Electromagnetics: Electromagnetic Field Theory

Electromagnetic Wave
Polarization

Lecture Outline

• Definition of Electromagnetic Wave Polarization
• Orthogonality & Handedness
• Polarization Classification: Linear, Circular & Elliptical
• Poincaré Sphere
• Polarization Explicit Form
• Dissection of a Plane Wave
What is Polarization?

Polarization is that property of an electromagnetic wave which describes the time-varying direction and relative magnitude of the electric field vector.

\[ \vec{E}(\vec{r}) = P e^{-j\vec{k} \cdot \vec{r}} \]

Polarization Vector
Why is Polarization Important?

• Different polarizations can behave differently in a device
• Orthogonal polarizations will not interfere with each other
• Polarization becomes critical when analyzing devices on the scale of a wavelength
• Focusing properties of lenses are different
• Reflection/transmission can be different
• Frequency of resonators can be different
• Cutoff conditions for filters, waveguides, etc.

Orthogonality & Handedness
Plane Waves in LHI Media are TEM

In linear, homogeneous and isotropic (LHI) media, the electric field $\vec{E}$, magnetic field $\vec{H}$, and direction of the wave $\vec{k}$ are all perpendicular to each other. These are called transverse electromagnetic (TEM) waves.

$\vec{E} \perp \vec{k} \perp \vec{H}$

Further $\vec{E}$, $\vec{H}$, and $\vec{k}$ follow a right-hand rule.

Proof That $\vec{E} \perp \vec{k}$

Start with Gauss' law.
$$\nabla \cdot \vec{D} = 0$$

Put this in terms of the electric field intensity $\vec{E}$.
$$\nabla \cdot (\varepsilon \vec{E}) = 0 \quad \rightarrow \quad \varepsilon \left( \nabla \cdot \vec{E} \right) = 0 \quad \rightarrow \quad \nabla \cdot \vec{E} = 0$$

Substitute in expression for a plane wave.
$$\nabla \cdot \left( \vec{P} e^{-j\vec{k} \cdot \vec{r}} \right) = 0 \quad \rightarrow \quad -j\vec{k} \cdot \vec{P} e^{-j\vec{k} \cdot \vec{r}} = 0 \quad \rightarrow \quad \vec{k} \cdot \vec{P} = 0$$

According to the dot-product test, if $\vec{k} \cdot \vec{P} = 0$ then $\vec{k} \perp \vec{P}$.

$\vec{E} \perp \vec{k}$
Proof That $\vec{H} \perp \vec{k}$

Start with Gauss’ law for magnetic fields.

\[
\nabla \cdot \vec{B} = 0
\]

Put this in terms of the magnetic field intensity $\vec{H}$.

\[
\nabla \cdot (\mu \vec{H}) = 0 \rightarrow \mu (\nabla \cdot \vec{H}) = 0 \rightarrow \nabla \cdot \vec{H} = 0
\]

Substitute in expression for a plane wave.

\[
\nabla \cdot (\vec{H}_0 e^{-j\vec{k} \cdot \vec{r}}) = 0 \rightarrow -j\vec{k} \cdot \vec{H}_0 e^{-j\vec{k} \cdot \vec{r}} = 0 \rightarrow \vec{k} \cdot \vec{H}_0 = 0
\]

According to the dot-product test, if $\vec{k} \cdot \vec{H}_0 = 0$ then $\vec{k} \perp \vec{H}_0$.

$\vec{H} \perp \vec{k}$

Proof That $\vec{E} \perp \vec{H}$

Start with Faraday’s law.

\[
\nabla \times \vec{E} = -j\omega \mu \vec{H}
\]

Substitute in expression for a plane wave.

\[
\nabla \times (\vec{E}_0 e^{-j\vec{k} \cdot \vec{r}}) = -j\omega \mu (\vec{H}_0 e^{-j\vec{k} \cdot \vec{r}})
\]

\[
(-j\vec{k}) \times (\vec{E}_0 e^{-j\vec{k} \cdot \vec{r}}) = -j\omega \mu (\vec{H}_0 e^{-j\vec{k} \cdot \vec{r}})
\]

\[
\vec{k} \times \vec{E}_0 = \omega \mu \vec{H}_0
\]

\[
\vec{H}_0 = \frac{\vec{k} \times \vec{E}_0}{\omega \mu}
\]

According to the properties of cross products, $\vec{H}$ will be perpendicular to both $\vec{k}$ and $\vec{E}$. 
Polarization Classification: 

*Linear, Circular & Elliptical*

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Linear Polarization (LP)

An electromagnetic wave has linear polarization if the electric field oscillation is confined to a single plane.

\[
\vec{E} = (E_0 \hat{a}_x) e^{-jkz}
\]
Linear Polarization (LP)

An electromagnetic wave has linear polarization if the electric field oscillation is confined to a single plane.

\[ \vec{E} = (E_0 \hat{a}_y) e^{-j kz} \]

Linear Polarization (LP)

An electromagnetic wave has linear polarization if the electric field oscillation is confined to a single plane.

\[ \vec{E} = E_0 \left( \cos \theta \hat{a}_x + \sin \theta \hat{a}_y \right) e^{-j kz} \]
Circular Polarization (CP)

Suppose two LP waves are brought together...

\[ \vec{E}_1 = (E_0 \hat{\alpha}_x) e^{-j kz} \]
\[ \vec{E}_2 = (E_0 \hat{\alpha}_y) e^{-j kz} \]

...and then one LP wave is phase shifted by 90°.

\[ \vec{E}_1 + \vec{E}_2 = \begin{cases} 
(E_0 \hat{\alpha}_x) e^{-j kz} + (E_0 \hat{\alpha}_y) e^{-j kz} & \text{Before phase shift (LP)} \\
(E_0 \hat{\alpha}_x) e^{-j kz} + (jE_0 \hat{\alpha}_y) e^{-j kz} & \text{After phase shift (RCP)}
\end{cases} \]
Circular Polarization (CP)

The overall electric field will appear to rotate with respect to both time and position.

\[ \vec{E}_1 + \vec{E}_2 = E_0 (\hat{a}_x + j\hat{a}_y) e^{-j kz} \]

Circular Polarization (CP)

An electromagnetic wave has circular polarization if the direction of the electric field rotates with time and has a constant magnitude.

\[ \vec{E} = E_0 (\hat{a}_x \pm j\hat{a}_y) e^{-j kz} \]
Right Circular Polarization (RCP)

An electromagnetic wave has right circular polarization if the electric field rotates clockwise when viewed from behind.

\[ \vec{E} = E_0 \left( \hat{a}_x + j\hat{a}_y \right) e^{-jkz} \]

Left Circular Polarization (LCP)

An electromagnetic wave has left circular polarization if the electric field rotates counterclockwise when viewed from behind.

\[ \vec{E} = E_0 \left( \hat{a}_x - j\hat{a}_y \right) e^{-jkz} \]
Elliptical Polarization (EP)

An electromagnetic wave has elliptical polarization if the electric field rotates with time to form an ellipse.

Continuum of LP+LP Polarizations

As the phase difference $\delta$ between the two LP waves changes, the resulting polarization...

$$\text{LP}_x + \text{LP}_y = E_0 \hat{a}_x e^{-jkz} + E_0 e^{j\delta} \hat{a}_y e^{-jkz}$$
Continuum of LP+LP Polarizations

As the phase difference $\delta$ between the two LP waves changes, the resulting polarization oscillates between RCP, LCP, LP$_{45^\circ}$, LP$_{-45^\circ}$, and EP.

$$\text{LP}_x + \text{LP}_y = E_0 \hat{a}_x e^{-jkz} + E_0 e^{j\delta} \hat{a}_y e^{-jkz}$$

$$= E_0 \left( \hat{a}_x + e^{j\delta} \hat{a}_y \right) e^{-jkz}$$

Continuum of CP+CP Polarizations

As the phase difference $\theta$ between the two CP waves changes, the resulting polarization is...

$$\text{RCP} + \text{LCP} = E_0 \left( \hat{a}_x + j\hat{a}_y \right) e^{-jkz} + E_0 \left( \hat{a}_x - j\hat{a}_y \right) e^{j\theta} e^{-jkz}$$
Continuum of CP+CP Polarizations

As the phase difference $\theta$ between the two CP waves changes, the resulting polarization is LP where the phase $\theta$ translates into a tilt angle.

$$\text{RCP} + \text{LCP} = E_0 (\hat{\lambda}_x + j\hat{\lambda}_y) e^{-jkz} + E_0 (\hat{\lambda}_x - j\hat{\lambda}_y) e^{j\theta} e^{-jkz}$$

$$= E_0 \left[ \cos \theta \hat{\lambda}_x + \sin \theta \hat{\lambda}_y \right] e^{-jkz}$$

Poincaré Sphere
The polarization of a wave can be mapped to a unique point on the Poincaré sphere. Points on opposite sides of the sphere are orthogonal.

See Balanis, Chap. 4.
Possibilities for Wave Polarization

Recall that $\vec{E} \perp \vec{k}$ so the polarization vector $\vec{P}$ must fall within the plane perpendicular to $\vec{k}$.

The polarization can be decomposed into two orthogonal directions, $\hat{a}$ and $\hat{b}$.

$\vec{P} = p_a \hat{a} + p_b \hat{b}$

Rules for Choosing $\hat{a}$ and $\hat{b}$

1. Orthogonality $\Rightarrow \hat{a} \perp \hat{b}$
2. Handedness $\Rightarrow \hat{a} \times \hat{b} = \hat{k}$
Explicit Form to Convey Polarization

The expression for the electromagnetic wave can now be written as

\[ \vec{E}(\vec{r}) = \vec{P}e^{-jk\vec{r}} = (p_a \hat{a} + p_b \hat{b})e^{-jk\vec{r}} \]

\( p_a \) and \( p_b \) are in general complex numbers in order to convey the relative phase of each of these components.

Substituting \( p_a \) and \( p_b \) into the wave expression gives

\[ \vec{E}(\vec{r}) = \left[ E_a e^{i\phi_a} \hat{a} + E_b e^{i\phi_b} \hat{b} \right] e^{-jk\vec{r}} = \left[ E_a \hat{a} + E_b e^{i(\phi_b - \phi_a)} \hat{b} \right] e^{i\theta} e^{-jk\vec{r}} \]

Interpret \( \phi_b - \phi_a \) as the phase difference between \( p_a \) and \( p_b \).
\[ \delta = \phi_b - \phi_a \]

Interpret \( \phi_a \) as the phase common to both \( p_a \) and \( p_b \).
\[ \theta = \phi_a \]

The final expression is:

\[ \vec{E}(\vec{r}) = \left( E_a \hat{a} + E_b e^{i\delta} \hat{b} \right) e^{i\theta} e^{-jk\vec{r}} \]

Determining Polarization of a Wave

To determine polarization, it is most convenient to write the expression for the wave that makes polarization explicit.

\[ \vec{E}(\vec{r}) = \left( E_a \hat{a} + E_b e^{i\delta} \hat{b} \right) e^{i\theta} e^{-jk\vec{r}} \]

Given \( E_a \), \( E_b \), and \( \delta \) the following table can be used to identify the polarization of the wave...

<table>
<thead>
<tr>
<th>Polarization Designation</th>
<th>Mathematical Definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>Linear Polarization (LP)</td>
<td>( \delta = 0^\circ )</td>
</tr>
<tr>
<td>Circular Polarization (CP)</td>
<td>( \delta = \pm 90^\circ ) and ( E_a = E_b )</td>
</tr>
<tr>
<td>Right-Hand CP (RCP)</td>
<td>( \delta = +90^\circ ) and ( E_a = E_b )</td>
</tr>
<tr>
<td>Left-Hand CP (LCP)</td>
<td>( \delta = -90^\circ ) and ( E_a = E_b )</td>
</tr>
<tr>
<td>Elliptical Polarization</td>
<td>Everything else</td>
</tr>
</tbody>
</table>
Dissection of a Plane Wave

Complex Amplitudes
\[ \hat{H}_0 = H_0 (k \times \hat{P}) \]
\[ \hat{E}_0 = E_0 \hat{P} \]

Impedance
\[ \eta = \frac{E_0}{H_0} \]

Ordinary Frequency
\[ \omega = 2\pi f \]

Free Space Wavelength
\[ c_0 = \frac{v_0}{f} \]

Direction
\[ \hat{k} = \frac{\hat{P}}{k} \]

Wavelength in Medium
\[ |k'| = \frac{2\pi}{\lambda} \]

Refractive Index
\[ n = \frac{c_0}{\lambda} = \frac{c_e}{\lambda} = \frac{n_0}{\eta_0} \]

Constitutive Parameters
\[ n = \sqrt{\frac{\mu_0}{\varepsilon_0}} \]
\[ \eta = \sqrt{\frac{\mu_0}{\varepsilon_0}} \]
\[ \varepsilon_n = n \eta_n \]
\[ \mu_n = n \eta_n \]

Speed
\[ v = f \lambda \]
Reading Off the Parameters

\[
\vec{E}(\vec{r}, t) = \vec{P} \cos(\omega t - \vec{k} \cdot \vec{r}) \text{ V/m}
\]

\[
\vec{E}(\vec{r}, t) = (3\hat{\alpha}_x - 7\hat{\alpha}_y) \cos(2.4 \times 10^9 t + 5.1x - 3.5z) \text{ V/m}
\]

- **Polarization Vector**: \( \vec{P} = 3\hat{\alpha}_x - 7\hat{\alpha}_y \text{ V m} \)
- **Angular Frequency**: \( \omega = 2.4 \times 10^9 \text{ rad/sec} \)
- **Wave Vector**
  - **Step 1** – Factor out negative sign.
    \[
    \cos[2.4 \times 10^9 t + 5.1x - 3.5z] = \cos[2.4 \times 10^9 t - (-5.1x + 3.5z)]
    \]
  - **Step 2** – Set expression in parentheses equal to \( \vec{k} \cdot \vec{r} = k_x x + k_y y + k_z z \)
    \[
    k_x + k_y + k_z = (-5.1x + 3.5z)
    \]
  - **Step 3** – Read off components of \( \vec{k} \)
    
    \[
    k_x = -5.1, k_y = 3.5 \Rightarrow \vec{k} = -5.1\hat{\alpha}_x + 3.5\hat{\alpha}_z \text{ m}^{-1}
    \]