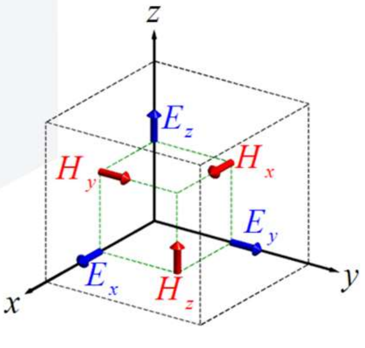
Advanced Computation:
Computational Electromagnetics

Maxwell's Equations in Matrix Form

Outline

- Finite-Difference Equations for Two-Dimensional Analysis
- Derivative Matrices for the Yee Grid
- Matrix form of Maxwell's equations



Finite-Difference Equations for Two-Dimensional Analysis

Summary of Finite-Difference Approximations of Maxwell's Equations

$$\frac{\partial E_z}{\partial y'} - \frac{\partial E_y}{\partial z'} = \mu_{xx}\tilde{H}_x$$

$$\frac{\partial E_x}{\partial z'} - \frac{\partial E_z}{\partial x'} = \mu_{yy}\tilde{H}_y$$

$$\frac{\partial E_y}{\partial x'} - \frac{\partial E_x}{\partial y'} = \mu_{zz}\tilde{H}_z$$

$$\frac{\partial \tilde{H}_z}{\partial y'} - \frac{\partial \tilde{H}_y}{\partial z'} = \varepsilon_{xx}E_x$$

$$\frac{\partial \tilde{H}_x}{\partial z'} - \frac{\partial \tilde{H}_z}{\partial x'} = \varepsilon_{yy}E_y$$

$$\frac{\partial \tilde{H}_y}{\partial x'} - \frac{\partial \tilde{H}_x}{\partial y'} = \varepsilon_{zz}E_z$$

$\Rightarrow$

$$\frac{E_z^{i,j+1,k} - E_z^{i,j,k}}{\Delta y'} - \frac{E_y^{i,j,k+1} - E_y^{i,j,k}}{\Delta z'} = \mu_{xx}^{i,j,k}\tilde{H}_x^{i,j,k}$$

$$\frac{E_x^{i,j,k+1} - E_x^{i,j,k}}{\Delta z'} - \frac{E_z^{i+1,j,k} - E_z^{i,j,k}}{\Delta x'} = \mu_{yy}^{i,j,k}\tilde{H}_y^{i,j,k}$$

$$\frac{E_y^{i+1,j,k} - E_y^{i,j,k}}{\Delta x'} - \frac{E_x^{i,j+1,k} - E_x^{i,j,k}}{\Delta y'} = \mu_{zz}^{i,j,k}\tilde{H}_z^{i,j,k}$$

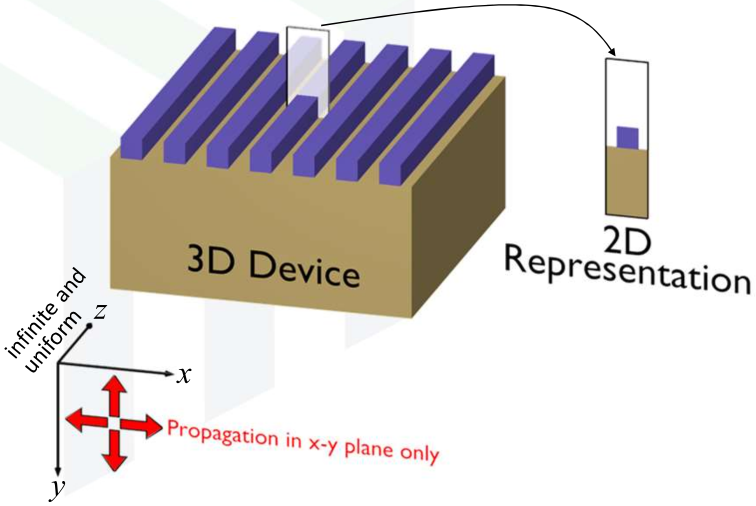
$$\frac{\tilde{H}_z^{i,j,k} - \tilde{H}_z^{i,j-1,k}}{\Delta y'} - \frac{\tilde{H}_y^{i,j,k} - \tilde{H}_y^{i,j,k-1}}{\Delta z'} = \varepsilon_{xx}^{i,j,k}E_x^{i,j,k}$$

$$\frac{\tilde{H}_x^{i,j,k} - \tilde{H}_x^{i,j,k-1}}{\Delta z'} - \frac{\tilde{H}_z^{i,j,k} - \tilde{H}_z^{i-1,j,k}}{\Delta x'} = \varepsilon_{yy}^{i,j,k}E_y^{i,j,k}$$

$$\frac{\tilde{H}_y^{i,j,k} - \tilde{H}_y^{i-1,j,k}}{\Delta x'} - \frac{\tilde{H}_x^{i,j,k} - \tilde{H}_x^{i,j-1,k}}{\Delta y'} = \varepsilon_{zz}^{i,j,k}E_z^{i,j,k}$$

Two-Dimensional Analysis

For 2D problems, let the device be uniform and infinite in the z-direction and let wave propagation be restricted to the x-y plane.



Maxwell's Equations for 2D Analysis

Let the uniform direction be the z-axis and wave propagation be restricted to the x-y plane.

For this case, $\partial/\partial z' = 0$ and the finite-difference equations simplify.

$$\frac{E_z^{i,j+1,k} - E_z^{i,j,k}}{\Delta y'} - \cancel{\frac{E_y^{i,j,k+1} - E_y^{i,j,k}}{\Delta z'}} = \mu_{xx}^{i,j,k} \tilde{H}_x^{i,j,k} \qquad \frac{\tilde{H}_z^{i,j,k} - \tilde{H}_z^{i,j-1,k}}{\Delta y'} - \cancel{\frac{\tilde{H}_y^{i,j,k} - \tilde{H}_y^{i,j,k-1}}{\Delta z'}} = \varepsilon_{xx}^{i,j,k} E_x^{i,j,k}$$

$$\cancel{\frac{E_x^{i,j,k+1} - E_x^{i,j,k}}{\Delta z'}} - \frac{E_z^{i+1,j,k} - E_z^{i,j,k}}{\Delta x'} = \mu_{yy}^{i,j,k} \tilde{H}_y^{i,j,k} \qquad \cancel{\frac{\tilde{H}_x^{i,j,k} - \tilde{H}_x^{i,j,k-1}}{\Delta z'}} - \frac{\tilde{H}_z^{i,j,k} - \tilde{H}_z^{i-1,j,k}}{\Delta x'} = \varepsilon_{yy}^{i,j,k} E_y^{i,j,k}$$

$$\frac{E_y^{i+1,j,k} - E_y^{i,j,k}}{\Delta x'} - \frac{E_x^{i,j+1,k} - E_x^{i,j,k}}{\Delta y'} = \mu_{zz}^{i,j,k} \tilde{H}_z^{i,j,k} \qquad \frac{\tilde{H}_y^{i,j,k} - \tilde{H}_y^{i-1,j,k}}{\Delta x'} - \frac{\tilde{H}_x^{i,j,k} - \tilde{H}_x^{i,j-1,k}}{\Delta y'} = \varepsilon_{zz}^{i,j,k} E_z^{i,j,k}$$

$$\Downarrow \qquad\qquad\qquad \Downarrow$$

$$\frac{E_z^{i,j+1,k} - E_z^{i,j,k}}{\Delta y'} = \mu_{xx}^{i,j,k} \tilde{H}_x^{i,j,k} \qquad \frac{\tilde{H}_z^{i,j,k} - \tilde{H}_z^{i,j-1,k}}{\Delta y'} = \varepsilon_{xx}^{i,j,k} E_x^{i,j,k}$$

$$-\frac{E_z^{i+1,j,k} - E_z^{i,j,k}}{\Delta x'} = \mu_{yy}^{i,j,k} \tilde{H}_y^{i,j,k} \qquad -\frac{\tilde{H}_z^{i,j,k} - \tilde{H}_z^{i-1,j,k}}{\Delta x'} = \varepsilon_{yy}^{i,j,k} E_y^{i,j,k}$$

$$\frac{E_y^{i+1,j,k} - E_y^{i,j,k}}{\Delta x'} - \frac{E_x^{i,j+1,k} - E_x^{i,j,k}}{\Delta y'} = \mu_{zz}^{i,j,k} \tilde{H}_z^{i,j,k} \qquad \frac{\tilde{H}_y^{i,j,k} - \tilde{H}_y^{i-1,j,k}}{\Delta x'} - \frac{\tilde{H}_x^{i,j,k} - \tilde{H}_x^{i,j-1,k}}{\Delta y'} = \varepsilon_{zz}^{i,j,k} E_z^{i,j,k}$$

Two Modes for 2D Analysis

Maxwell's equations have split into two sets of three coupled equations.

$$\frac{E_z^{i,j+1,k}-E_z^{i,j,k}}{\Delta y'}=\mu_{xx}^{i,j,k}\tilde{H}_x^{i,j,k}$$

$$-\frac{E_z^{i+1,j,k}-E_z^{i,j,k}}{\Delta x'}=\mu_{yy}^{i,j,k}\tilde{H}_y^{i,j,k}$$

$$\frac{E_y^{i+1,j,k}-E_y^{i,j,k}}{\Delta x'}-\frac{E_x^{i,j+1,k}-E_x^{i,j,k}}{\Delta y'}=\mu_{zz}^{i,j,k}\tilde{H}_z^{i,j,k}$$

$$\nabla'\times\vec{E}=[\mu]\vec{\tilde{H}}$$

$$\frac{\tilde{H}_z^{i,j,k}-\tilde{H}_z^{i,j-1,k}}{\Delta y'}=\varepsilon_{xx}^{i,j,k}E_x^{i,j,k}$$

$$-\frac{\tilde{H}_z^{i,j,k}-\tilde{H}_z^{i-1,j,k}}{\Delta x'}=\varepsilon_{yy}^{i,j,k}E_y^{i,j,k}$$

$$\frac{\tilde{H}_y^{i,j,k}-\tilde{H}_y^{i-1,j,k}}{\Delta x'}-\frac{\tilde{H}_x^{i,j,k}-\tilde{H}_x^{i,j-1,k}}{\Delta y'}=\varepsilon_{zz}^{i,j,k}E_z^{i,j,k}$$

$$\nabla'\times\vec{\tilde{H}}=[\varepsilon]\vec{E}$$

→

H Mode

E Mode

EMPossible

Two Modes for 2D Analysis

Maxwell's equations have split into two sets of three coupled equations.

$$\frac{E_z^{i,j+1,k}-E_z^{i,j,k}}{\Delta y'}=\mu_{xx}^{i,j,k}\tilde{H}_x^{i,j,k}$$

$$-\frac{E_z^{i+1,j,k}-E_z^{i,j,k}}{\Delta x'}=\mu_{yy}^{i,j,k}\tilde{H}_y^{i,j,k}$$

$$\frac{E_y^{i+1,j,k}-E_y^{i,j,k}}{\Delta x'}-\frac{E_x^{i,j+1,k}-E_x^{i,j,k}}{\Delta y'}=\mu_{zz}^{i,j,k}\tilde{H}_z^{i,j,k}$$

$$\nabla'\times\vec{E}=[\mu]\vec{\tilde{H}}$$

$$\frac{\tilde{H}_z^{i,j,k}-\tilde{H}_z^{i,j-1,k}}{\Delta y'}=\varepsilon_{xx}^{i,j,k}E_x^{i,j,k}$$

$$-\frac{\tilde{H}_z^{i,j,k}-\tilde{H}_z^{i-1,j,k}}{\Delta x'}=\varepsilon_{yy}^{i,j,k}E_y^{i,j,k}$$

$$\frac{\tilde{H}_y^{i,j,k}-\tilde{H}_y^{i-1,j,k}}{\Delta x'}-\frac{\tilde{H}_x^{i,j,k}-\tilde{H}_x^{i,j-1,k}}{\Delta y'}=\varepsilon_{zz}^{i,j,k}E_z^{i,j,k}$$

$$\nabla'\times\vec{\tilde{H}}=[\varepsilon]\vec{E}$$

→

$$\frac{E_y^{i+1,j,k}-E_y^{i,j,k}}{\Delta x'}-\frac{E_x^{i,j+1,k}-E_x^{i,j,k}}{\Delta y'}=\mu_{zz}^{i,j,k}\tilde{H}_z^{i,j,k}$$

$$\frac{\tilde{H}_z^{i,j,k}-\tilde{H}_z^{i,j-1,k}}{\Delta y'}=\varepsilon_{xx}^{i,j,k}E_x^{i,j,k}$$

$$-\frac{\tilde{H}_z^{i,j,k}-\tilde{H}_z^{i-1,j,k}}{\Delta x'}=\varepsilon_{yy}^{i,j,k}E_y^{i,j,k}$$

H Mode

$$\frac{\tilde{H}_y^{i,j,k}-\tilde{H}_y^{i-1,j,k}}{\Delta x'}-\frac{\tilde{H}_x^{i,j,k}-\tilde{H}_x^{i,j-1,k}}{\Delta y'}=\varepsilon_{zz}^{i,j,k}E_z^{i,j,k}$$

$$\frac{E_z^{i,j+1,k}-E_z^{i,j,k}}{\Delta y'}=\mu_{xx}^{i,j,k}\tilde{H}_x^{i,j,k}$$

$$-\frac{E_z^{i+1,j,k}-E_z^{i,j,k}}{\Delta x'}=\mu_{yy}^{i,j,k}\tilde{H}_y^{i,j,k}$$

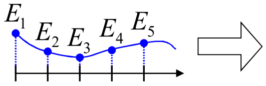
E Mode

EMPossible

Derivative Matrices for the Yee Grid

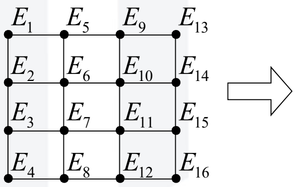
Recall That Fields are Stored in Column Vectors

1-D Systems

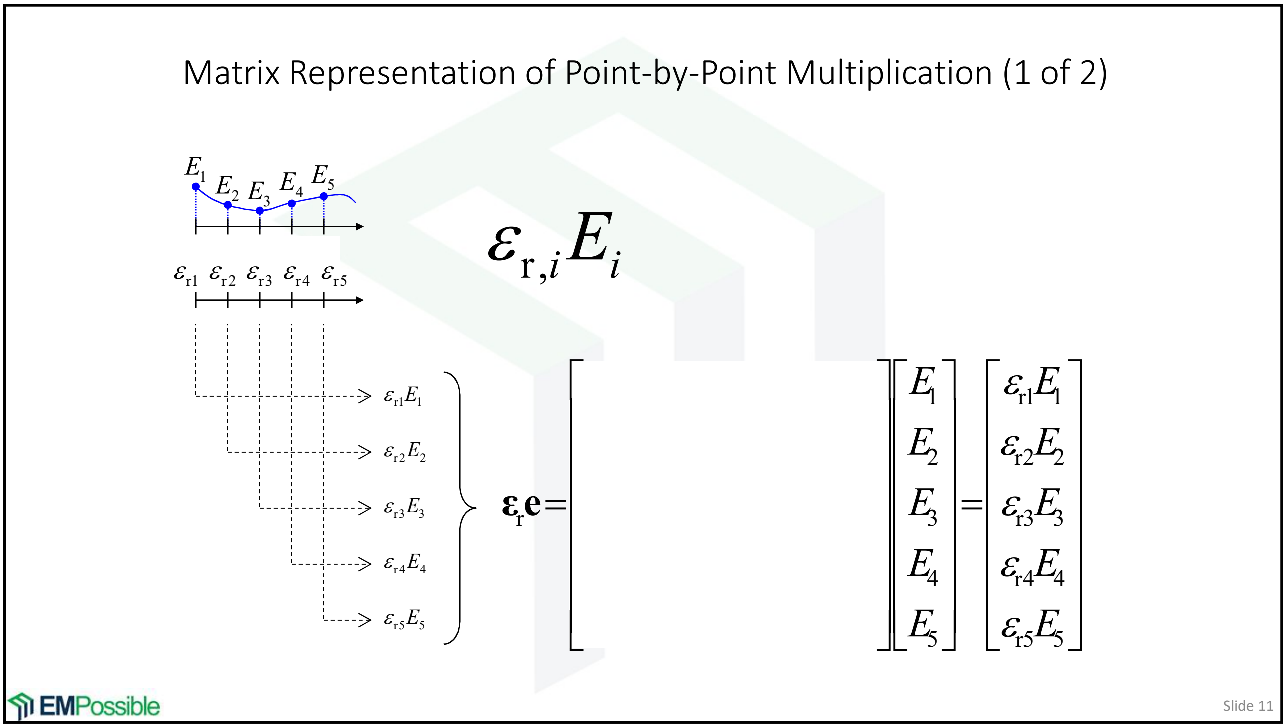


$$\mathbf{e} = \begin{bmatrix} E_1 \\ E_2 \\ E_3 \\ E_4 \\ E_5 \end{bmatrix}$$

2-D Systems



$$\mathbf{e} = \begin{bmatrix} E_1 \\ E_2 \\ E_3 \\ E_4 \\ E_5 \\ E_6 \\ E_7 \\ E_8 \\ E_9 \\ E_{10} \\ E_{11} \\ E_{12} \\ E_{13} \\ E_{14} \\ E_{15} \\ E_{16} \end{bmatrix}$$

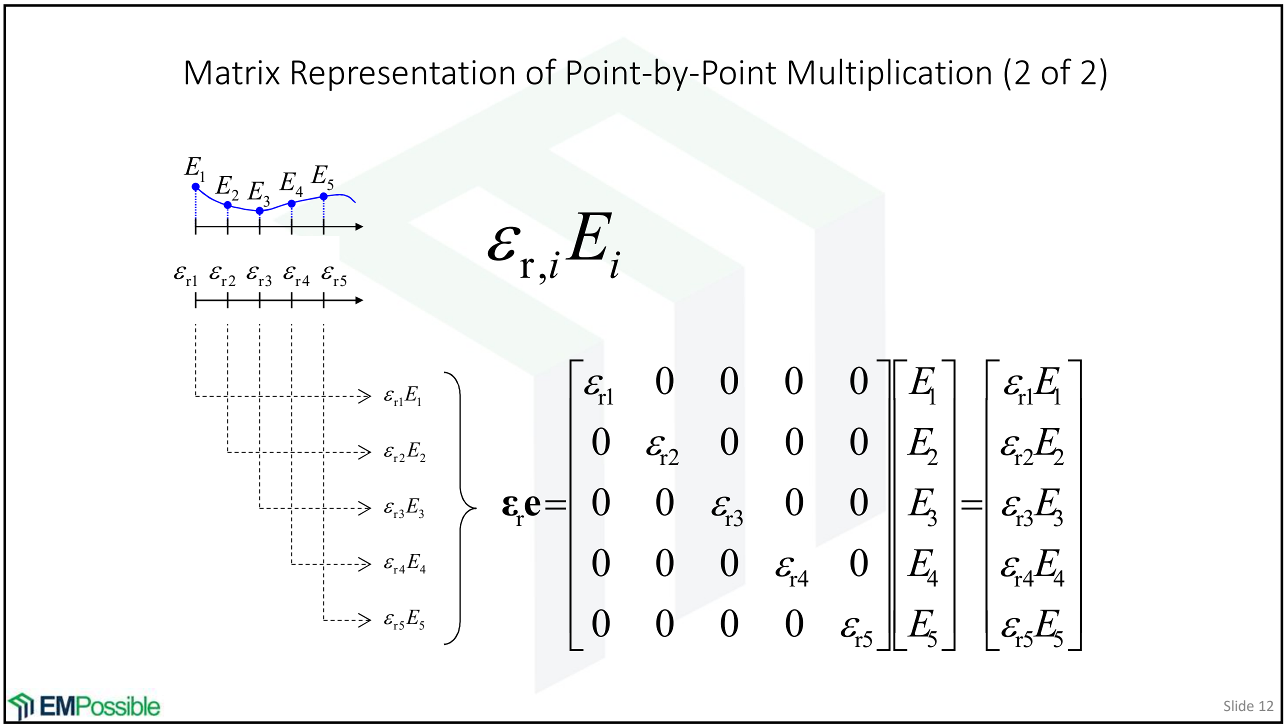


Matrix Representation of Point-by-Point Multiplication (1 of 2)

$$\varepsilon_{r,i}E_i$$

$$\boldsymbol{\varepsilon}_r\mathbf{e}=\begin{bmatrix} & & & & \\ & & & & \\ & & & & \\ & & & & \\ & & & & \end{bmatrix}\begin{bmatrix} E_1 \\ E_2 \\ E_3 \\ E_4 \\ E_5 \end{bmatrix}=\begin{bmatrix} \varepsilon_{r1}E_1 \\ \varepsilon_{r2}E_2 \\ \varepsilon_{r3}E_3 \\ \varepsilon_{r4}E_4 \\ \varepsilon_{r5}E_5 \end{bmatrix}$$

EMPossible



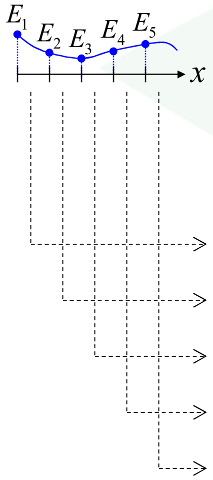
Matrix Representation of Point-by-Point Multiplication (2 of 2)

$$\varepsilon_{r,i}E_i$$

$$\boldsymbol{\varepsilon}_r\mathbf{e}=\begin{bmatrix} \varepsilon_{r1} & 0 & 0 & 0 & 0 \\ 0 & \varepsilon_{r2} & 0 & 0 & 0 \\ 0 & 0 & \varepsilon_{r3} & 0 & 0 \\ 0 & 0 & 0 & \varepsilon_{r4} & 0 \\ 0 & 0 & 0 & 0 & \varepsilon_{r5} \end{bmatrix}\begin{bmatrix} E_1 \\ E_2 \\ E_3 \\ E_4 \\ E_5 \end{bmatrix}=\begin{bmatrix} \varepsilon_{r1}E_1 \\ \varepsilon_{r2}E_2 \\ \varepsilon_{r3}E_3 \\ \varepsilon_{r4}E_4 \\ \varepsilon_{r5}E_5 \end{bmatrix}$$

EMPossible

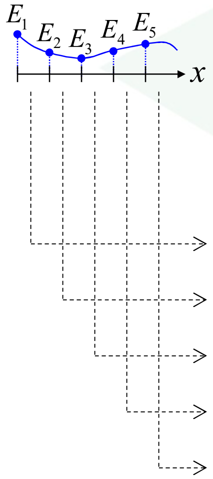
Derivative Matrices for Electric Fields (1 of 2)



$$\left.\frac{\partial E}{\partial x}\right|_{i+\frac{1}{2}} \cong \frac{E_{i+1}-E_i}{\Delta x}$$

$$\left.\begin{matrix} \dfrac{E_2-E_1}{\Delta x} \\ \dfrac{E_3-E_2}{\Delta x} \\ \dfrac{E_4-E_3}{\Delta x} \\ \dfrac{E_5-E_4}{\Delta x} \\ \dfrac{E_6-E_5}{\Delta x} \end{matrix}\right\} \mathbf{D}_x^e\mathbf{e} = \frac{1}{\Delta x}\begin{bmatrix} \quad & \quad & \quad & \quad & \quad \\ \\ \\ \\ \\ \end{bmatrix}\begin{bmatrix} E_1 \\ E_2 \\ E_3 \\ E_4 \\ E_5 \end{bmatrix} = \begin{bmatrix} \frac{\partial}{\partial x}E_{1.5} \\ \frac{\partial}{\partial x}E_{2.5} \\ \frac{\partial}{\partial x}E_{3.5} \\ \frac{\partial}{\partial x}E_{4.5} \\ \frac{\partial}{\partial x}E_{5.5} \end{bmatrix}$$

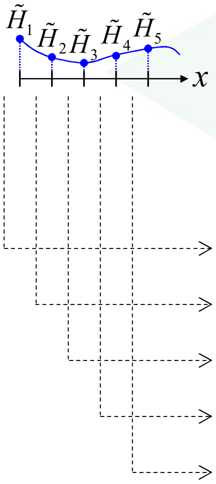
Derivative Matrices for Electric Fields (2 of 2)



$$\left.\frac{\partial E}{\partial x}\right|_{i+\frac{1}{2}} \cong \frac{E_{i+1}-E_i}{\Delta x}$$

$$\left.\begin{matrix} \dfrac{E_2-E_1}{\Delta x} \\ \dfrac{E_3-E_2}{\Delta x} \\ \dfrac{E_4-E_3}{\Delta x} \\ \dfrac{E_5-E_4}{\Delta x} \\ \dfrac{E_6-E_5}{\Delta x} \end{matrix}\right\} \mathbf{D}_x^e\mathbf{e} = \frac{1}{\Delta x}\begin{bmatrix} -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix}\begin{bmatrix} E_1 \\ E_2 \\ E_3 \\ E_4 \\ E_5 \end{bmatrix} = \begin{bmatrix} \frac{\partial}{\partial x}E_{1.5} \\ \frac{\partial}{\partial x}E_{2.5} \\ \frac{\partial}{\partial x}E_{3.5} \\ \frac{\partial}{\partial x}E_{4.5} \\ \frac{\partial}{\partial x}E_{5.5} \end{bmatrix}$$

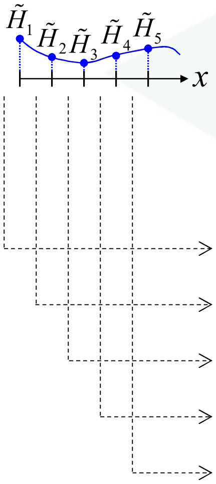
Derivative Matrices for Magnetic Fields (2 of 2)



$$\left.\frac{\partial \tilde{H}}{\partial x}\right|_{i-\frac{1}{2}} \cong \frac{\tilde{H}_i - \tilde{H}_{i-1}}{\Delta x}$$

$$\left.\begin{matrix} \frac{\tilde{H}_1 - \tilde{H}_0}{\Delta x} \\ \frac{\tilde{H}_2 - \tilde{H}_1}{\Delta x} \\ \frac{\tilde{H}_3 - \tilde{H}_2}{\Delta x} \\ \frac{\tilde{H}_4 - \tilde{H}_3}{\Delta x} \\ \frac{\tilde{H}_5 - \tilde{H}_4}{\Delta x} \end{matrix}\right\} \mathbf{D}_x^h \tilde{\mathbf{h}} = \frac{1}{\Delta x}\begin{bmatrix} \quad & \quad & \quad & \quad & \quad \\ \\ \\ \\ \\ \end{bmatrix} \begin{bmatrix} \tilde{H}_1 \\ \tilde{H}_2 \\ \tilde{H}_3 \\ \tilde{H}_4 \\ \tilde{H}_5 \end{bmatrix} = \begin{bmatrix} \frac{\partial}{\partial x}\tilde{H}_{0.5} \\ \frac{\partial}{\partial x}\tilde{H}_{1.5} \\ \frac{\partial}{\partial x}\tilde{H}_{2.5} \\ \frac{\partial}{\partial x}\tilde{H}_{3.5} \\ \frac{\partial}{\partial x}\tilde{H}_{4.5} \end{bmatrix}$$

Derivative Matrices for Magnetic Fields (2 of 2)



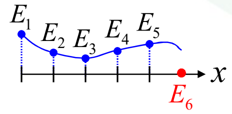
$$\left.\frac{\partial \tilde{H}}{\partial x}\right|_{i-\frac{1}{2}} \cong \frac{\tilde{H}_i - \tilde{H}_{i-1}}{\Delta x}$$

$$\left.\begin{matrix} \frac{\tilde{H}_1 - \tilde{H}_0}{\Delta x} \\ \frac{\tilde{H}_2 - \tilde{H}_1}{\Delta x} \\ \frac{\tilde{H}_3 - \tilde{H}_2}{\Delta x} \\ \frac{\tilde{H}_4 - \tilde{H}_3}{\Delta x} \\ \frac{\tilde{H}_5 - \tilde{H}_4}{\Delta x} \end{matrix}\right\} \mathbf{D}_x^h \tilde{\mathbf{h}} = \frac{1}{\Delta x}\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} \tilde{H}_1 \\ \tilde{H}_2 \\ \tilde{H}_3 \\ \tilde{H}_4 \\ \tilde{H}_5 \end{bmatrix} = \begin{bmatrix} \frac{\partial}{\partial x}\tilde{H}_{0.5} \\ \frac{\partial}{\partial x}\tilde{H}_{1.5} \\ \frac{\partial}{\partial x}\tilde{H}_{2.5} \\ \frac{\partial}{\partial x}\tilde{H}_{3.5} \\ \frac{\partial}{\partial x}\tilde{H}_{4.5} \end{bmatrix}$$

Simplest Boundary Conditions

Dirichlet Boundary Conditions

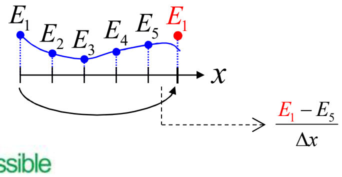
Assume $E_6 = 0$



$$\mathbf{D}_x^{\mathrm{e}}\mathbf{e} = \frac{1}{\Delta x}\begin{bmatrix} -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix}\begin{bmatrix} E_1 \\ E_2 \\ E_3 \\ E_4 \\ E_5 \end{bmatrix} = \frac{1}{\Delta x}\begin{bmatrix} E_2 - E_1 \\ E_3 - E_2 \\ E_4 - E_3 \\ E_5 - E_4 \\ -E_5 \end{bmatrix}$$

Periodic Boundary Conditions

Assume $E_6 = E_1$



$$\mathbf{D}_x^{\mathrm{e}}\mathbf{e} = \frac{1}{\Delta x}\begin{bmatrix} -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 \\ 1 & 0 & 0 & 0 & -1 \end{bmatrix}\begin{bmatrix} E_1 \\ E_2 \\ E_3 \\ E_4 \\ E_5 \end{bmatrix} = \frac{1}{\Delta x}\begin{bmatrix} E_2 - E_1 \\ E_3 - E_2 \\ E_4 - E_3 \\ E_5 - E_4 \\ E_1 - E_5 \end{bmatrix}$$

Derivative Matrices on a 3×3 Grid Using Dirichlet Boundary Conditions

$$\mathbf{D}_x^{\mathrm{e}} = \frac{1}{\Delta_x}\begin{bmatrix} -1 & 1 & & & & & & & \\ & -1 & 1 & & & & & & \\ & & -1 & 0 & & & & & \\ & & & -1 & 1 & & & & \\ & & & & -1 & 1 & & & \\ & & & & & -1 & 0 & & \\ & & & & & & -1 & 1 & \\ & & & & & & & -1 & 1 \\ & & & & & & & & -1 \end{bmatrix}$$

Dirichlet boundary conditions

$$\mathbf{D}_y^{\mathrm{e}} = \frac{1}{\Delta_y}\begin{bmatrix} -1 & 0 & 0 & 1 & & & & & \\ & -1 & 0 & 0 & 1 & & & & \\ & & -1 & 0 & 0 & 1 & & & \\ & & & -1 & 0 & 0 & 1 & & \\ & & & & -1 & 0 & 0 & 1 & \\ & & & & & -1 & 0 & 0 & 1 \\ & & & & & & -1 & 0 & 0 \\ & & & & & & & -1 & 0 \\ & & & & & & & & -1 \end{bmatrix}$$

N_x

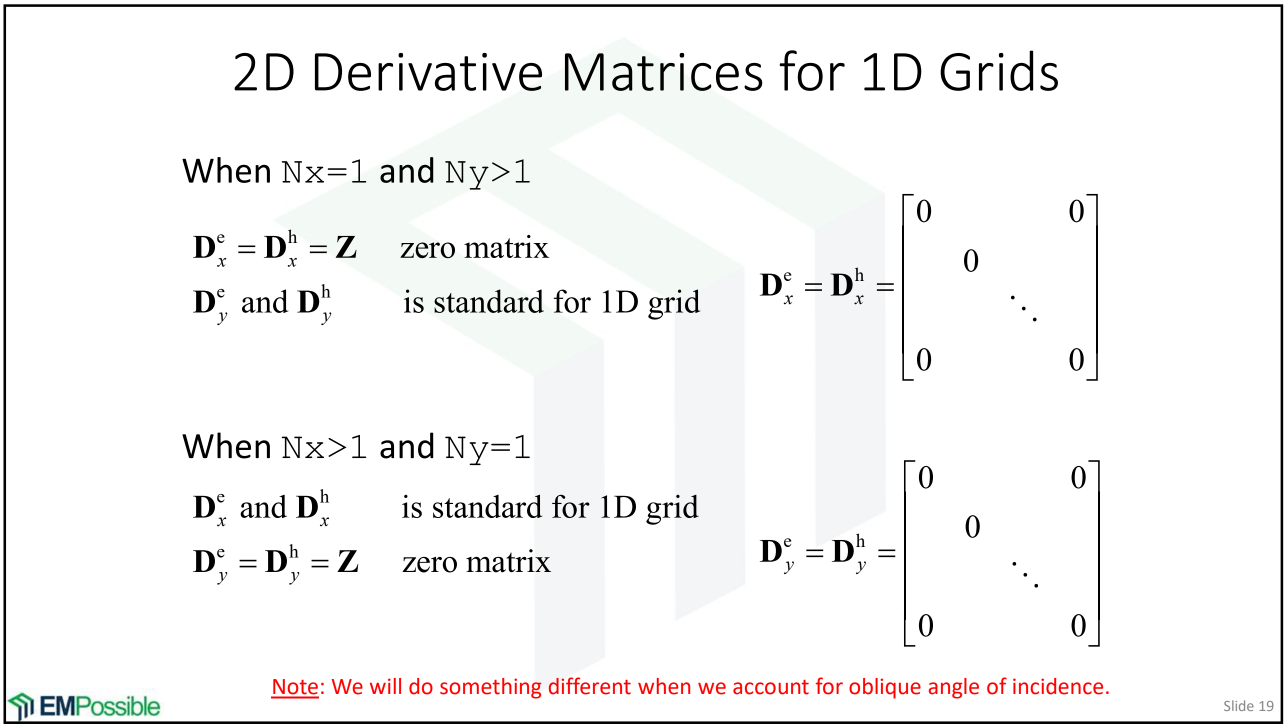
$$\mathbf{D}_x^{\mathrm{h}} = \frac{1}{\Delta_x}\begin{bmatrix} 1 & & & & & & & & \\ -1 & 1 & & & & & & & \\ & -1 & 1 & & & & & & \\ & & 0 & 1 & & & & & \\ & & & -1 & 1 & & & & \\ & & & & -1 & 1 & & & \\ & & & & & 0 & 1 & & \\ & & & & & & -1 & 1 & \\ & & & & & & & -1 & 1 \end{bmatrix}$$

Dirichlet boundary conditions

N_x

$$\mathbf{D}_y^{\mathrm{h}} = \frac{1}{\Delta_y}\begin{bmatrix} 1 & & & & & & & & \\ 0 & 1 & & & & & & & \\ 0 & 0 & 1 & & & & & & \\ -1 & 0 & 0 & 1 & & & & & \\ & -1 & 0 & 0 & 1 & & & & \\ & & -1 & 0 & 0 & 1 & & & \\ & & & -1 & 0 & 0 & 1 & & \\ & & & & -1 & 0 & 0 & 1 & \\ & & & & & -1 & 0 & 0 & 1 \end{bmatrix}$$

Note: All of these matrices have only two diagonals so they are very easy to construct!



2D Derivative Matrices for 1D Grids

When `Nx=1` and `Ny>1`

$\mathbf{D}_x^{\mathrm{e}} = \mathbf{D}_x^{\mathrm{h}} = \mathbf{Z}$ zero matrix

$\mathbf{D}_y^{\mathrm{e}}$ and $\mathbf{D}_y^{\mathrm{h}}$ is standard for 1D grid

$$\mathbf{D}_x^{\mathrm{e}} = \mathbf{D}_x^{\mathrm{h}} = \begin{bmatrix} 0 & & & 0 \\ & 0 & & \\ & & \ddots & \\ 0 & & & 0 \end{bmatrix}$$

When `Nx>1` and `Ny=1`

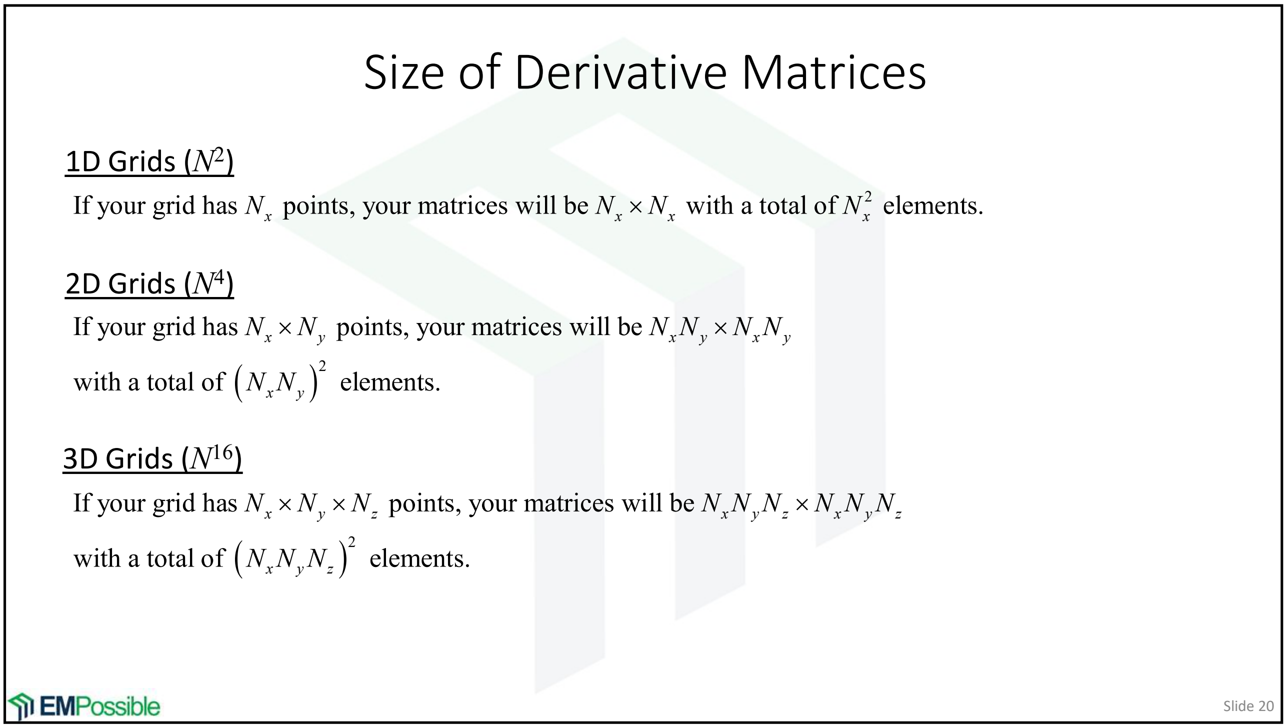
$\mathbf{D}_x^{\mathrm{e}}$ and $\mathbf{D}_x^{\mathrm{h}}$ is standard for 1D grid

$\mathbf{D}_y^{\mathrm{e}} = \mathbf{D}_y^{\mathrm{h}} = \mathbf{Z}$ zero matrix

$$\mathbf{D}_y^{\mathrm{e}} = \mathbf{D}_y^{\mathrm{h}} = \begin{bmatrix} 0 & & & 0 \\ & 0 & & \\ & & \ddots & \\ 0 & & & 0 \end{bmatrix}$$

Note: We will do something different when we account for oblique angle of incidence.

EMPossible



Size of Derivative Matrices

1D Grids (N^2)

If your grid has N_x points, your matrices will be $N_x \times N_x$ with a total of N_x^2 elements.

2D Grids (N^4)

If your grid has $N_x \times N_y$ points, your matrices will be $N_x N_y \times N_x N_y$ with a total of $\left(N_x N_y\right)^2$ elements.

3D Grids (N^{16})

If your grid has $N_x \times N_y \times N_z$ points, your matrices will be $N_x N_y N_z \times N_x N_y N_z$ with a total of $\left(N_x N_y N_z\right)^2$ elements.

EMPossible

USE SPARSE MATRICES*!!!!!!!*

WARNING !!

The derivative matrices will be **EXTREMELY** large matrices.

For a small grid that is just 100×200 points:

Total Number of Points:	20,000
Size of Derivate Matrices:	20,000 × 20,000
Total Elements in Matrices:	400,000,000
Memory to Store One Full Matrix:	6 Gb
Memory to Store One Sparse Matrix:	1 Mb

NEVER AT ANY POINT should you use **FULL MATRICES** in the finite-difference method. Not even for intermediate steps. **NEVER*!***

Derivative Matrices on a 4×4 Grid Using Dirichlet Boundary Conditions

$$\mathbf{D}_x^{\mathrm{e}} = \frac{1}{\Delta x}\left[\begin{array}{cccccccccccccccc}
-1 & +1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & +1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & +1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & +1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & +1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & +1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & +1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & +1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & +1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & +1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & +1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & +1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1
\end{array}\right]$$

$$\mathbf{D}_y^{\mathrm{e}} = \frac{1}{\Delta y}\left[\begin{array}{cccccccccccccccc}
-1 & 0 & 0 & 0 & +1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & +1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & +1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 & +1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & +1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & +1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & +1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & +1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & +1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & +1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & +1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & +1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1
\end{array}\right]$$

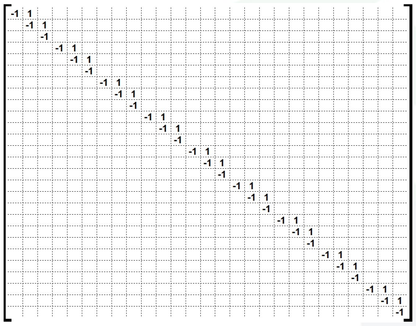
$$\mathbf{D}_x^{\mathrm{h}} = \frac{1}{\Delta x}\left[\begin{array}{cccccccccccccccc}
+1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & +1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & +1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & +1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & +1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & +1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & +1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & +1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & +1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & +1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & +1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & +1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & +1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & +1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & +1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & +1
\end{array}\right]$$

$$\mathbf{D}_y^{\mathrm{h}} = \frac{1}{\Delta y}\left[\begin{array}{cccccccccccccccc}
+1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & +1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & +1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & +1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & +1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & +1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & +1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 & +1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & +1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & +1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & +1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & +1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & +1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & +1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & +1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & +1
\end{array}\right]$$

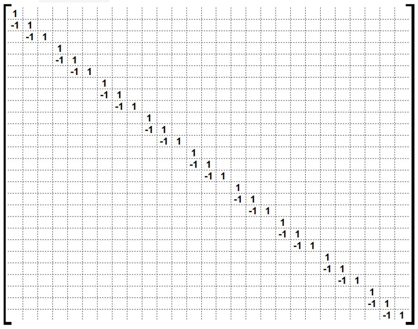
Note: These matrices have only two diagonals so they are very easy to construct!

$\mathbf{D}_x^{\mathrm{e}}$ and $\mathbf{D}_x^{\mathrm{h}}$ on a 3×3×3 Yee Grid

$$\mathbf{D}_x^{\mathrm{e}} = \frac{1}{\Delta x}$$

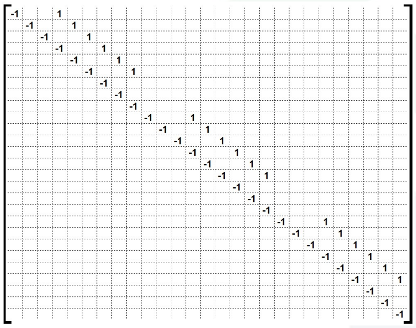


$$\mathbf{D}_x^{\mathrm{h}} = \frac{1}{\Delta x}$$

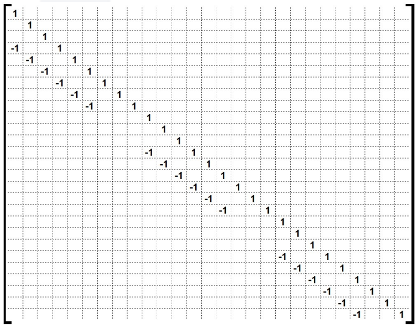


$\mathbf{D}_y^{\mathrm{e}}$ and $\mathbf{D}_y^{\mathrm{h}}$ on a 3×3×3 Yee Grid

$$\mathbf{D}_y^{\mathrm{e}} = \frac{1}{\Delta y}$$

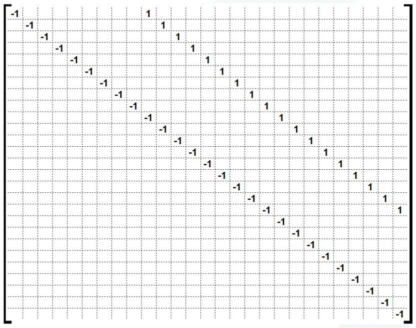


$$\mathbf{D}_y^{\mathrm{h}} = \frac{1}{\Delta y}$$

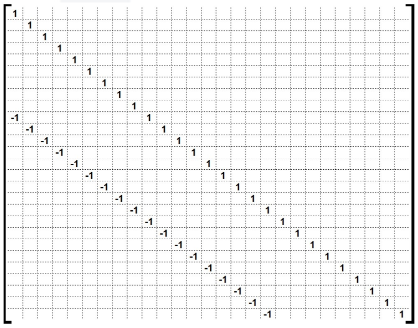


$\mathbf{D}_z^{\mathrm{e}}$ and $\mathbf{D}_z^{\mathrm{h}}$ on a 3×3×3 Yee Grid

$$\mathbf{D}_z^{\mathrm{e}} = \frac{1}{\Delta z}$$



$$\mathbf{D}_z^{\mathrm{h}} = \frac{1}{\Delta z}$$



Relationship Between the Derivative Matrices

Transpose Operation

$$\left[\mathbf{A}^{\mathrm{T}}\right]_{i,j} = \left[\mathbf{A}\right]_{j,i}$$

$$\mathbf{A} = \begin{bmatrix}1 & 2\end{bmatrix}, \mathbf{A}^{\mathrm{T}} = \begin{bmatrix}1\\2\end{bmatrix} \qquad \mathbf{A} = \begin{bmatrix}1 & 2\\3 & 4\end{bmatrix}, \mathbf{A}^{\mathrm{T}} = \begin{bmatrix}1 & 3\\2 & 4\end{bmatrix} \qquad \mathbf{A} = \begin{bmatrix}a_{11} & a_{12}\\a_{21} & a_{22}\end{bmatrix}, \mathbf{A}^{\mathrm{T}} = \begin{bmatrix}a_{11} & a_{21}\\a_{22} & a_{22}\end{bmatrix}$$

```
AT = transpose(A);        AT = A.';
```

Hermitian (Conjugate) Transpose Operation

$$\left[\mathbf{A}^{\mathrm{H}}\right]_{i,j} = \left[\mathbf{A}\right]_{j,i}^{*}$$

$$\mathbf{A} = \begin{bmatrix}a_{11} & a_{12}\\a_{21} & a_{22}\end{bmatrix}, \mathbf{A}^{\mathrm{H}} = \begin{bmatrix}a_{11}^{*} & a_{21}^{*}\\a_{12}^{*} & a_{22}^{*}\end{bmatrix} \qquad \mathbf{A} = \begin{bmatrix}(1+4j) & (2-3j)\\(3-2j) & (4+j)\end{bmatrix}, \mathbf{A}^{\mathrm{H}} = \begin{bmatrix}(1-4j) & (3+2j)\\(2+3j) & (4-j)\end{bmatrix}$$

```
AT = ctranspose(A);       AT = A';
```

Relationship Between the Derivative Operators

$$\boxed{\mathbf{D}_x^h = -\left[\mathbf{D}_x^e\right]^H} \qquad \boxed{\mathbf{D}_y^h = -\left[\mathbf{D}_y^e\right]^H}$$

```
DHX = -DEX';              DHY = -DEY';
```

This means you only have to construct derivative operators for the electric field. The derivative operators for the magnetic field can be computed directly from the electric field derivative operators.

This relation does not hold for some boundary conditions such as Neumann.

What About the Second-Order Derivatives?

Recall from Lecture 5, Slide 27 that $\mathbf{D}^{(1)}\mathbf{D}^{(1)}$ was a poor approximation of $\mathbf{D}^{(2)}$.

$$\mathbf{D}_x^{(1)}\mathbf{D}_x^{(1)} = \frac{1}{(2\Delta_x)^2}\begin{bmatrix} -1 & 0 & 1 & 0 & 0 \\ 0 & -2 & 0 & 1 & 0 \\ 1 & 0 & -2 & 0 & 1 \\ 0 & 1 & 0 & -2 & 0 \\ 0 & 0 & 1 & 0 & -1 \end{bmatrix} \qquad \mathbf{D}_x^{(2)} = \frac{1}{\Delta_x^2}\begin{bmatrix} -2 & 1 & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 1 & -2 \end{bmatrix}$$

What about the derivative operators derived from the Yee grid?

$$\mathbf{D}_x^{e} = \frac{1}{\Delta_x}\begin{bmatrix} -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix} \qquad \mathbf{D}_x^{h} = \frac{1}{\Delta_x}\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix}$$

$$\mathbf{D}_x^{e} \cdot \mathbf{D}_x^{h} = \frac{1}{(\Delta_x)^2}\begin{bmatrix} -2 & 1 & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 1 & \color{red}{-1} \end{bmatrix}$$

The numbers is this matrix may differ slightly from the "ideal" 2nd-order derivate operator due to the boundary conditions.

Matrix Form of Maxwell's Equations

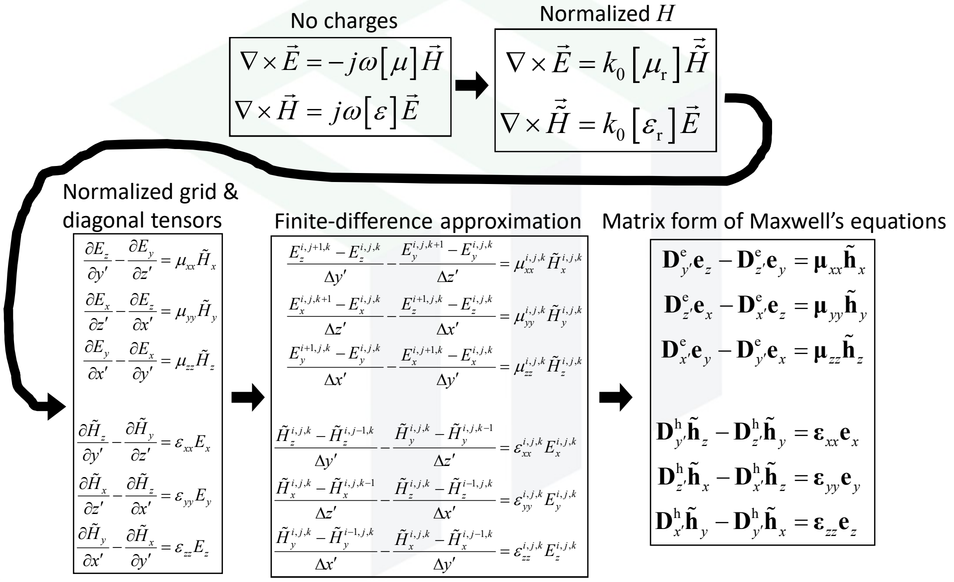
Maxwell's Equations in Matrix Form

$$
\begin{aligned}
\frac{\partial}{\partial y'}E_z-\frac{\partial}{\partial z'}E_y&=\mu_{xx}\tilde{H}_x\\
\frac{\partial}{\partial z'}E_x-\frac{\partial}{\partial x'}E_z&=\mu_{yy}\tilde{H}_y\\
\frac{\partial}{\partial x'}E_y-\frac{\partial}{\partial y'}E_x&=\mu_{zz}\tilde{H}_z
\end{aligned}
\Rightarrow
\begin{aligned}
\frac{E_z^{i,j+1,k}-E_z^{i,j,k}}{\Delta y'}-\frac{E_y^{i,j,k+1}-E_y^{i,j,k}}{\Delta z'}&=\mu_{xx}^{i,j,k}\tilde{H}_x^{i,j,k}\\
\frac{E_x^{i,j,k+1}-E_x^{i,j,k}}{\Delta z'}-\frac{E_z^{i+1,j,k}-E_z^{i,j,k}}{\Delta x'}&=\mu_{yy}^{i,j,k}\tilde{H}_y^{i,j,k}\\
\frac{E_y^{i+1,j,k}-E_y^{i,j,k}}{\Delta x'}-\frac{E_x^{i,j+1,k}-E_x^{i,j,k}}{\Delta y'}&=\mu_{zz}^{i,j,k}\tilde{H}_z^{i,j,k}
\end{aligned}
\Rightarrow
\begin{aligned}
\mathbf{D}_{y'}^{\mathrm{e}}\mathbf{e}_z-\mathbf{D}_{z'}^{\mathrm{e}}\mathbf{e}_y&=\boldsymbol{\mu}_{xx}\tilde{\mathbf{h}}_x\\
\mathbf{D}_{z'}^{\mathrm{e}}\mathbf{e}_x-\mathbf{D}_{x'}^{\mathrm{e}}\mathbf{e}_z&=\boldsymbol{\mu}_{yy}\tilde{\mathbf{h}}_y\\
\mathbf{D}_{x'}^{\mathrm{e}}\mathbf{e}_y-\mathbf{D}_{y'}^{\mathrm{e}}\mathbf{e}_x&=\boldsymbol{\mu}_{zz}\tilde{\mathbf{h}}_z
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial}{\partial y'}\tilde{H}_z-\frac{\partial}{\partial z'}\tilde{H}_y&=\varepsilon_{xx}E_x\\
\frac{\partial}{\partial z'}\tilde{H}_x-\frac{\partial}{\partial x'}\tilde{H}_z&=\varepsilon_{yy}E_y\\
\frac{\partial}{\partial x'}\tilde{H}_y-\frac{\partial}{\partial y'}\tilde{H}_x&=\varepsilon_{zz}E_z
\end{aligned}
\Rightarrow
\begin{aligned}
\frac{\tilde{H}_z^{i,j,k}-\tilde{H}_z^{i,j-1,k}}{\Delta y'}-\frac{\tilde{H}_y^{i,j,k}-\tilde{H}_y^{i,j,k-1}}{\Delta z'}&=\varepsilon_{xx}^{i,j,k}E_x^{i,j,k}\\
\frac{\tilde{H}_x^{i,j,k}-\tilde{H}_x^{i,j,k-1}}{\Delta z'}-\frac{\tilde{H}_z^{i,j,k}-\tilde{H}_z^{i-1,j,k}}{\Delta x'}&=\varepsilon_{yy}^{i,j,k}E_y^{i,j,k}\\
\frac{\tilde{H}_y^{i,j,k}-\tilde{H}_y^{i-1,j,k}}{\Delta x'}-\frac{\tilde{H}_x^{i,j,k}-\tilde{H}_x^{i,j-1,k}}{\Delta y'}&=\varepsilon_{zz}^{i,j,k}E_z^{i,j,k}
\end{aligned}
\Rightarrow
\begin{aligned}
\mathbf{D}_{y'}^{\mathrm{h}}\tilde{\mathbf{h}}_z-\mathbf{D}_{z'}^{\mathrm{h}}\tilde{\mathbf{h}}_y&=\boldsymbol{\varepsilon}_{xx}\mathbf{e}_x\\
\mathbf{D}_{z'}^{\mathrm{h}}\tilde{\mathbf{h}}_x-\mathbf{D}_{x'}^{\mathrm{h}}\tilde{\mathbf{h}}_z&=\boldsymbol{\varepsilon}_{yy}\mathbf{e}_y\\
\mathbf{D}_{x'}^{\mathrm{h}}\tilde{\mathbf{h}}_y-\mathbf{D}_{y'}^{\mathrm{h}}\tilde{\mathbf{h}}_x&=\boldsymbol{\varepsilon}_{zz}\mathbf{e}_z
\end{aligned}
$$

EMPossible

Summary of Putting Maxwell's Equations in Matrix Form



No charges

$$
\nabla\times\vec{E}=-j\omega[\mu]\vec{H}\qquad
\nabla\times\vec{H}=j\omega[\varepsilon]\vec{E}
$$

Normalized H

$$
\nabla\times\vec{E}=k_0\left[\mu_{\mathrm{r}}\right]\vec{\tilde{H}}\qquad
\nabla\times\vec{\tilde{H}}=k_0\left[\varepsilon_{\mathrm{r}}\right]\vec{E}
$$

Normalized grid & diagonal tensors

$$
\begin{aligned}
\frac{\partial E_z}{\partial y'}-\frac{\partial E_y}{\partial z'}&=\mu_{xx}\tilde{H}_x\\
\frac{\partial E_x}{\partial z'}-\frac{\partial E_z}{\partial x'}&=\mu_{yy}\tilde{H}_y\\
\frac{\partial E_y}{\partial x'}-\frac{\partial E_x}{\partial y'}&=\mu_{zz}\tilde{H}_z\\
\frac{\partial \tilde{H}_z}{\partial y'}-\frac{\partial \tilde{H}_y}{\partial z'}&=\varepsilon_{xx}E_x\\
\frac{\partial \tilde{H}_x}{\partial z'}-\frac{\partial \tilde{H}_z}{\partial x'}&=\varepsilon_{yy}E_y\\
\frac{\partial \tilde{H}_y}{\partial x'}-\frac{\partial \tilde{H}_x}{\partial y'}&=\varepsilon_{zz}E_z
\end{aligned}
$$

Finite-difference approximation

$$
\begin{aligned}
\frac{E_z^{i,j+1,k}-E_z^{i,j,k}}{\Delta y'}-\frac{E_y^{i,j,k+1}-E_y^{i,j,k}}{\Delta z'}&=\mu_{xx}^{i,j,k}\tilde{H}_x^{i,j,k}\\
\frac{E_x^{i,j,k+1}-E_x^{i,j,k}}{\Delta z'}-\frac{E_z^{i+1,j,k}-E_z^{i,j,k}}{\Delta x'}&=\mu_{yy}^{i,j,k}\tilde{H}_y^{i,j,k}\\
\frac{E_y^{i+1,j,k}-E_y^{i,j,k}}{\Delta x'}-\frac{E_x^{i,j+1,k}-E_x^{i,j,k}}{\Delta y'}&=\mu_{zz}^{i,j,k}\tilde{H}_z^{i,j,k}\\
\frac{\tilde{H}_z^{i,j,k}-\tilde{H}_z^{i,j-1,k}}{\Delta y'}-\frac{\tilde{H}_y^{i,j,k}-\tilde{H}_y^{i,j,k-1}}{\Delta z'}&=\varepsilon_{xx}^{i,j,k}E_x^{i,j,k}\\
\frac{\tilde{H}_x^{i,j,k}-\tilde{H}_x^{i,j,k-1}}{\Delta z'}-\frac{\tilde{H}_z^{i,j,k}-\tilde{H}_z^{i-1,j,k}}{\Delta x'}&=\varepsilon_{yy}^{i,j,k}E_y^{i,j,k}\\
\frac{\tilde{H}_y^{i,j,k}-\tilde{H}_y^{i-1,j,k}}{\Delta x'}-\frac{\tilde{H}_x^{i,j,k}-\tilde{H}_x^{i,j-1,k}}{\Delta y'}&=\varepsilon_{zz}^{i,j,k}E_z^{i,j,k}
\end{aligned}
$$

Matrix form of Maxwell's equations

$$
\begin{aligned}
\mathbf{D}_{y'}^{\mathrm{e}}\mathbf{e}_z-\mathbf{D}_{z'}^{\mathrm{e}}\mathbf{e}_y&=\boldsymbol{\mu}_{xx}\tilde{\mathbf{h}}_x\\
\mathbf{D}_{z'}^{\mathrm{e}}\mathbf{e}_x-\mathbf{D}_{x'}^{\mathrm{e}}\mathbf{e}_z&=\boldsymbol{\mu}_{yy}\tilde{\mathbf{h}}_y\\
\mathbf{D}_{x'}^{\mathrm{e}}\mathbf{e}_y-\mathbf{D}_{y'}^{\mathrm{e}}\mathbf{e}_x&=\boldsymbol{\mu}_{zz}\tilde{\mathbf{h}}_z\\
\mathbf{D}_{y'}^{\mathrm{h}}\tilde{\mathbf{h}}_z-\mathbf{D}_{z'}^{\mathrm{h}}\tilde{\mathbf{h}}_y&=\boldsymbol{\varepsilon}_{xx}\mathbf{e}_x\\
\mathbf{D}_{z'}^{\mathrm{h}}\tilde{\mathbf{h}}_x-\mathbf{D}_{x'}^{\mathrm{h}}\tilde{\mathbf{h}}_z&=\boldsymbol{\varepsilon}_{yy}\mathbf{e}_y\\
\mathbf{D}_{x'}^{\mathrm{h}}\tilde{\mathbf{h}}_y-\mathbf{D}_{y'}^{\mathrm{h}}\tilde{\mathbf{h}}_x&=\boldsymbol{\varepsilon}_{zz}\mathbf{e}_z
\end{aligned}
$$

EMPossible