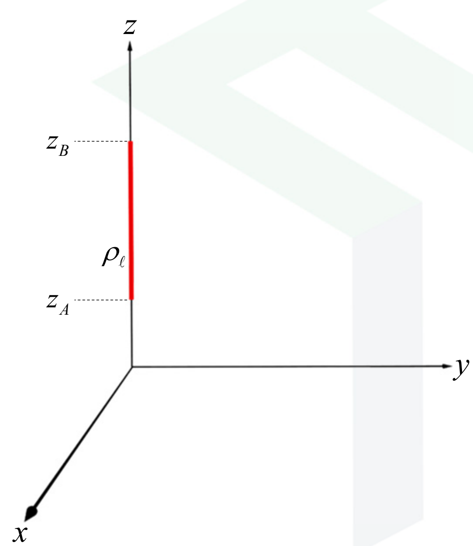
Electromagnetics:

Electromagnetic Field Theory

Example: Uniform Finite Line Charge

1

Total Charge Q_{Total}

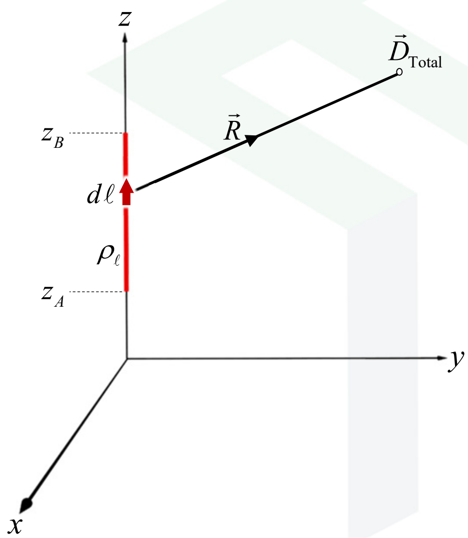


What is the total charge Q_{Total}?

1. Draw the problem.
2. Choose a coordinate system.
   Cartesian
3. Write general equation.
   $$Q_{\text{Total}} = \int_\ell \rho_\ell \, d\ell$$
4. Write expressions for each term.
   $$\rho_\ell = \rho_\ell \qquad d\ell = dz$$
5. Choose limits of integration.
   $$Q_{\text{Total}} = \int_{z_A}^{z_B} \rho_\ell \, dz$$
6. Solve the integral.
   $$Q_{\text{Total}} = \int_{z_A}^{z_B} \rho_\ell \, dz = \rho_\ell z \Big|_{z_A}^{z_B} = \rho_\ell \left(z_B - z_A \right)$$
7. Interpret the result.
   $\boxed{Q_{\text{Total}} = \rho_\ell L}$ *for uniform charge density*

2

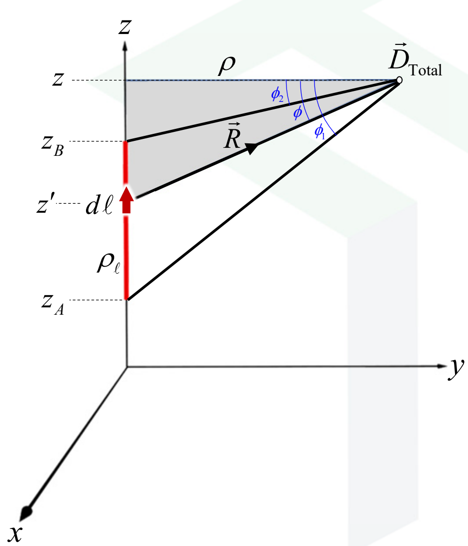
Total Field $\vec{D}_{\text{Total}}$



What is the total field $\vec{D}_{\text{Total}}$?

1. Draw the problem.
2. Choose a coordinate system.
   Cartesian
3. Write general equation.
   $$\vec{D}_{\text{Total}} = \int_\ell \frac{\rho_\ell d\ell}{4\pi R^2}\hat{a}_R$$
4. Write expressions for each term.
   $$\rho_\ell = \rho_\ell \qquad \frac{\hat{a}_R}{R^2} = \frac{\vec{R}}{\left|\vec{R}\right|^3} \qquad d\ell = dz'$$
5. Choose limits of integration.
   $$\vec{D}_{\text{Total}} = \frac{\rho_\ell}{4\pi}\int_{z_A}^{z_B} \frac{\vec{R}}{\left|\vec{R}\right|^3} dz'$$

Total Field $\vec{D}_{\text{Total}}$



What is the total field $\vec{D}_{\text{Total}}$?

6. Solve the integral.
   $$\vec{D}_{\text{Total}} = \frac{\rho_\ell}{4\pi}\int_{z_A}^{z_B} \frac{\vec{R}}{\left|\vec{R}\right|^3} dz'$$

Rather than integrate over z, integrate over the angle ϕ.

$$z_A \to \phi_1$$
$$z_B \to \phi_2$$
$$dz' \to ?$$
$$\vec{R}/\left|\vec{R}\right|^3 \to ?$$

$$\vec{D}_{\text{Total}} = \frac{\rho_\ell}{4\pi}\int_{\phi_1}^{\phi_2} ?$$

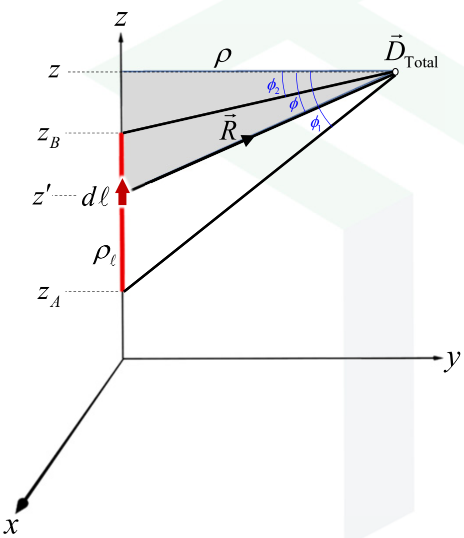
From the figure,

$$\tan\phi = \frac{z - z'}{\rho} \quad \to \quad z' = z - \rho\tan\phi$$

Differentiate to get

$$dz' = -\rho\sec^2\phi\, d\phi$$

Total Field $\vec{D}_{\text{Total}}$



What is the total field $\vec{D}_{\text{Total}}$?

6. Solve the integral cont'd…

Write an expression for $\vec{R}$

$$\begin{aligned}\vec{R} &= \rho\hat{a}_\rho + (z - z')\hat{a}_z \\ &= \rho\hat{a}_\rho + \rho\tan\phi\hat{a}_z \\ &= \frac{\rho}{\cos\phi}\left(\cos\phi\hat{a}_\rho + \sin\phi\hat{a}_z\right) \\ &= \rho\sec\phi\underbrace{\left(\cos\phi\hat{a}_\rho + \sin\phi\hat{a}_z\right)}_{\text{Magnitude is 1}}\end{aligned}$$

Magnitude is

$$\left|\vec{R}\right|^3 = \left(\rho\sec\phi\right)^3$$

Plug everything back into integral.

$$\vec{D}_{\text{Total}} = \frac{\rho_\ell}{4\pi}\int_{\phi_1}^{\phi_2}\frac{\overbrace{\rho\sec\phi\left(\cos\phi\hat{a}_\rho + \sin\phi\hat{a}_z\right)}^{\vec{R}}}{\underbrace{\left(\rho\sec\phi\right)^3}_{\left|\vec{R}\right|^3}}\overbrace{\left(-\rho\sec^2\phi\, d\phi\right)}^{dz'}$$

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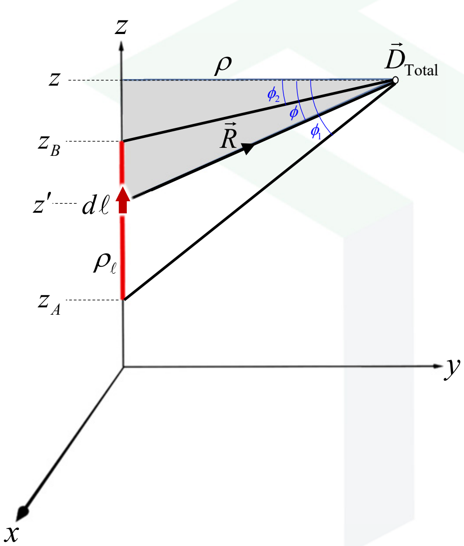
Total Field $\vec{D}_{\text{Total}}$



What is the total field $\vec{D}_{\text{Total}}$?

6. Solve the integral cont'd…

Simplify integral.

$$\vec{D}_{\text{Total}} = \frac{\rho_\ell}{4\pi}\int_{\phi_1}^{\phi_2}\frac{\cancel{\rho\sec\phi}\left(\cos\phi\hat{a}_\rho + \sin\phi\hat{a}_z\right)}{\left(\cancel{\rho\sec\phi}\right)^3}\left(-\cancel{\rho\sec^2\phi}\, d\phi\right)$$

$$= -\frac{\rho_\ell}{4\pi\rho}\int_{\phi_1}^{\phi_2}\left(\cos\phi\hat{a}_\rho + \sin\phi\hat{a}_z\right)d\phi$$

Perform integration.

$$\begin{aligned}\vec{D}_{\text{Total}} &= -\frac{\rho_\ell}{4\pi\rho}\left(\sin\phi\hat{a}_\rho - \cos\phi\hat{a}_z\right)\Big|_{\phi_1}^{\phi_2} \\ &= -\frac{\rho_\ell}{4\pi\rho}\begin{bmatrix}\left(\sin\phi_2\hat{a}_\rho - \cos\phi_2\hat{a}_z\right) \\ -\left(\sin\phi_1\hat{a}_\rho - \cos\phi_1\hat{a}_z\right)\end{bmatrix} \\ &= -\frac{\rho_\ell}{4\pi\rho}\begin{bmatrix}\left(\sin\phi_2 - \sin\phi_1\right)\hat{a}_\rho \\ +\left(\cos\phi_1 - \cos\phi_2\right)\hat{a}_z\end{bmatrix} \\ &= \frac{\rho_\ell}{4\pi\rho}\begin{bmatrix}\left(\sin\phi_1 - \sin\phi_2\right)\hat{a}_\rho \\ +\left(\cos\phi_2 - \cos\phi_1\right)\hat{a}_z\end{bmatrix}\end{aligned}$$

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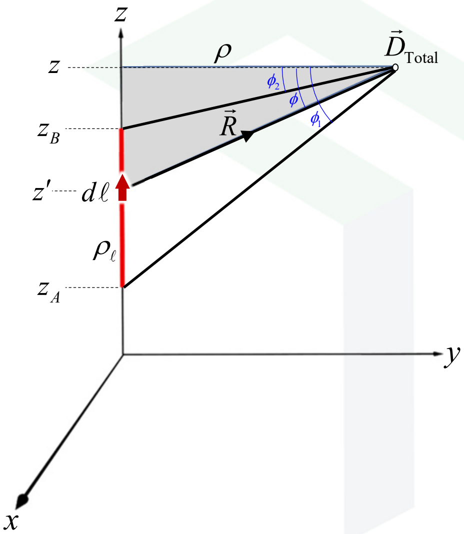
Total Field $\vec{D}_{\text{Total}}$



What is the total field $\vec{D}_{\text{Total}}$?

7. Interpret the result.

The final answer…

$$\boxed{\vec{D}_{\text{Total}} = \frac{\rho_\ell}{4\pi\rho}\left[\left(\sin\phi_1 - \sin\phi_2\right)\hat{a}_\rho + \left(\cos\phi_2 - \cos\phi_1\right)\hat{a}_z\right]}$$

$\left|\vec{D}\right|$ decays as $1/\rho$.

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