Lines:

Lossless Lines

Definition of Lossless TL

When we think about transmission lines, we tend to think of the special case of the lossless line because the equations simplify considerably.

For a transmission line to be lossless, it must have

$$R = G = 0$$

Parameters for Lossless TLs

Propagation Constant, γ

$$\gamma = \alpha + j\beta = j\omega\sqrt{LC}$$

Attenuation Coefficient, α

$$\alpha = 0$$

Phase Constant, β

$$\beta = \omega\sqrt{LC}$$

Characteristic Impedance, Z_0

$$Z_0 = R_0 + jX_0 = \sqrt{\frac{L}{C}}$$

$$R_0 = \sqrt{\frac{L}{C}} \qquad X_0 = 0$$

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Special Cases of Transmission Lines:

Weakly Absorbing Line

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Definition of Weakly Absorbing TL

Most practical transmission lines have loss, but very low loss making them weakly absorbing.

We will define a weakly absorbing line as

$$R \leq \omega L \quad \text{and} \quad G \ll \omega C$$

Ensures very little conduction between the lines through the dielectric.

Ensures low ohmic loss for signals propagating through the line.

Parameters for Weakly Absorbing TLs

Attenuation Coefficient, α

$$\alpha \approx \frac{1}{2}\left(\frac{R}{Z_0} + GZ_0\right)$$

Conductance through the dielectric dominates attenuation in high-impedance transmission lines.

Resistivity in the conductors dominates attenuation in low-impedance transmission lines.

In weakly absorbing transmission lines, there usually exists a “sweet spot” for the impedance where attenuation is minimized.

Special Cases of Transmission Lines:

Distortionless Lines

Definition of Distortionless TL

In a real transmission line, different frequencies will be attenuated differently because α is a function of ω. This causes distortion in the signals carried by the line.

$$\alpha(\omega)=\sqrt{\frac{\left(RG-\omega^2 LC\right)+\sqrt{\left(R^2+\omega^2 L^2\right)\left(G^2+\omega^2 C^2\right)}}{2}}$$

To be distortionless, there must be a choice of R, L, G, and C that eliminates ω from the expression of α, effectively making α independent of frequency ω.

The necessary condition to be distortionless is

$$\frac{R}{L}=\frac{G}{C} \quad \text{or} \quad RC=LG$$

Parameters for Distortionless TLs

Propagation Constant, γ

$$\gamma = \alpha + j\beta = \sqrt{RG} + j\omega\sqrt{LC}$$

Attenuation Coefficient, α

$$\alpha = \sqrt{RG}$$

Phase Constant, β

$$\beta = \omega\sqrt{LC}$$

To be distortionless, we must have $\beta \propto \omega$. β is a measure of how quickly a signal accumulates phase. Different frequencies have different wavelengths and therefore must accumulate different phase through the same length of line.

Characteristic Impedance, Z_0

$$Z_0 = R_0 + jX_0 = \sqrt{\frac{R}{G}} = \sqrt{\frac{L}{C}}$$

$$R_0 = \sqrt{\frac{R}{G}} = \sqrt{\frac{L}{C}} \qquad X_0 = 0$$

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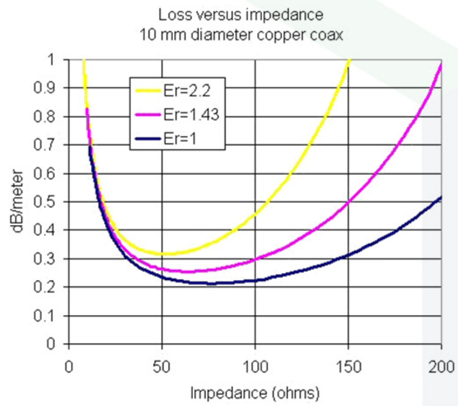
Why 50 Ω?

Cable Loss Vs. Characteristic Impedance

As we adjust the cable dimensions (i.e. b/a), we change both its impedance and its loss characteristics. This let's us plot the cable loss vs. characteristic impedance for a coax with different dielectric fills.



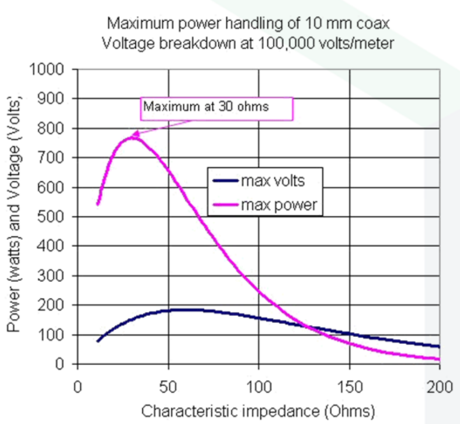
For the air-filled coax, we observe minimum loss at around 77 Ω, where $b/a \approx 3.5$.

A coaxial cable filled with polyethelene ($\varepsilon_r = 2.2$), the minimum loss occurs at 51.2 Ω ($b/a = 3.6$).

https://www.microwaves101.com/encyclopedias/why-fifty-ohms

Power Handling Vs. Characteristic Impedance

As we adjust the cable dimensions (i.e. b/a), we affect the peak voltage handling capability (breakdown) and its power handling capability (heat).



We observe the lowest peak voltage at just over 50 Ω which we interpret as the point of best voltage handling capability.

We observe the lowest peak current at around 30 Ω which we interpret as the point of best power handling capability.

https://www.microwaves101.com/encyclopedias/why-fifty-ohms

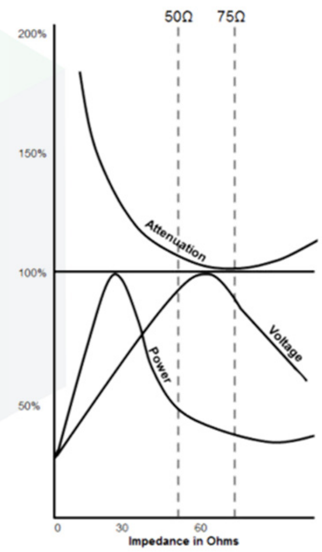
Why 50 Ω Impedance is Best?

Two researchers, Lloyd Espenscheid and Herman Affel, working at Bell Labs produced this graph in 1929. They needed to send 4 MHz signals hundreds of miles. Transmission lines capable of handling high voltage and high power were needed in order to accomplish this.

The data shown at right was generated for an air-filled coaxial cable.

Best for High Voltage: $Z_0 = 60\ \Omega$
Best for High Power: $Z_0 = 30\ \Omega$
Best for Low Attenuation: $Z_0 = 75\ \Omega$

50 Ω seemed like the best compromise.



Why 75 Ω Impedance Standard for Coax?

Nobody really knows!!

The ideal impedance is closer to 50 Ω, however this requires a thicker center conductor. Maybe 75 Ω is a compromise between low loss and mechanical flexibility?

Appendix

Simplification of α^2

$$\alpha^2 = \frac{-\left(\omega^2 LC - RG\right) \pm \sqrt{\left(\omega^2 LC - RG\right)^2 + 4\left[\frac{\omega}{2}\left(RC + LG\right)\right]^2}}{2}$$

Direct application of quadratic formula

$$= \frac{RG - \omega^2 LC \pm \sqrt{\omega^4 L^2 C^2 - 2\omega^2 RCLG + R^2 G^2 + \omega^2 R^2 C^2 + 2\omega^2 RCLG + \omega^2 L^2 G^2}}{2}$$

Expand all terms

$$= \frac{RG - \omega^2 LC \pm \sqrt{R^2 G^2 + \omega^2 R^2 C^2 + \omega^2 L^2 G^2 + \omega^4 L^2 C^2}}{2}$$

Collect terms

$$= \frac{RG - \omega^2 LC \pm \sqrt{\left(R^2 + \omega^2 L^2\right)\left(G^2 + \omega^2 C^2\right)}}{2}$$

Factor terms

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