Electromagnetics:
Electromagnetic Field Theory

Wave Examples

1

Lecture Outline

- Example #1 – Frequency and wavelength from $\vec{k}$
- Example #2 – Angle of linear polarization
- Example #3 – Speed of a wave
- Example #4 – Quick identification of polarization type
- Example #5 – World's hardest wave dissection

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2

Example #1

Determining frequency and wavelength from $\vec{k}$

Example #1

A wave propagating through free space has the following wave vector. What is the frequency and wavelength of the wave?

$$\vec{k} = 25.1327\hat{x} + 25.1327\hat{y} - 35.5431\hat{z}\ (\text{m}^{-1})$$

Solution

The wavelength is determined from the magnitude of the wave vector.

$$\left|\vec{k}\right| = \frac{2\pi}{\lambda}$$

Since the wave is propagating in free space, the wavelength above must be the free space wavelength.

$$\left|\vec{k}\right| = \frac{2\pi}{\lambda_0} \quad \rightarrow \lambda_0 = \frac{2\pi}{\left|\vec{k}\right|} = \frac{2\pi}{\sqrt{25.1327^2 + 25.1327^2 + 35.5431^2}} = \boxed{0.125\ \text{m}}$$

Example #1 cont'd

A wave propagating through free space has the following wave vector. What is the frequency and wavelength of the wave?

$$\vec{k} = 25.1327\hat{x} + 25.1327\hat{y} - 35.5431\hat{z}\ (\mathrm{m}^{-1})$$

<u>Solution cont'd</u>

The frequency is determined from the free space wavelength.

$$c_0 = f\lambda_0 \quad \rightarrow f = \frac{c_0}{\lambda_0} = \frac{3\times10^8\ \mathrm{m/s}}{0.125\ \mathrm{m}} = 2.4\times10^9\ \mathrm{Hz} = \boxed{2.4\ \mathrm{GHz}}$$

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Slide 5

5

Example #2

Angle of linear polarization

Slide 6

6

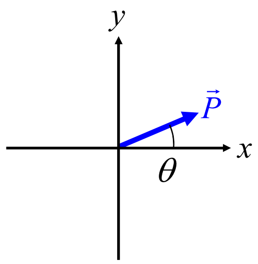
Example #2

A wave has the following polarization vector for a linearly polarized wave. Calculate the angle θ the electric field is from the x-axis.

$$\vec{P} = 1.23\hat{a}_x + 0.68\hat{a}_y$$

Solution

A rough sketch of the electric field in the xy plane is



The angle θ is then

$$\tan\theta = \frac{P_y}{P_x} \quad \rightarrow \theta = \tan^{-1}\left(\frac{P_y}{P_x}\right)$$

$$\rightarrow \theta = \tan^{-1}\left(\frac{0.68}{1.23}\right) = \boxed{28.9^\circ}$$

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Slide 7

7

Example #3

Speed of a wave

Slide 8

8

Example #3

Two waves at 5.0 GHz are propagating through two different media. Given the wave expressions below, determine which is propagating the fastest.

$$\vec{E}_1 = \left(5.4\hat{a} - 3.9\hat{b}\right)e^{-j209.582z} \qquad \vec{E}_2 = \left(10.2\hat{a} + 0.8\hat{b}\right)e^{-j419.170z}$$

<u>Solution</u>

The speed of the waves can be assessed through the refractive index of the mediums. The refractive indices can be determined from the magnitude of the wave vectors since frequency is known.

$$\left|\vec{k}\right| = \frac{2\pi n}{\lambda_0} \rightarrow n = \frac{\lambda_0\left|\vec{k}\right|}{2\pi}$$

The free space wavelength is determined from the frequency.

$$c_0 = f\lambda_0 \rightarrow \lambda_0 = \frac{c_0}{f} = \frac{3\times10^8 \text{ m/s}}{5.0\times10^9 \text{ s}^{-1}} = \boxed{0.06 \text{ m}}$$

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Slide 9

9

Example #3 cont'd

Two waves at 5.0 GHz are propagating through two different media. Given the wave expressions below, determine which is propagating the fastest.

$$\vec{E}_1 = \left(5.4\hat{a} - 3.9\hat{b}\right)e^{-j209.582z} \qquad \vec{E}_2 = \left(10.2\hat{a} + 0.8\hat{b}\right)e^{-j419.170z}$$

<u>Solution cont'd</u>

The refractive indices are

$$n = \frac{\lambda_0\left|\vec{k}\right|}{2\pi} \rightarrow \begin{aligned} n_1 &= \frac{\lambda_0}{2\pi}\left|\vec{k}_1\right| = \frac{(0.06 \text{ m})}{2\pi}\cdot 209.582 = 2.0 \\ n_2 &= \frac{\lambda_0}{2\pi}\left|\vec{k}_2\right| = \frac{(0.06 \text{ m})}{2\pi}\cdot 419.170 = 4.0 \end{aligned}$$

The <u>first wave</u> is propagating at a higher speed because refractive index is the smallest.

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Slide 10

10

Example #4

Quick Identification of Polarization Type

Example #4 – Polarization

Determine the polarization of the following waves:

$\vec{E}(\vec{r},t) = 3\cos(3t - 4z)\hat{a}_x$	LP
$\vec{E}(\vec{r},t) = 2\sin(5\times10^6 t - 289x)\hat{a}_y - 10\sin(5\times10^6 t - 289x)\hat{a}_z$	LP
$\vec{E}(\vec{r},t) = 2\sin(5\times10^6 t - 289x)\hat{a}_y + 10\cos(5\times10^6 t - 289x)\hat{a}_z$	EP
$\vec{E}(\vec{r},t) = 2\sin(5\times10^6 t - 289x)\hat{a}_y + 2\cos(5\times10^6 t - 289x)\hat{a}_z$	LCP
$\vec{E}(\vec{r}) = 7\hat{a}_y e^{-jkz}$	LP
$\vec{E}(\vec{r}) = (8\hat{a}_x + 3j\hat{a}_y)e^{-jkz}$	EP
$\vec{E}(\vec{r}) = (3\hat{a}_x + 3j\hat{a}_y)e^{-jkz}$	RCP

Example #4

World's Hardest Wave Dissection

Example #5 – Dissect a Wave (1 of 9)

The electric field component of a 5.6 GHz plane wave is given by:

$$\begin{aligned}\vec{E}(\vec{r}) = &\hat{a}_x (0.4915 + j0.8550) e^{-j573.0795x} e^{-j330.8676y} e^{-j240.8519z} \\ &+\hat{a}_y (-1.4224 - j0.4702) e^{-j573.0795x} e^{-j330.8676y} e^{-j240.8519z} \\ &+\hat{a}_z (0.7844 - j1.3885) e^{-j573.0795x} e^{-j330.8676y} e^{-j240.8519z}\end{aligned}$$

1. Determine the wave vector.
2. Determine the wavelength inside of the medium.
3. Determine the free space wavelength.
4. Determine refractive index of the medium.
5. Determine the dielectric constant of the medium.
6. Determine the polarization of the wave.
7. Determine the magnitude of the wave.

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Example #5 – Dissect a Wave (2 of 9)

Solution – Part 1 – Determine Wave Vector

The standard form for a plane wave is

$$\vec{E}(\vec{r}) = \vec{P}e^{-j\vec{k}\bullet\vec{r}}$$

Comparing this to the expression for the electric field shows that

$$\vec{P} = \hat{a}_x(0.4915 + j0.8550) + \hat{a}_y(-1.4224 - j0.4702) + \hat{a}_z(0.7844 - j1.3885)$$

$$e^{-j\vec{k}\bullet\vec{r}} = e^{-j573.0795x}e^{-j330.8676y}e^{-j240.8519z}$$

The polarization vector $\vec{P}$ will be used later. The wave vector $\vec{k}$ is determined from the second expression above to be

$$e^{-j\vec{k}\bullet\vec{r}} = e^{-jk_x x}e^{-jk_y y}e^{-jk_z z} = e^{-j573.0795x}e^{-j330.8676y}e^{-j240.8519z}$$

$$\boxed{\vec{k} = 573.0795\hat{a}_x + 330.8676\hat{a}_y + 240.8519\hat{a}_z \ \text{m}^{-1}}$$

Example #5 – Dissect a Wave (3 of 9)

Solution – Part 2 – Wavelength inside the medium

The wavelength inside the medium is related to the magnitude of the wave vector through

$$|\vec{k}| = \frac{2\pi}{\lambda} \quad \rightarrow \quad \lambda = \frac{2\pi}{|\vec{k}|}$$

The magnitude of the wave vector is

$$\begin{aligned} |\vec{k}| &= \sqrt{k_x^2 + k_y^2 + k_z^2} \\ &= \sqrt{(573.0795\ \text{m}^{-1})^2 + (330.8676\ \text{m}^{-1})^2 + (240.8519\ \text{m}^{-1})^2} \\ &= 704.239\ \text{m}^{-1} \end{aligned}$$

The wavelength is therefore

$$\lambda = \frac{2\pi}{704.239\ \text{m}^{-1}} = \boxed{8.9224\ \text{cm}}$$

Example #5 – Dissect a Wave (4 of 9)

Solution – Part 3 – Free space wavelength

The free space wavelength is

$$c_0 = f\lambda_0 \quad \rightarrow \quad \lambda_0 = \frac{c_0}{f} = \frac{3\times10^8 \text{ m/s}}{5.6\times10^9 \text{ s}^{-1}} = \boxed{53.5344 \text{ cm}}$$

Solution – Part 4 – Refractive index

It follows that the refractive index of the medium is

$$\lambda = \frac{\lambda_0}{n} \quad \rightarrow \quad n = \frac{\lambda_0}{\lambda} = \frac{53.5344 \text{ cm}}{8.9224 \text{ cm}} = \boxed{6.0}$$

Alternatively, we could determine the refractive index through $|\vec{k}|$

$$|\vec{k}| = k_0 n \quad \rightarrow \quad n = \frac{|\vec{k}|}{k_0} = \frac{|\vec{k}|}{\omega/c_0} = \frac{c_0|\vec{k}|}{2\pi f} = \frac{(3\times10^8 \text{ m/s})(704.239 \text{ m}^{-1})}{2\pi(5.6\times10^9 \text{ s}^{-1})} = \boxed{6.0}$$

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Example #5 – Dissect a Wave (5 of 9)

Solution – Part 5 – Dielectric constant

Assuming the medium has no magnetic response,

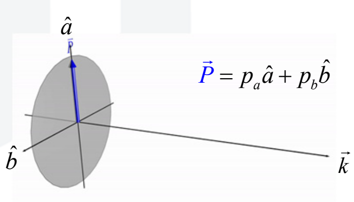
$$n = \sqrt{\varepsilon_r} \quad \rightarrow \quad \varepsilon_r = n^2 = (6.0)^2 = \boxed{36}$$

Solution – Part 6 – Wave Polarization

To determine the polarization, the electric field is written in the form that makes polarization explicit.

$$\vec{E}(\vec{r}) = \left(E_a\hat{a} + E_b e^{j\delta}\hat{b}\right)e^{j\theta}e^{-j\vec{k}\bullet\vec{r}}$$

The choice for $\hat{a}$ and $\hat{b}$ is arbitrary, but they must both be perpendicular to $\vec{k}$.



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Example #5 – Dissect a Wave (6 of 9)

Solution – Part 6 – Wave polarization (cont'd)

We determine a valid choice for $\hat{a}$ by first picking any vector that is not in the same direction as $\vec{k}$

$$\vec{v} = 1\hat{a}_x + 2\hat{a}_y + 3\hat{a}_z$$

The cross product will give us a vector perpendicular to $\vec{k}$

$$\hat{a} = \frac{\vec{k} \times \vec{v}}{\left|\vec{k} \times \vec{v}\right|} = 0.2896\hat{a}_x - 0.8381\hat{a}_y + 0.4622\hat{a}_z$$

We determine a valid choice for $\hat{b}$ using the cross product so that it is perpendicular to both $\hat{a}$ and $\vec{k}$

$$\hat{b} = \frac{\vec{k} \times \hat{a}}{\left|\vec{k} \times \hat{a}\right|} = 0.5038\hat{a}_x - 0.2771\hat{a}_y - 0.8182\hat{a}_z$$

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Slide 19

19

Example #5 – Dissect a Wave (7 of 9)

Solution – Part 6 – Wave polarization (cont'd)

To determine the component of the polarization vector $\vec{P}$ in the $\hat{a}$ and $\hat{b}$ directions, use the dot product.

$$p_a = \vec{P} \bullet \hat{a} = 1.6971 \text{ V/m}$$

$$p_b = \vec{P} \bullet \hat{b} = -j1.6971 \text{ V/m}$$

We can now write E_a and E_b from p_a and p_b by incorporating the phase difference into the parameter δ. By inspection of the above, we get

$$E_a = 1.6971 \text{ V/m}$$

$$E_b = 1.6971 \text{ V/m}$$

$$\delta = -90°$$

The common phase between p_a and p_b is simply 0°.

$$\theta = 0°$$

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Slide 20

20

Example #5 – Dissect a Wave (8 of 9)

Solution – Part 6 – Wave polarization (cont'd)

Finally, we have

$$\vec{E}(\vec{r}) = \left(E_a\hat{a} + E_b e^{j\delta}\hat{b}\right)e^{j\theta}e^{-j\vec{k}\bullet\vec{r}}$$

$$E_a = 1.6971 \text{ V/m}$$
$$E_b = 1.6971 \text{ V/m}$$
$$\delta = -90°$$
$$\theta = 0°$$
$$\vec{k} = 573.0795\hat{a}_x + 330.8676\hat{a}_y + 240.8519\hat{a}_z \text{ m}^{-1}$$

From this, we determine that we have circular polarization (CP) because $E_a = E_b$ and $\delta = \pm 90°$.

More specifically, this is left-hand circular polarization (LCP) because δ = -90°.

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Slide 21

21

Example #5 – Dissect a Wave (9 of 9)

Solution – Part 7 – Magnitude of electric field

The magnitude of the wave is simply the magnitude of the polarization vector

$$\begin{aligned}\left|\vec{E}(\vec{r})\right| &= \left|\vec{P}\right| \\ &= \sqrt{E_a^2 + E_b^2} \\ &= \left|1.6971 \text{ V/m}\right|^2 + \left|1.6971 \text{ V/m}\right|^2 \\ &= \boxed{2.4 \text{ V/m}}\end{aligned}$$

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Slide 22

22