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Computational Science:
Introduction to Finite-Difference Time-Domain

Derivation of 3D Update Equations With a UPML

Lecture Outline

- Review
- Conversion to the time-domain
- Numerical approximations
- Derivation of the update equations
- Summary of all update equations

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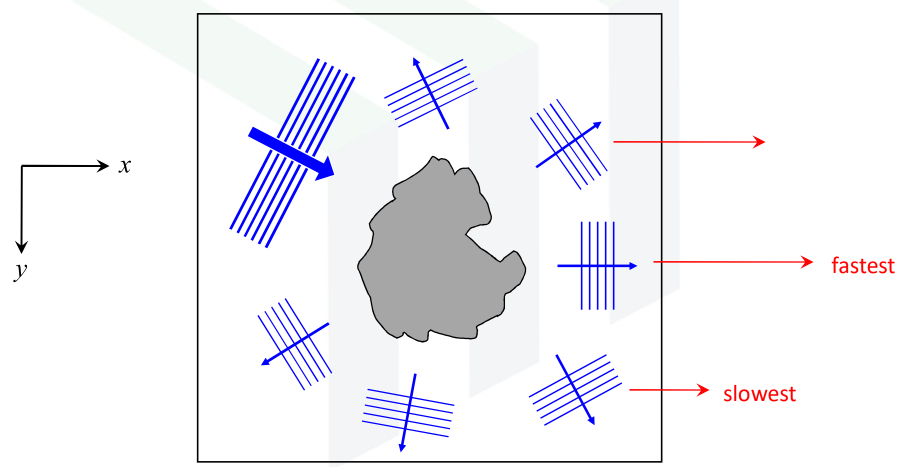
Review

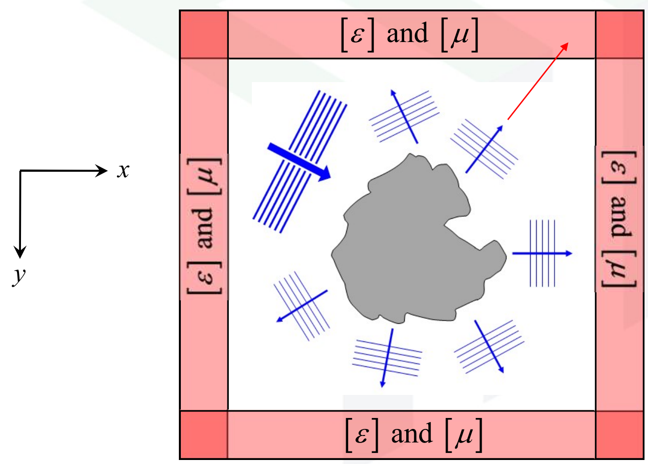
No PAB in Two Dimensions

If we simulate a wave hitting some device or object, it will scatter the applied wave into potentially many directions. Waves at different angles travel at different speeds through a boundary. Therefore, the PAB condition can only be satisfied for one direction, not all directions.



Anisotropy to the Rescue!!

It turns out reflections can be eliminated at all angles and for all polarizations if the absorbing material is made to be doubly-diagonally anisotropic.



The PML Parameters (1 of 2)

The PML must be generalized to handle waves incident on all boundaries.

$$[S]=\begin{bmatrix} \dfrac{s_y s_z}{s_x} & 0 & 0 \\ 0 & \dfrac{s_x s_z}{s_y} & 0 \\ 0 & 0 & \dfrac{s_x s_y}{s_z} \end{bmatrix}$$

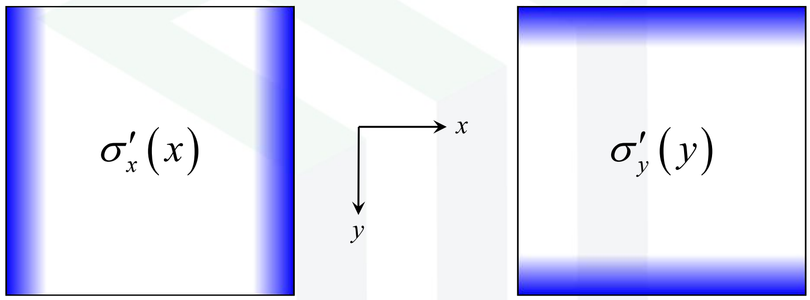
$$s_x(x)=1+\frac{\sigma'_x(x)}{j\omega\varepsilon_0}$$

$$s_y(y)=1+\frac{\sigma'_y(y)}{j\omega\varepsilon_0}$$

$$s_z(z)=1+\frac{\sigma'_z(z)}{j\omega\varepsilon_0}$$

The PML Parameters (2 of 2)

The loss terms should increase gradually into the PMLs.



$$\sigma_x'(x) = \frac{\varepsilon_0}{2\Delta t}\left(\frac{x}{L_x}\right)^3 \qquad \sigma_y'(y) = \frac{\varepsilon_0}{2\Delta t}\left(\frac{y}{L_y}\right)^3 \qquad \sigma_z'(z) = \frac{\varepsilon_0}{2\Delta t}\left(\frac{z}{L_z}\right)^3$$

$$L_? \equiv \text{length of the PML}$$

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Maxwell's Equations with a UPML

$$\nabla \times \tilde{\vec{E}}(\omega) = -j\omega \frac{[\mu_r]}{c_0}[S]\vec{H}(\omega) \quad \Rightarrow$$

$$j\omega\left(1+\frac{\sigma_x'}{j\omega\varepsilon_0}\right)^{-1}\left(1+\frac{\sigma_y'}{j\omega\varepsilon_0}\right)\left(1+\frac{\sigma_z'}{j\omega\varepsilon_0}\right)H_x(\omega) = -\frac{c_0}{\mu_{xx}}\left[\frac{\partial \tilde{E}_z(\omega)}{\partial y} - \frac{\partial \tilde{E}_y(\omega)}{\partial z}\right]$$

$$j\omega\left(1+\frac{\sigma_x'}{j\omega\varepsilon_0}\right)\left(1+\frac{\sigma_y'}{j\omega\varepsilon_0}\right)^{-1}\left(1+\frac{\sigma_z'}{j\omega\varepsilon_0}\right)H_y(\omega) = -\frac{c_0}{\mu_{yy}}\left[\frac{\partial \tilde{E}_x(\omega)}{\partial z} - \frac{\partial \tilde{E}_z(\omega)}{\partial x}\right]$$

$$j\omega\left(1+\frac{\sigma_x'}{j\omega\varepsilon_0}\right)\left(1+\frac{\sigma_y'}{j\omega\varepsilon_0}\right)\left(1+\frac{\sigma_z'}{j\omega\varepsilon_0}\right)^{-1}H_z(\omega) = -\frac{c_0}{\mu_{zz}}\left[\frac{\partial \tilde{E}_y(\omega)}{\partial x} - \frac{\partial \tilde{E}_x(\omega)}{\partial y}\right]$$

$$\nabla \times \vec{H}(\omega) = \eta_0\sigma\tilde{\vec{E}}(\omega) + \frac{j\omega}{c_0}[S]\tilde{\vec{D}}(\omega) \quad \Rightarrow$$

$$j\omega\left(1+\frac{\sigma_x'}{j\omega\varepsilon_0}\right)^{-1}\left(1+\frac{\sigma_y'}{j\omega\varepsilon_0}\right)\left(1+\frac{\sigma_z'}{j\omega\varepsilon_0}\right)\tilde{D}_x(\omega) = c_0\left[\frac{\partial H_z(\omega)}{\partial y} - \frac{\partial H_y(\omega)}{\partial z}\right] - \frac{\sigma_{xx}}{\varepsilon_0}\tilde{E}_x(\omega)$$

$$j\omega\left(1+\frac{\sigma_x'}{j\omega\varepsilon_0}\right)\left(1+\frac{\sigma_y'}{j\omega\varepsilon_0}\right)^{-1}\left(1+\frac{\sigma_z'}{j\omega\varepsilon_0}\right)\tilde{D}_y(\omega) = c_0\left[\frac{\partial H_x(\omega)}{\partial z} - \frac{\partial H_z(\omega)}{\partial x}\right] - \frac{\sigma_{yy}}{\varepsilon_0}\tilde{E}_y(\omega)$$

$$j\omega\left(1+\frac{\sigma_x'}{j\omega\varepsilon_0}\right)\left(1+\frac{\sigma_y'}{j\omega\varepsilon_0}\right)\left(1+\frac{\sigma_z'}{j\omega\varepsilon_0}\right)^{-1}\tilde{D}_z(\omega) = c_0\left[\frac{\partial H_y(\omega)}{\partial x} - \frac{\partial H_x(\omega)}{\partial y}\right] - \frac{\sigma_{zz}}{\varepsilon_0}\tilde{E}_z(\omega)$$

$$\tilde{\vec{D}}(\omega) = [\varepsilon_r]\tilde{\vec{E}}(\omega) \quad \Rightarrow$$

$$\tilde{D}_x(\omega) = \varepsilon_{xx}\tilde{E}_x(\omega)$$

$$\tilde{D}_y(\omega) = \varepsilon_{yy}\tilde{E}_y(\omega)$$

$$\tilde{D}_z(\omega) = \varepsilon_{zz}\tilde{E}_z(\omega)$$

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Conversion to the Time-Domain

Assume No Conductivity (σ=0)

$$\nabla \times \tilde{\vec{E}}(\omega) = -j\omega \frac{[\mu_r]}{c_0}[s]\vec{H}(\omega) \quad \Rightarrow \quad \begin{aligned} j\omega\left(1+\frac{\sigma_x'}{j\omega\varepsilon_0}\right)^{-1}\left(1+\frac{\sigma_y'}{j\omega\varepsilon_0}\right)\left(1+\frac{\sigma_z'}{j\omega\varepsilon_0}\right)H_x(\omega) &= -\frac{c_0}{\mu_{xx}}\left[\frac{\partial \tilde{E}_z(\omega)}{\partial y} - \frac{\partial \tilde{E}_y(\omega)}{\partial z}\right] \\ j\omega\left(1+\frac{\sigma_x'}{j\omega\varepsilon_0}\right)\left(1+\frac{\sigma_y'}{j\omega\varepsilon_0}\right)^{-1}\left(1+\frac{\sigma_z'}{j\omega\varepsilon_0}\right)H_y(\omega) &= -\frac{c_0}{\mu_{yy}}\left[\frac{\partial \tilde{E}_x(\omega)}{\partial z} - \frac{\partial \tilde{E}_z(\omega)}{\partial x}\right] \\ j\omega\left(1+\frac{\sigma_x'}{j\omega\varepsilon_0}\right)\left(1+\frac{\sigma_y'}{j\omega\varepsilon_0}\right)\left(1+\frac{\sigma_z'}{j\omega\varepsilon_0}\right)^{-1}H_z(\omega) &= -\frac{c_0}{\mu_{zz}}\left[\frac{\partial \tilde{E}_y(\omega)}{\partial x} - \frac{\partial \tilde{E}_x(\omega)}{\partial y}\right] \end{aligned}$$

$$\nabla \times \vec{H}(\omega) = \eta_0\sigma\tilde{\vec{E}}(\omega) + \frac{j\omega}{c_0}[s]\tilde{\vec{D}}(\omega) \quad \Rightarrow \quad \begin{aligned} j\omega\left(1+\frac{\sigma_x'}{j\omega\varepsilon_0}\right)^{-1}\left(1+\frac{\sigma_y'}{j\omega\varepsilon_0}\right)\left(1+\frac{\sigma_z'}{j\omega\varepsilon_0}\right)\tilde{D}_x(\omega) &= c_0\left[\frac{\partial H_z(\omega)}{\partial y} - \frac{\partial H_y(\omega)}{\partial z}\right] \\ j\omega\left(1+\frac{\sigma_x'}{j\omega\varepsilon_0}\right)\left(1+\frac{\sigma_y'}{j\omega\varepsilon_0}\right)^{-1}\left(1+\frac{\sigma_z'}{j\omega\varepsilon_0}\right)\tilde{D}_y(\omega) &= c_0\left[\frac{\partial H_x(\omega)}{\partial z} - \frac{\partial H_z(\omega)}{\partial x}\right] \\ j\omega\left(1+\frac{\sigma_x'}{j\omega\varepsilon_0}\right)\left(1+\frac{\sigma_y'}{j\omega\varepsilon_0}\right)\left(1+\frac{\sigma_z'}{j\omega\varepsilon_0}\right)^{-1}\tilde{D}_z(\omega) &= c_0\left[\frac{\partial H_y(\omega)}{\partial x} - \frac{\partial H_x(\omega)}{\partial y}\right] \end{aligned}$$

$$\tilde{\vec{D}}(\omega) = [\varepsilon_r]\tilde{\vec{E}}(\omega) \quad \Rightarrow \quad \begin{aligned} \tilde{D}_x(\omega) &= \varepsilon_{xx}\tilde{E}_x(\omega) \\ \tilde{D}_y(\omega) &= \varepsilon_{yy}\tilde{E}_y(\omega) \\ \tilde{D}_z(\omega) &= \varepsilon_{zz}\tilde{E}_z(\omega) \end{aligned}$$

Recall Fourier Transform Properties

$$\mathbf{F}\left\{g(t)\right\} = G(\omega)$$

$$\mathbf{F}\left\{a g(t)\right\} = aG(\omega)$$

$$\mathbf{F}\left\{\frac{d^a}{dt^a} g(t)\right\} = (j\omega)^a G(\omega)$$

$$\mathbf{F}\left\{\int_{-\infty}^{t} g(\tau)\, d\tau\right\} = \frac{1}{j\omega} G(\omega)$$

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Prepare Maxwell's Equations for Conversion (1 of 2)

Start with the following equation. The procedure for all other equations has the same steps.

$$j\omega\left(1+\frac{\sigma'_x}{j\omega\varepsilon_0}\right)^{-1}\left(1+\frac{\sigma'_y}{j\omega\varepsilon_0}\right)\left(1+\frac{\sigma'_z}{j\omega\varepsilon_0}\right)H_x(\omega) = -\frac{c_0}{\mu_{xx}}\left[\frac{\partial \tilde{E}_z(\omega)}{\partial y} - \frac{\partial \tilde{E}_y(\omega)}{\partial z}\right]$$

First, let

$$C_x^E(\omega) = \frac{\partial \tilde{E}_z(\omega)}{\partial y} - \frac{\partial \tilde{E}_y(\omega)}{\partial z}$$

The equation can now be written more compactly as

$$j\omega\left(1+\frac{\sigma'_x}{j\omega\varepsilon_0}\right)^{-1}\left(1+\frac{\sigma'_y}{j\omega\varepsilon_0}\right)\left(1+\frac{\sigma'_z}{j\omega\varepsilon_0}\right)H_x(\omega) = -\frac{c_0}{\mu_{xx}} C_x^E(\omega)$$

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Prepare Maxwell's Equations for Conversion (2 of 2)

The inverse term on the left is brought to the right.

$$j\omega\left(1+\frac{\sigma'_x}{j\omega\varepsilon_0}\right)^{-1}\left(1+\frac{\sigma'_y}{j\omega\varepsilon_0}\right)\left(1+\frac{\sigma'_z}{j\omega\varepsilon_0}\right)H_x(\omega)=-\frac{c_0}{\mu_{xx}}C_x^E(\omega)$$

$$\downarrow$$

$$j\omega\left(1+\frac{\sigma'_y}{j\omega\varepsilon_0}\right)\left(1+\frac{\sigma'_z}{j\omega\varepsilon_0}\right)H_x(\omega)=-\frac{c_0}{\mu_{xx}}\left(1+\frac{\sigma'_x}{j\omega\varepsilon_0}\right)C_x^E(\omega)$$

The equation is then expanded by multiplying all of the terms.

$$j\omega H_x(\omega)+\frac{\sigma'_y+\sigma'_z}{\varepsilon_0}H_x(\omega)+\frac{1}{j\omega}\frac{\sigma'_y\sigma'_z}{\varepsilon_0^2}H_x(\omega)=-\frac{c_0}{\mu_{xx}}C_x^E(\omega)-\frac{1}{j\omega}\frac{c_0\sigma'_x}{\varepsilon_0\mu_{xx}}C_x^E(\omega)$$

For easiest conversion to the time-domain, all the terms should be multiplied by $(j\omega)^a$.

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Convert Each Term to Time-Domain Individually

$$j\omega H_x(\omega)+\frac{\sigma'_y+\sigma'_z}{\varepsilon_0}H_x(\omega)+\frac{1}{j\omega}\frac{\sigma'_y\sigma'_z}{\varepsilon_0^2}H_x(\omega)=-\frac{c_0}{\mu_{xx}}C_x^E(\omega)-\frac{1}{j\omega}\frac{c_0\sigma'_x}{\varepsilon_0\mu_{xx}}C_x^E(\omega)$$

$$j\omega H_x(\omega)\Leftrightarrow\frac{\partial H_x(t)}{\partial t}$$

$$\frac{\sigma'_y+\sigma'_z}{\varepsilon_0}H_x(\omega)\Leftrightarrow\frac{\sigma'_y+\sigma'_z}{\varepsilon_0}H_x(t)$$

$$\frac{1}{j\omega}\frac{\sigma'_y\sigma'_z}{\varepsilon_0^2}H_x(\omega)\Leftrightarrow\int_{-\infty}^{t}\frac{\sigma'_y\sigma'_z}{\varepsilon_0^2}H_x(\tau)\,d\tau$$

$$\frac{c_0}{\mu_{xx}}C_x^E(\omega)\Leftrightarrow\frac{c_0}{\mu_{xx}}C_x^E(t)$$

$$\frac{1}{j\omega}\frac{c_0\sigma'_x}{\varepsilon_0\mu_{xx}}C_x^E(\omega)\Leftrightarrow\int_{-\infty}^{t}\frac{c_0\sigma'_x}{\varepsilon_0\mu_{xx}}C_x^E(\tau)\,d\tau$$

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The Time-Domain Equation with UPML

$$j\omega H_x(\omega)+\frac{\sigma_y'+\sigma_z'}{\varepsilon_0}H_x(\omega)+\frac{1}{j\omega}\frac{\sigma_y'\sigma_z'}{\varepsilon_0^2}H_x(\omega)=-\frac{c_0}{\mu_{xx}}C_x^E(\omega)-\frac{1}{j\omega}\frac{c_0\sigma_x'}{\varepsilon_0\mu_{xx}}C_x^E(\omega)$$

$$\frac{\partial H_x(t)}{\partial t}+\frac{\sigma_y'+\sigma_z'}{\varepsilon_0}H_x(t)+\int_{-\infty}^{t}\frac{\sigma_y'\sigma_z'}{\varepsilon_0^2}H_x(\tau)d\tau=-\frac{c_0}{\mu_{xx}}C_x^E(t)-\int_{-\infty}^{t}\frac{c_0\sigma_x'}{\varepsilon_0\mu_{xx}}C_x^E(\tau)d\tau$$

$$\begin{aligned}\frac{\partial H_x(t)}{\partial t}+\frac{\sigma_y'+\sigma_z'}{\varepsilon_0}H_x(t)+\int_{-\infty}^{t}\frac{\sigma_y'\sigma_z'}{\varepsilon_0^2}H_x(\tau)d\tau\\=-\frac{c_0}{\mu_{xx}}C_x^E(t)-\int_{-\infty}^{t}\frac{c_0\sigma_x'}{\varepsilon_0\mu_{xx}}C_x^E(\tau)d\tau\end{aligned}$$

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Summary of All Time-Domain Equations

$$\frac{\partial}{\partial t}H_x(t)+\frac{\sigma_y'+\sigma_z'}{\varepsilon_0}H_x(t)+\frac{\sigma_y'\sigma_z'}{\varepsilon_0^2}\int_{-\infty}^{t}H_x(\tau)d\tau=-\frac{c_0}{\mu_{xx}}\left[\frac{\partial\tilde{E}_z(t)}{\partial y}-\frac{\partial\tilde{E}_y(t)}{\partial z}\right]-\frac{c_0\sigma_x'}{\varepsilon_0\mu_{xx}}\int_{-\infty}^{t}\left[\frac{\partial\tilde{E}_z(\tau)}{\partial y}-\frac{\partial\tilde{E}_y(\tau)}{\partial z}\right]d\tau$$

$$\frac{\partial}{\partial t}H_y(t)+\frac{\sigma_x'+\sigma_z'}{\varepsilon_0}H_y(t)+\frac{\sigma_x'\sigma_z'}{\varepsilon_0^2}\int_{-\infty}^{t}H_y(\tau)d\tau=-\frac{c_0}{\mu_{yy}}\left[\frac{\partial\tilde{E}_x(t)}{\partial z}-\frac{\partial\tilde{E}_z(t)}{\partial x}\right]-\frac{c_0\sigma_y'}{\varepsilon_0\mu_{yy}}\int_{-\infty}^{t}\left[\frac{\partial\tilde{E}_x(\tau)}{\partial z}-\frac{\partial\tilde{E}_z(\tau)}{\partial x}\right]d\tau$$

$$\frac{\partial}{\partial t}H_z(t)+\frac{\sigma_x'+\sigma_y'}{\varepsilon_0}H_z(t)+\frac{\sigma_x'\sigma_y'}{\varepsilon_0^2}\int_{-\infty}^{t}H_z(\tau)d\tau=-\frac{c_0}{\mu_{zz}}\left[\frac{\partial\tilde{E}_y(t)}{\partial x}-\frac{\partial\tilde{E}_x(t)}{\partial y}\right]-\frac{c_0\sigma_z'}{\varepsilon_0\mu_{zz}}\int_{-\infty}^{t}\left[\frac{\partial\tilde{E}_y(\tau)}{\partial x}-\frac{\partial\tilde{E}_x(\tau)}{\partial y}\right]d\tau$$

$$\frac{\partial}{\partial t}\tilde{D}_x(t)+\frac{\sigma_y'+\sigma_z'}{\varepsilon_0}\tilde{D}_x(t)+\frac{\sigma_y'\sigma_z'}{\varepsilon_0^2}\int_{-\infty}^{t}\tilde{D}_x(\tau)d\tau=c_0\left[\frac{\partial H_z(t)}{\partial y}-\frac{\partial H_y(t)}{\partial z}\right]+\frac{c_0\sigma_x'}{\varepsilon_0}\int_{-\infty}^{t}\left[\frac{\partial H_z(\tau)}{\partial y}-\frac{\partial H_y(\tau)}{\partial z}\right]d\tau$$

$$\frac{\partial}{\partial t}\tilde{D}_y(t)+\frac{\sigma_x'+\sigma_z'}{\varepsilon_0}\tilde{D}_y(t)+\frac{\sigma_x'\sigma_z'}{\varepsilon_0^2}\int_{-\infty}^{t}\tilde{D}_y(\tau)d\tau=c_0\left[\frac{\partial H_x(t)}{\partial z}-\frac{\partial H_z(t)}{\partial x}\right]+\frac{c_0\sigma_y'}{\varepsilon_0}\int_{-\infty}^{t}\left[\frac{\partial H_x(\tau)}{\partial z}-\frac{\partial H_z(\tau)}{\partial x}\right]d\tau$$

$$\frac{\partial}{\partial t}\tilde{D}_z(t)+\frac{\sigma_x'+\sigma_y'}{\varepsilon_0}\tilde{D}_z(t)+\frac{\sigma_x'\sigma_y'}{\varepsilon_0^2}\int_{-\infty}^{t}\tilde{D}_z(\tau)d\tau=c_0\left[\frac{\partial H_y(t)}{\partial x}-\frac{\partial H_x(t)}{\partial y}\right]+\frac{c_0\sigma_z'}{\varepsilon_0}\int_{-\infty}^{t}\left[\frac{\partial H_y(\tau)}{\partial x}-\frac{\partial H_x(\tau)}{\partial y}\right]d\tau$$

$$\tilde{D}_x(t)=\varepsilon_{xx}\tilde{E}_x(t)$$
$$\tilde{D}_y(t)=\varepsilon_{yy}\tilde{E}_y(t)$$
$$\tilde{D}_z(t)=\varepsilon_{zz}\tilde{E}_z(t)$$

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Numerical Approximations

Handle Each Term Separately

The first governing equation in the time-domain was

$$\frac{\partial H_x(t)}{\partial t}+\frac{\sigma'_y+\sigma'_z}{\varepsilon_0}H_x(t)+\int_{-\infty}^{t}\frac{\sigma'_y\sigma'_z}{\varepsilon_0^2}H_x(\tau)\,d\tau=-\frac{c_0}{\mu_{xx}}C_x^E(t)-\int_{-\infty}^{t}\frac{c_0\sigma'_x}{\varepsilon_0\mu_{xx}}C_x^E(\tau)\,d\tau$$

The terms listed separately are:

$$\text{Term 1:}\quad \frac{\partial H_x(t)}{\partial t}$$

$$\text{Term 2:}\quad \frac{\sigma'_y+\sigma'_z}{\varepsilon_0}H_x(t)$$

$$\text{Term 3:}\quad \int_{-\infty}^{t}\frac{\sigma'_y\sigma'_z}{\varepsilon_0^2}H_x(\tau)\,d\tau$$

$$\text{Term 4:}\quad -\frac{c_0}{\mu_{xx}}C_x^E(t)$$

$$\text{Term 5:}\quad -\int_{-\infty}^{t}\frac{c_0\sigma'_x}{\varepsilon_0\mu_{xx}}C_x^E(\tau)\,d\tau$$

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Term 1

This term was approximated previously using finite-differences as follows.

$$\frac{\partial H_x(t)}{\partial t} \cong \frac{H_x\big|_{t+\Delta t/2}^{i,j,k} - H_x\big|_{t-\Delta t/2}^{i,j,k}}{\Delta t}$$

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Term 2

The trick here is that H_x has to be interpolated at time t. This is accomplished by averaging the values at time $t + \Delta t/2$ and $t - \Delta t/2$.

$$\frac{\sigma_y^H + \sigma_z'^H}{\varepsilon_0} H_x(t) \cong \frac{\sigma_y^H\big|^{i,j,k} + \sigma_z^H\big|^{i,j,k}}{\varepsilon_0} \frac{H_x\big|_{t+\Delta t/2}^{i,j,k} + H_x\big|_{t-\Delta t/2}^{i,j,k}}{2}$$

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Term 3 (1 of 3)

Approximate the integral in this equation with a summation.

$$\int_{-\infty}^{t} \frac{\sigma'_y \sigma'_z}{\varepsilon_0^2} H_x(\tau)\, d\tau = \frac{\sigma'_y \sigma'_z}{\varepsilon_0^2} \int_{-\infty}^{t} H_x(\tau)\, d\tau$$

$$\cong \frac{\sigma_y^H \sigma_z^H}{\varepsilon_0^2} \sum_{T=\Delta t/2}^{t+\Delta t/2} H_x\big|_T^{i,j,k} \Delta t$$

There is a problem. The summation includes up to $t + \Delta t/2$, but the integral only goes up to t. This accidentally integrates too far. This happened because the summation is integrating a term that exists at the half time steps.

Term 3 (2 of 3)

Pull out the last term in the summation.

$$\int_{-\infty}^{t} \frac{\sigma'_y \sigma'_z}{\varepsilon_0^2} H_x(\tau)\, d\tau \cong \frac{\sigma_y^H \sigma_z^H}{\varepsilon_0^2} \sum_{T=\Delta t/2}^{t+\Delta t/2} H_x\big|_T^{i,j,k} \Delta t$$

$$\cong \frac{\sigma_y^H \sigma_z^H}{\varepsilon_0^2} \left(H_x\big|_{t+\Delta t/2}^{i,j,k} \frac{\Delta t}{2} + \sum_{T=\Delta t/2}^{t-\Delta t/2} H_x\big|_T^{i,j,k} \Delta t \right)$$

Force the extracted term to only integrate over half of a time step.

$$\int_{-\infty}^{t} \frac{\sigma'_y \sigma'_z}{\varepsilon_0^2} H_x(\tau)\, d\tau \cong \frac{\sigma_y^H \sigma_z^H}{\varepsilon_0^2} \left(\underbrace{\frac{H_x\big|_{t+\Delta t/2}^{i,j,k} + H_x\big|_{t-\Delta t/2}^{i,j,k}}{2}}_{H \text{ field interpolated at time } t} \frac{\Delta t}{2} + \sum_{T=\Delta t/2}^{t-\Delta t/2} H_x\big|_T^{i,j,k} \Delta t \right)$$

$\frac{\Delta t}{2}$: Half time step.

Term 3 (3 of 3)

Factor out the Δt term.

$$\int_{-\infty}^{t} \frac{\sigma'_y \sigma'_z}{\varepsilon_0^2} H_x(\tau)\, d\tau \cong \frac{\sigma_y^H \sigma_z^H \Delta t}{\varepsilon_0^2} \left(\frac{H_x\big|_{t-\Delta t/2}^{i,j,k} + H_x\big|_{t+\Delta t/2}^{i,j,k}}{4} + \sum_{T=\Delta t/2}^{t-\Delta t/2} H_x\big|_{T}^{i,j,k} \right)$$

★ The end-time on this summation implies that the summation will be update before updating H_x.

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Term 4

This term was approximated using finite-differences in a previous lecture.

$$-\frac{c_0}{\mu_{xx}} C_x^E(t) \cong -\frac{c_0}{\mu_{xx}\big|^{i,j,k}} C_x^E\big|_t^{i,j,k}$$

Recall that the curl term is

$$C_x^E\big|_t^{i,j,k} = \frac{\tilde{E}_z\big|_t^{i,j+1,k} - \tilde{E}_z\big|_t^{i,j,k}}{\Delta y} - \frac{\tilde{E}_y\big|_t^{i,j,k+1} - \tilde{E}_y\big|_t^{i,j,k}}{\Delta z}$$

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Term 5

The integral in this equation is approximated with a summation.

$$-\int_{-\infty}^{t}\frac{c_0\sigma'_x}{\varepsilon_0\mu_{xx}}C_x^E(\tau)d\tau=-\frac{c_0\sigma'_x}{\varepsilon_0\mu_{xx}}\int_{-\infty}^{t}C_x^E(\tau)d\tau$$

$$\cong-\frac{c_0\,\sigma_x^H\big|^{i,j,k}}{\varepsilon_0\,\mu_{xx}\big|^{i,j,k}}\sum_{T=0}^{t}C_x^E\big|_T^{i,j,k}\,\Delta t$$

The end-time on this summation implies that the summation should be updated before updating H_x.

$$\cong-\frac{c_0\Delta t\,\sigma_x^H\big|^{i,j,k}}{\varepsilon_0\,\mu_{xx}\big|^{i,j,k}}\sum_{T=0}^{t}C_x^E\big|_T^{i,j,k}$$

There are no problems with this summation because it is integrating a term that exists at integer time steps.

Putting it All Together

Putting all the terms together, the equation is

$$\frac{\partial H_x(t)}{\partial t}+\frac{\sigma'_y+\sigma'_z}{\varepsilon_0}H_x(t)+\int_{-\infty}^{t}\frac{\sigma'_y\sigma'_z}{\varepsilon_0^2}H_x(\tau)d\tau=-\frac{c_0}{\mu_{xx}}C_x^E(t)-\int_{-\infty}^{t}\frac{c_0\sigma'_x}{\varepsilon_0\mu_{xx}}C_x^E(\tau)d\tau$$

$$\Downarrow$$

$$\frac{H_x\big|_{t+\Delta t/2}^{i,j,k}-H_x\big|_{t-\Delta t/2}^{i,j,k}}{\Delta t}+\frac{\sigma_y^H\big|^{i,j,k}+\sigma_z^H\big|^{i,j,k}}{\varepsilon_0}\frac{H_x\big|_{t+\Delta t/2}^{i,j,k}+H_x\big|_{t-\Delta t/2}^{i,j,k}}{2}+\frac{\sigma_y^H\sigma_z^H\Delta t}{\varepsilon_0^2}\left(\frac{H_x\big|_{t-\Delta t/2}^{i,j,k}+H_x\big|_{t+\Delta t/2}^{i,j,k}}{4}+\sum_{T=\Delta t/2}^{t-\Delta t/2}H_x\big|_T^{i,j,k}\right)$$

$$=-\frac{c_0}{\mu_{xx}\big|^{i,j,k}}C_x^E\big|_t^{i,j,k}-\frac{c_0\Delta t\,\sigma_x^H\big|^{i,j,k}}{\varepsilon_0\,\mu_{xx}\big|^{i,j,k}}\sum_{T=0}^{t}C_x^E\big|_T^{i,j,k}$$

Summary of All Numerical Equations

$$\frac{H_x\big|_{t+\Delta t/2}^{i,j,k} - H_x\big|_{t-\Delta t/2}^{i,j,k}}{\Delta t} + \frac{\sigma_y^H\big|^{i,j,k} + \sigma_z^H\big|^{i,j,k}}{\varepsilon_0}\left(\frac{H_x\big|_{t+\frac{\Delta t}{2}}^{i,j,k} + H_x\big|_{t-\frac{\Delta t}{2}}^{i,j,k}}{2}\right) + \frac{\left(\sigma_y^H\big|^{i,j,k}\right)\left(\sigma_z^H\big|^{i,j,k}\right)\Delta t}{\varepsilon_0^2}\left[\frac{1}{4}\left(H_x\big|_{t+\frac{\Delta t}{2}}^{i,j,k} + H_x\big|_{t-\frac{\Delta t}{2}}^{i,j,k}\right) + \sum_{T=\Delta t/2}^{t-\frac{\Delta t}{2}} H_x\big|_T^{i,j,k}\right] = -\frac{c_0}{\mu_{xx}\big|^{i,j,k}} C_x^E\big|_t^{i,j,k} - \frac{c_0 \Delta t\, \sigma_x^H\big|^{i,j,k}}{\varepsilon_0\, \mu_{xx}\big|^{i,j,k}} \sum_{T=0}^{t} C_x^E\big|_T^{i,j,k}$$

$$\frac{H_y\big|_{t+\Delta t/2}^{i,j,k} - H_y\big|_{t-\Delta t/2}^{i,j,k}}{\Delta t} + \frac{\sigma_x^H\big|^{i,j,k} + \sigma_z^H\big|^{i,j,k}}{\varepsilon_0}\left(\frac{H_y\big|_{t+\frac{\Delta t}{2}}^{i,j,k} + H_y\big|_{t-\frac{\Delta t}{2}}^{i,j,k}}{2}\right) + \frac{\left(\sigma_x^H\big|^{i,j,k}\right)\left(\sigma_z^H\big|^{i,j,k}\right)\Delta t}{\varepsilon_0^2}\left[\frac{1}{4}\left(H_y\big|_{t+\frac{\Delta t}{2}}^{i,j,k} + H_y\big|_{t-\frac{\Delta t}{2}}^{i,j,k}\right) + \sum_{T=\Delta t/2}^{t-\frac{\Delta t}{2}} H_y\big|_T^{i,j,k}\right] = -\frac{c_0}{\mu_{yy}\big|^{i,j,k}} C_y^E\big|_t^{i,j,k} - \frac{c_0 \Delta t\, \sigma_y^H\big|^{i,j,k}}{\varepsilon_0\, \mu_{yy}\big|^{i,j,k}} \sum_{T=0}^{t} C_y^E\big|_T^{i,j,k}$$

$$\frac{H_z\big|_{t+\Delta t/2}^{i,j,k} - H_z\big|_{t-\Delta t/2}^{i,j,k}}{\Delta t} + \frac{\sigma_x^H\big|^{i,j,k} + \sigma_y^H\big|^{i,j,k}}{\varepsilon_0}\left(\frac{H_z\big|_{t+\frac{\Delta t}{2}}^{i,j,k} + H_z\big|_{t-\frac{\Delta t}{2}}^{i,j,k}}{2}\right) + \frac{\left(\sigma_x^H\big|^{i,j,k}\right)\left(\sigma_y^H\big|^{i,j,k}\right)\Delta t}{\varepsilon_0^2}\left[\frac{1}{4}\left(H_z\big|_{t+\frac{\Delta t}{2}}^{i,j,k} + H_z\big|_{t-\frac{\Delta t}{2}}^{i,j,k}\right) + \sum_{T=\Delta t/2}^{t-\frac{\Delta t}{2}} H_z\big|_T^{i,j,k}\right] = -\frac{c_0}{\mu_{zz}\big|^{i,j,k}} C_z^E\big|_t^{i,j,k} - \frac{c_0 \Delta t\, \sigma_z^H\big|^{i,j,k}}{\varepsilon_0\, \mu_{zz}\big|^{i,j,k}} \sum_{T=0}^{t} C_z^E\big|_T^{i,j,k}$$

$$\frac{\tilde{D}_x\big|_{t+\Delta t}^{i,j,k} - \tilde{D}_x\big|_t^{i,j,k}}{\Delta t} + \frac{\sigma_y^D\big|^{i,j,k} + \sigma_z^D\big|^{i,j,k}}{\varepsilon_0}\left(\frac{\tilde{D}_x\big|_{t+\Delta t}^{i,j,k} + \tilde{D}_x\big|_t^{i,j,k}}{2}\right) + \frac{\left(\sigma_y^D\big|^{i,j,k}\right)\left(\sigma_z^D\big|^{i,j,k}\right)\Delta t}{\varepsilon_0^2}\left[\frac{1}{4}\left(\tilde{D}_x\big|_{t+\Delta t}^{i,j,k} + \tilde{D}_x\big|_t^{i,j,k}\right) + \sum_{T=0}^{t} \tilde{D}_x\big|_T^{i,j,k}\right] = c_0\, C_x^H\big|_t^{i,j,k} + \frac{c_0 \Delta t\, \sigma_x^D\big|^{i,j,k}}{\varepsilon_0} \sum_{T=\Delta t/2}^{t-\frac{\Delta t}{2}} C_x^H\big|_T^{i,j,k}$$

$$\frac{\tilde{D}_y\big|_{t+\Delta t}^{i,j,k} - \tilde{D}_y\big|_t^{i,j,k}}{\Delta t} + \frac{\sigma_x^D\big|^{i,j,k} + \sigma_z^D\big|^{i,j,k}}{\varepsilon_0}\left(\frac{\tilde{D}_y\big|_{t+\Delta t}^{i,j,k} + \tilde{D}_y\big|_t^{i,j,k}}{2}\right) + \frac{\left(\sigma_x^D\big|^{i,j,k}\right)\left(\sigma_z^D\big|^{i,j,k}\right)\Delta t}{\varepsilon_0^2}\left[\frac{1}{4}\left(\tilde{D}_y\big|_{t+\Delta t}^{i,j,k} + \tilde{D}_y\big|_t^{i,j,k}\right) + \sum_{T=0}^{t} \tilde{D}_y\big|_T^{i,j,k}\right] = c_0\, C_y^H\big|_t^{i,j,k} + \frac{c_0 \Delta t\, \sigma_y^D\big|^{i,j,k}}{\varepsilon_0} \sum_{T=\Delta t/2}^{t-\frac{\Delta t}{2}} C_y^H\big|_T^{i,j,k}$$

$$\frac{\tilde{D}_z\big|_{t+\Delta t}^{i,j,k} - \tilde{D}_z\big|_t^{i,j,k}}{\Delta t} + \frac{\sigma_x^D\big|^{i,j,k} + \sigma_y^D\big|^{i,j,k}}{\varepsilon_0}\left(\frac{\tilde{D}_z\big|_{t+\Delta t}^{i,j,k} + \tilde{D}_z\big|_t^{i,j,k}}{2}\right) + \frac{\left(\sigma_x^D\big|^{i,j,k}\right)\left(\sigma_y^D\big|^{i,j,k}\right)\Delta t}{\varepsilon_0^2}\left[\frac{1}{4}\left(\tilde{D}_z\big|_{t+\Delta t}^{i,j,k} + \tilde{D}_z\big|_t^{i,j,k}\right) + \sum_{T=0}^{t} \tilde{D}_z\big|_T^{i,j,k}\right] = c_0\, C_z^H\big|_t^{i,j,k} + \frac{c_0 \Delta t\, \sigma_z^D\big|^{i,j,k}}{\varepsilon_0} \sum_{T=\Delta t/2}^{t-\frac{\Delta t}{2}} C_z^H\big|_T^{i,j,k}$$

$$\tilde{D}_x\big|_{t+\Delta t}^{i,j,k} = \left(\varepsilon_{xx}\big|^{i,j,k}\right)\tilde{E}_x\big|_{t+\Delta t}^{i,j,k}$$

$$\tilde{D}_y\big|_{t+\Delta t}^{i,j,k} = \left(\varepsilon_{yy}\big|^{i,j,k}\right)\tilde{E}_y\big|_{t+\Delta t}^{i,j,k}$$

$$\tilde{D}_z\big|_{t+\Delta t}^{i,j,k} = \left(\varepsilon_{zz}\big|^{i,j,k}\right)\tilde{E}_z\big|_{t+\Delta t}^{i,j,k}$$

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Derivation of the Update Equations

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The Starting Point

Just as before, the numerical equation is solved for the future time value of H_x. This term appears three times so there is more algebra to perform in order to collect the terms.

$$\frac{H_x\big|_{t+\Delta t/2}^{i,j,k} - H_x\big|_{t-\Delta t/2}^{i,j,k}}{\Delta t} + \frac{\sigma_y^H\big|^{i,j,k} + \sigma_z^H\big|^{i,j,k}}{\varepsilon_0}\left(\frac{H_x\big|_{t+\Delta t/2}^{i,j,k} + H_x\big|_{t-\Delta t/2}^{i,j,k}}{2}\right)$$

$$+\frac{\left(\sigma_y^H\big|^{i,j,k}\right)\left(\sigma_z^H\big|^{i,j,k}\right)\Delta t}{\varepsilon_0^2}\left[\frac{1}{4}\left(H_x\big|_{t+\Delta t/2}^{i,j,k} + H_x\big|_{t-\Delta t/2}^{i,j,k}\right) + \sum_{T=\Delta t/2}^{t-\frac{\Delta t}{2}} H_x\big|_T^{i,j,k}\right]$$

$$= -\frac{c_0}{\mu_{xx}\big|^{i,j,k}} C_x^E\big|_t^{i,j,k} - \frac{c_0 \Delta t}{\varepsilon_0}\frac{\sigma_x^H\big|^{i,j,k}}{\mu_{xx}\big|^{i,j,k}}\sum_{T=0}^{t} C_x^E\big|_t^{i,j,k}$$

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Expand the Equation

Start by expanding the equation by multiplying all of the terms.

$$\frac{H_x\big|_{t+\Delta t/2}^{i,j,k} - H_x\big|_{t-\Delta t/2}^{i,j,k}}{\Delta t} + \frac{\sigma_y^H\big|^{i,j,k} + \sigma_z^H\big|^{i,j,k}}{\varepsilon_0}\left(\frac{H_x\big|_{t+\frac{\Delta t}{2}}^{i,j,k} + H_x\big|_{t-\frac{\Delta t}{2}}^{i,j,k}}{2}\right) + \frac{\left(\sigma_y^H\big|^{i,j,k}\right)\left(\sigma_z^H\big|^{i,j,k}\right)\Delta t}{\varepsilon_0^2}\left[\frac{1}{4}\left(H_x\big|_{t+\frac{\Delta t}{2}}^{i,j,k} + H_x\big|_{t-\frac{\Delta t}{2}}^{i,j,k}\right) + \sum_{T=\Delta t/2}^{t-\frac{\Delta t}{2}} H_x\big|_T^{i,j,k}\right]$$

$$= -\frac{c_0}{\mu_{xx}\big|^{i,j,k}} C_x^E\big|_t^{i,j,k} - \frac{c_0 \Delta t}{\varepsilon_0}\frac{\sigma_x^H\big|^{i,j,k}}{\mu_{xx}\big|^{i,j,k}}\sum_{T=0}^{t} C_x^E\big|_t^{i,j,k}$$

$$\Downarrow$$

$$\frac{1}{\Delta t}H_x\big|_{t+\Delta t/2}^{i,j,k} - \frac{1}{\Delta t}H_x\big|_{t-\Delta t/2}^{i,j,k} + \left(\frac{\sigma_y^H\big|^{i,j,k} + \sigma_z^H\big|^{i,j,k}}{2\varepsilon_0}\right)H_x\big|_{t+\frac{\Delta t}{2}}^{i,j,k} + \left(\frac{\sigma_y^H\big|^{i,j,k} + \sigma_z^H\big|^{i,j,k}}{2\varepsilon_0}\right)H_x\big|_{t-\frac{\Delta t}{2}}^{i,j,k}$$

$$+\frac{\left(\sigma_y^H\big|^{i,j,k}\right)\left(\sigma_z^H\big|^{i,j,k}\right)\Delta t}{4\varepsilon_0^2}H_x\big|_{t+\frac{\Delta t}{2}}^{i,j,k} + \frac{\left(\sigma_y^H\big|^{i,j,k}\right)\left(\sigma_z^H\big|^{i,j,k}\right)\Delta t}{4\varepsilon_0^2}H_x\big|_{t-\frac{\Delta t}{2}}^{i,j,k} + \frac{\left(\sigma_y^H\big|^{i,j,k}\right)\left(\sigma_z^H\big|^{i,j,k}\right)\Delta t}{\varepsilon_0^2}\sum_{T=\Delta t/2}^{t-\frac{\Delta t}{2}} H_x\big|_T^{i,j,k}$$

$$= -\frac{c_0}{\mu_{xx}\big|^{i,j,k}} C_x^E\big|_t^{i,j,k} - \frac{c_0 \Delta t}{\varepsilon_0}\frac{\sigma_x^H\big|^{i,j,k}}{\mu_{xx}\big|^{i,j,k}}\sum_{T=0}^{t} C_x^E\big|_t^{i,j,k}$$

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Collect Common Terms

Next, collect the coefficients of the common field terms.

$$\frac{1}{\Delta t} H_x\big|_{t+\Delta t/2}^{i,j,k} - \frac{1}{\Delta t} H_x\big|_{t-\Delta t/2}^{i,j,k} + \left(\frac{\sigma_y^H\big|^{i,j,k} + \sigma_z^H\big|^{i,j,k}}{2\varepsilon_0}\right) H_x\big|_{t+\frac{\Delta t}{2}}^{i,j,k} + \left(\frac{\sigma_y^H\big|^{i,j,k} + \sigma_z^H\big|^{i,j,k}}{2\varepsilon_0}\right) H_x\big|_{t-\frac{\Delta t}{2}}^{i,j,k}$$

$$+ \frac{\left(\sigma_y^H\big|^{i,j,k}\right)\left(\sigma_z^H\big|^{i,j,k}\right)\Delta t}{4\varepsilon_0^2} H_x\big|_{t+\frac{\Delta t}{2}}^{i,j,k} + \frac{\left(\sigma_y^H\big|^{i,j,k}\right)\left(\sigma_z^H\big|^{i,j,k}\right)\Delta t}{4\varepsilon_0^2} H_x\big|_{t-\frac{\Delta t}{2}}^{i,j,k} + \frac{\left(\sigma_y^H\big|^{i,j,k}\right)\left(\sigma_z^H\big|^{i,j,k}\right)\Delta t}{\varepsilon_0^2} \sum_{T=\Delta t/2}^{t-\frac{\Delta t}{2}} H_x\big|_T^{i,j,k} = -\frac{c_0}{\mu_{xx}\big|^{i,j,k}} C_x^E\big|_t^{i,j,k} - \frac{c_0 \Delta t}{\varepsilon_0} \frac{\sigma_x^H\big|^{i,j,k}}{\mu_{xx}\big|^{i,j,k}} \sum_{T=0}^{t} C_x^E\big|_t^{i,j,k}$$

⇩

$$\left[\frac{1}{\Delta t} + \left(\frac{\sigma_y^H\big|^{i,j,k} + \sigma_z^H\big|^{i,j,k}}{2\varepsilon_0}\right) + \frac{\left(\sigma_y^H\big|^{i,j,k}\right)\left(\sigma_z^H\big|^{i,j,k}\right)\Delta t}{4\varepsilon_0^2}\right] H_x\big|_{t+\frac{\Delta t}{2}}^{i,j,k} - \left[\frac{1}{\Delta t} - \left(\frac{\sigma_y^H\big|^{i,j,k} + \sigma_z^H\big|^{i,j,k}}{2\varepsilon_0}\right) - \frac{\left(\sigma_y^H\big|^{i,j,k}\right)\left(\sigma_z^H\big|^{i,j,k}\right)\Delta t}{4\varepsilon_0^2}\right] H_x\big|_{t-\frac{\Delta t}{2}}^{i,j,k}$$

$$= \left[-\frac{\left(\sigma_y^H\big|^{i,j,k}\right)\left(\sigma_z^H\big|^{i,j,k}\right)\Delta t}{\varepsilon_0^2}\right] \sum_{T=\Delta t/2}^{t-\frac{\Delta t}{2}} H_x\big|_T^{i,j,k} + \left[-\frac{c_0}{\mu_{xx}\big|^{i,j,k}}\right] C_x^E\big|_t^{i,j,k} + \left[-\frac{c_0 \Delta t}{\varepsilon_0} \frac{\sigma_x^H\big|^{i,j,k}}{\mu_{xx}\big|^{i,j,k}}\right] \sum_{T=0}^{t} C_x^E\big|_t^{i,j,k}$$

Solve for the Future H Field

Solving the numerical equation for H_x field at $t + \Delta t/2$ yields

$$H_x\big|_{t+\frac{\Delta t}{2}}^{i,j,k} = \left[\frac{\frac{1}{\Delta t} - \left(\frac{\sigma_y^H\big|^{i,j,k} + \sigma_z^H\big|^{i,j,k}}{2\varepsilon_0}\right) - \frac{\left(\sigma_y^H\big|^{i,j,k}\right)\left(\sigma_z^H\big|^{i,j,k}\right)\Delta t}{4\varepsilon_0^2}}{\frac{1}{\Delta t} + \left(\frac{\sigma_y^H\big|^{i,j,k} + \sigma_z^H\big|^{i,j,k}}{2\varepsilon_0}\right) + \frac{\left(\sigma_y^H\big|^{i,j,k}\right)\left(\sigma_z^H\big|^{i,j,k}\right)\Delta t}{4\varepsilon_0^2}}\right] H_x\big|_{t-\frac{\Delta t}{2}}^{i,j,k} + \left[-\frac{c_0/\mu_{xx}\big|^{i,j,k}}{\frac{1}{\Delta t} + \left(\frac{\sigma_y^H\big|^{i,j,k} + \sigma_z^H\big|^{i,j,k}}{2\varepsilon_0}\right) + \frac{\left(\sigma_y^H\big|^{i,j,k}\right)\left(\sigma_z^H\big|^{i,j,k}\right)\Delta t}{4\varepsilon_0^2}}\right] C_x^E\big|_t^{i,j,k}$$

$$+ \left[-\frac{\frac{c_0 \Delta t}{\varepsilon_0} \frac{\sigma_x^H\big|^{i,j,k}}{\mu_{xx}\big|^{i,j,k}}}{\frac{1}{\Delta t} + \left(\frac{\sigma_y^H\big|^{i,j,k} + \sigma_z^H\big|^{i,j,k}}{2\varepsilon_0}\right) + \frac{\left(\sigma_y^H\big|^{i,j,k}\right)\left(\sigma_z^H\big|^{i,j,k}\right)\Delta t}{4\varepsilon_0^2}}\right] \sum_{T=0}^{t} C_x^E\big|_t^{i,j,k} + \left[-\frac{\frac{\Delta t}{\varepsilon_0^2}\left(\sigma_y^H\big|^{i,j,k}\right)\left(\sigma_z^H\big|^{i,j,k}\right)}{\frac{1}{\Delta t} + \left(\frac{\sigma_y^H\big|^{i,j,k} + \sigma_z^H\big|^{i,j,k}}{2\varepsilon_0}\right) + \frac{\left(\sigma_y^H\big|^{i,j,k}\right)\left(\sigma_z^H\big|^{i,j,k}\right)\Delta t}{4\varepsilon_0^2}}\right] \sum_{T=\Delta t/2}^{t-\frac{\Delta t}{2}} H_x\big|_T^{i,j,k}$$

This is the largest equation in the entire course! ☺

Final Form of the Update Equation for H_x

The update coefficients are computed before the main FDTD loop.

$$m_{Hx0}\Big|^{i,j,k} = \frac{1}{\Delta t} + \left(\frac{\sigma_y^H\big|^{i,j,k} + \sigma_z^H\big|^{i,j,k}}{2\varepsilon_0} \right) + \frac{\left(\sigma_y^H\big|^{i,j,k}\right)\left(\sigma_z^H\big|^{i,j,k}\right)\Delta t}{4\varepsilon_0^2} \qquad m_{Hx1}\Big|^{i,j,k} = \frac{1}{m_{Hx0}\big|^{i,j,k}} \left[\frac{1}{\Delta t} - \left(\frac{\sigma_y^H\big|^{i,j,k} + \sigma_z^H\big|^{i,j,k}}{2\varepsilon_0} \right) - \frac{\left(\sigma_y^H\big|^{i,j,k}\right)\left(\sigma_z^H\big|^{i,j,k}\right)\Delta t}{4\varepsilon_0^2} \right]$$

$$m_{Hx2}\Big|^{i,j,k} = -\frac{1}{m_{Hx0}\big|^{i,j,k}} \frac{c_0}{\mu_{xx}\big|^{i,j,k}} \qquad m_{Hx3}\Big|^{i,j,k} = -\frac{1}{m_{Hx0}\big|^{i,j,k}} \frac{c_0 \Delta t}{\varepsilon_0} \frac{\sigma_x^H\big|^{i,j,k}}{\mu_{xx}\big|^{i,j,k}} \qquad m_{Hx4}\Big|^{i,j,k} = -\frac{1}{m_{Hx0}\big|^{i,j,k}} \frac{\Delta t}{\varepsilon_0^2} \left(\sigma_y^H\big|^{i,j,k}\right)\left(\sigma_z^H\big|^{i,j,k}\right)$$

The integration terms are computed inside the main FDTD loop, but before the update equation.

$$I_{CEx}\Big|_t^{i,j,k} = \sum_{T=0}^{t} C_x^E\Big|_t^{i,j,k} \qquad I_{Hx}\Big|_{t-\frac{\Delta t}{2}}^{i,j,k} = \sum_{T=\Delta t/2}^{t-\frac{\Delta t}{2}} H_x\Big|_T^{i,j,k} \qquad C_x^E\Big|_t^{i,j,k} = \frac{\tilde{E}_z\big|_t^{i,j+1,k} - \tilde{E}_z\big|_t^{i,j,k}}{\Delta y} - \frac{\tilde{E}_y\big|_t^{i,j,k+1} - \tilde{E}_y\big|_t^{i,j,k}}{\Delta z}$$

The update equation is computed inside the main FDTD loop immediately after the integration terms are updated.

$$H_x\Big|_{t+\Delta t/2}^{i,j,k} = \left(m_{Hx1}\big|^{i,j,k}\right) H_x\Big|_{t-\Delta t/2}^{i,j,k} + \left(m_{Hx2}\big|^{i,j,k}\right) C_x^E\Big|_t^{i,j,k} + \left(m_{Hx3}\big|^{i,j,k}\right) I_{CEx}\Big|_t^{i,j,k} + \left(m_{Hx4}\big|^{i,j,k}\right) I_{Hx}\Big|_t^{i,j,k}$$

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Are These Equations Correct?

A simple way to test these update equations is to set the PML parameters to zero and see if the update equations reduce to what was derived in a previous lecture without the PML present.

The update coefficients reduce to

$$m_{Hx1}\Big|^{i,j,k} = 1 \qquad m_{Hx2}\Big|^{i,j,k} = -\frac{c_0 \Delta t}{\mu_{xx}\big|^{i,j,k}} \qquad m_{Hx3}\Big|^{i,j,k} = 0 \qquad m_{Hx4}\Big|^{i,j,k} = 0$$

The update equation reduces to

$$H_x\Big|_{t+\Delta t/2}^{i,j,k} = H_x\Big|_{t-\Delta t/2}^{i,j,k} + \left(-\frac{c_0 \Delta t}{\mu_{xx}\big|^{i,j,k}} \right) \left(\frac{\tilde{E}_z\big|_t^{i,j+1,k} - \tilde{E}_z\big|_t^{i,j,k}}{\Delta y} - \frac{\tilde{E}_y\big|_t^{i,j,k+1} - \tilde{E}_y\big|_t^{i,j,k}}{\Delta z} \right)$$

For problems uniform in the z direction, the z derivative is zero. This equation reduces to exactly what was derived in a previous lecture.

$$H_x\Big|_{t+\Delta t/2}^{i,j} = H_x\Big|_{t-\Delta t/2}^{i,j} + \left(-\frac{c_0 \Delta t}{\mu_{xx}\big|^{i,j}} \right) \left(\frac{\tilde{E}_z\big|_t^{i,j+1} - \tilde{E}_z\big|_t^{i,j}}{\Delta y} \right)$$

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Summary of the Update Equations

Final Form of the Update Equation for H_x

The update coefficients are computed before the main FDTD loop.

$$m_{Hx0}\Big|^{i,j,k} = \frac{1}{\Delta t} + \left(\frac{\sigma_y^H\Big|^{i,j,k} + \sigma_z^H\Big|^{i,j,k}}{2\varepsilon_0} \right) + \frac{\left(\sigma_y^H\Big|^{i,j,k}\right)\left(\sigma_z^H\Big|^{i,j,k}\right)\Delta t}{4\varepsilon_0^2}$$

$$m_{Hx1}\Big|^{i,j,k} = \frac{1}{m_{Hx0}\Big|^{i,j,k}} \left[\frac{1}{\Delta t} - \left(\frac{\sigma_y^H\Big|^{i,j,k} + \sigma_z^H\Big|^{i,j,k}}{2\varepsilon_0} \right) - \frac{\left(\sigma_y^H\Big|^{i,j,k}\right)\left(\sigma_z^H\Big|^{i,j,k}\right)\Delta t}{4\varepsilon_0^2} \right]$$

$$m_{Hx2}\Big|^{i,j,k} = -\frac{1}{m_{Hx0}\Big|^{i,j,k}} \frac{c_0}{\mu_{xx}\Big|^{i,j,k}}$$

$$m_{Hx3}\Big|^{i,j,k} = -\frac{1}{m_{Hx0}\Big|^{i,j,k}} \frac{c_0 \Delta t}{\varepsilon_0} \frac{\sigma_x^H\Big|^{i,j,k}}{\mu_{xx}\Big|^{i,j,k}}$$

$$m_{Hx4}\Big|^{i,j,k} = -\frac{1}{m_{Hx0}\Big|^{i,j,k}} \frac{\Delta t}{\varepsilon_0^2} \left(\sigma_y^H\Big|^{i,j,k}\right)\left(\sigma_z^H\Big|^{i,j,k}\right)$$

The integration terms are computed inside the main FDTD loop, but before the update equation.

$$I_{CEx}\Big|_t^{i,j,k} = \sum_{T=0}^{t} C_x^E\Big|_t^{i,j,k}$$

$$I_{Hx}\Big|_{t-\frac{\Delta t}{2}}^{i,j,k} = \sum_{T=\Delta t/2}^{t-\frac{\Delta t}{2}} H_x\Big|_T^{i,j,k}$$

$$C_x^E\Big|_t^{i,j,k} = \frac{\tilde{E}_z\Big|_t^{i,j+1,k} - \tilde{E}_z\Big|_t^{i,j,k}}{\Delta y} - \frac{\tilde{E}_y\Big|_t^{i,j,k+1} - \tilde{E}_y\Big|_t^{i,j,k}}{\Delta z}$$

The update equation is computed inside the main FDTD loop immediately after the integration terms are updated.

$$H_x\Big|_{t+\Delta t/2}^{i,j,k} = \left(m_{Hx1}\Big|^{i,j,k}\right) H_x\Big|_{t-\Delta t/2}^{i,j,k} + \left(m_{Hx2}\Big|^{i,j,k}\right) C_x^E\Big|_t^{i,j,k} + \left(m_{Hx3}\Big|^{i,j,k}\right) I_{CEx}\Big|_t^{i,j,k} + \left(m_{Hx4}\Big|^{i,j,k}\right) I_{Hx}\Big|_{t-\frac{\Delta t}{2}}^{i,j,k}$$

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Final Form of the Update Equation for H_y

The update coefficients are computed before the main FDTD loop.

$$m_{Hy0}\Big|^{i,j,k} = \frac{1}{\Delta t} + \left(\frac{\sigma_x^H\Big|^{i,j,k} + \sigma_z^H\Big|^{i,j,k}}{2\varepsilon_0}\right) + \frac{\left(\sigma_x^H\Big|^{i,j,k}\right)\left(\sigma_z^H\Big|^{i,j,k}\right)\Delta t}{4\varepsilon_0^2}$$

$$m_{Hy1}\Big|^{i,j,k} = \frac{1}{m_{Hy0}\Big|^{i,j,k}}\left[\frac{1}{\Delta t} - \left(\frac{\sigma_x^H\Big|^{i,j,k} + \sigma_z^H\Big|^{i,j,k}}{2\varepsilon_0}\right) - \frac{\left(\sigma_x^H\Big|^{i,j,k}\right)\left(\sigma_z^H\Big|^{i,j,k}\right)\Delta t}{4\varepsilon_0^2}\right]$$

$$m_{Hy2}\Big|^{i,j,k} = -\frac{1}{m_{Hy0}\Big|^{i,j,k}}\frac{c_0}{\mu_{yy}\Big|^{i,j,k}} \qquad m_{Hy3}\Big|^{i,j,k} = -\frac{1}{m_{Hy0}\Big|^{i,j,k}}\frac{c_0\Delta t}{\varepsilon_0}\frac{\sigma_y^H\Big|^{i,j,k}}{\mu_{yy}\Big|^{i,j,k}} \qquad m_{Hy4}\Big|^{i,j,k} = -\frac{1}{m_{Hy0}\Big|^{i,j,k}}\frac{\Delta t}{\varepsilon_0^2}\left(\sigma_x^H\Big|^{i,j,k}\right)\left(\sigma_z^H\Big|^{i,j,k}\right)$$

The integration terms are computed inside the main FDTD loop, but before the update equation.

$$I_{CEy}\Big|_t^{i,j,k} = \sum_{T=0}^{t} C_y^E\Big|_t^{i,j,k} \qquad I_{Hy}\Big|_{t-\frac{\Delta t}{2}}^{i,j,k} = \sum_{T=\Delta t/2}^{t-\frac{\Delta t}{2}} H_y\Big|_T^{i,j,k} \qquad C_y^E\Big|_t^{i,j,k} = \frac{\tilde{E}_x\Big|_t^{i,j,k+1} - \tilde{E}_x\Big|_t^{i,j,k}}{\Delta z} - \frac{\tilde{E}_z\Big|_t^{i+1,j,k} - \tilde{E}_z\Big|_t^{i,j,k}}{\Delta x}$$

The update equation is computed inside the main FDTD loop immediately after the integration terms are updated.

$$H_y\Big|_{t+\frac{\Delta t}{2}}^{i,j,k} = \left(m_{Hy1}\Big|^{i,j,k}\right)H_y\Big|_{t-\frac{\Delta t}{2}}^{i,j,k} + \left(m_{Hy2}\Big|^{i,j,k}\right)C_y^E\Big|_t^{i,j,k} + \left(m_{Hy3}\Big|^{i,j,k}\right)I_{CEy}\Big|_t^{i,j,k} + \left(m_{Hy4}\Big|^{i,j,k}\right)I_{Hy}\Big|_{t-\frac{\Delta t}{2}}^{i,j,k}$$

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Final Form of the Update Equation for H_z

The update coefficients are computed before the main FDTD loop.

$$m_{Hz0}\Big|^{i,j,k} = \frac{1}{\Delta t} + \left(\frac{\sigma_x^H\Big|^{i,j,k} + \sigma_y^H\Big|^{i,j,k}}{2\varepsilon_0}\right) + \frac{\left(\sigma_x^H\Big|^{i,j,k}\right)\left(\sigma_y^H\Big|^{i,j,k}\right)\Delta t}{4\varepsilon_0^2}$$

$$m_{Hz1}\Big|^{i,j,k} = \frac{1}{m_{Hz0}\Big|^{i,j,k}}\left[\frac{1}{\Delta t} - \left(\frac{\sigma_x^H\Big|^{i,j,k} + \sigma_y^H\Big|^{i,j,k}}{2\varepsilon_0}\right) - \frac{\left(\sigma_x^H\Big|^{i,j,k}\right)\left(\sigma_y^H\Big|^{i,j,k}\right)\Delta t}{4\varepsilon_0^2}\right]$$

$$m_{Hz2}\Big|^{i,j,k} = -\frac{1}{m_{Hz0}\Big|^{i,j,k}}\frac{c_0}{\mu_{zz}\Big|^{i,j,k}} \qquad m_{Hz3}\Big|^{i,j,k} = -\frac{1}{m_{Hz0}\Big|^{i,j,k}}\frac{c_0\Delta t}{\varepsilon_0}\frac{\sigma_z^H\Big|^{i,j,k}}{\mu_{zz}\Big|^{i,j,k}} \qquad m_{Hz4}\Big|^{i,j,k} = -\frac{1}{m_{Hz0}\Big|^{i,j,k}}\frac{\Delta t}{\varepsilon_0^2}\left(\sigma_x^H\Big|^{i,j,k}\right)\left(\sigma_y^H\Big|^{i,j,k}\right)$$

The integration terms are computed inside the main FDTD loop, but before the update equation.

$$I_{CEz}\Big|_t^{i,j,k} = \sum_{T=0}^{t} C_z^E\Big|_t^{i,j,k} \qquad I_{Hz}\Big|_{t-\frac{\Delta t}{2}}^{i,j,k} = \sum_{T=\Delta t/2}^{t-\frac{\Delta t}{2}} H_z\Big|_T^{i,j,k} \qquad C_z^E\Big|_t^{i,j,k} = \frac{\tilde{E}_y\Big|_t^{i+1,j,k} - \tilde{E}_y\Big|_t^{i,j,k}}{\Delta x} - \frac{\tilde{E}_x\Big|_t^{i,j+1,k} - \tilde{E}_x\Big|_t^{i,j,k}}{\Delta y}$$

The update equation is computed inside the main FDTD loop immediately after the integration terms are updated.

$$H_z\Big|_{t+\frac{\Delta t}{2}}^{i,j,k} = \left(m_{Hz1}\Big|^{i,j,k}\right)H_z\Big|_{t-\frac{\Delta t}{2}}^{i,j,k} + \left(m_{Hz2}\Big|^{i,j,k}\right)C_z^E\Big|_t^{i,j,k} + \left(m_{Hz3}\Big|^{i,j,k}\right)I_{CEz}\Big|_t^{i,j,k} + \left(m_{Hz4}\Big|^{i,j,k}\right)I_{Hz}\Big|_{t-\frac{\Delta t}{2}}^{i,j,k}$$

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Final Form of the Update Equation for D_x

The update coefficients are computed before the main FDTD loop.

$$m_{Dx0}\Big|^{i,j,k} = \frac{1}{\Delta t} + \frac{\sigma_y^D\Big|^{i,j,k} + \sigma_z^D\Big|^{i,j,k}}{2\varepsilon_0} + \frac{\left(\sigma_y^D\Big|^{i,j,k}\right)\left(\sigma_z^D\Big|^{i,j,k}\right)\Delta t}{4\varepsilon_0^2} \qquad m_{Dx1}\Big|^{i,j,k} = \frac{1}{m_{Dx0}\Big|^{i,j,k}}\left[\frac{1}{\Delta t} - \frac{\sigma_y^D\Big|^{i,j,k} + \sigma_z^D\Big|^{i,j,k}}{2\varepsilon_0} - \frac{\left(\sigma_y^D\Big|^{i,j,k}\right)\left(\sigma_z^D\Big|^{i,j,k}\right)\Delta t}{4\varepsilon_0^2}\right]$$

$$m_{Dx2}\Big|^{i,j,k} = \frac{c_0}{m_{Dx0}\Big|^{i,j,k}} \qquad m_{Dx3}\Big|^{i,j,k} = \frac{1}{m_{Dx0}\Big|^{i,j,k}}\frac{c_0 \Delta t\, \sigma_x^D\Big|^{i,j,k}}{\varepsilon_0} \qquad m_{Dx4}\Big|^{i,j,k} = -\frac{1}{m_{Dx0}\Big|^{i,j,k}}\frac{\Delta t}{\varepsilon_0^2}\left(\sigma_y^D\Big|^{i,j,k}\right)\left(\sigma_z^D\Big|^{i,j,k}\right)$$

The integration terms are computed inside the main FDTD loop, but before the update equation.

$$I_{CHx}\Big|_{t-\frac{\Delta t}{2}}^{i,j,k} = \sum_{T=\Delta t/2}^{t-\frac{\Delta t}{2}} C_x^H\Big|_T^{i,j,k} \qquad I_{Dx}\Big|_t^{i,j,k} = \sum_{T=0}^{t} \tilde{D}_x\Big|_T^{i,j,k} \qquad C_x^H\Big|_{t+\frac{\Delta t}{2}}^{i,j,k} = \frac{H_z\Big|_{t+\frac{\Delta t}{2}}^{i,j,k} - H_z\Big|_{t+\frac{\Delta t}{2}}^{i,j-1,k}}{\Delta y} - \frac{H_y\Big|_{t+\frac{\Delta t}{2}}^{i,j,k} - H_y\Big|_{t+\frac{\Delta t}{2}}^{i,j,k-1}}{\Delta z}$$

The update equation is computed inside the main FDTD loop immediately after the integration terms are updated.

$$\tilde{D}_x\Big|_{t+\Delta t}^{i,j,k} = \left(m_{Dx1}\Big|^{i,j,k}\right)\tilde{D}_x\Big|_t^{i,j,k} + \left(m_{Dx2}\Big|^{i,j,k}\right)C_x^H\Big|_{t+\frac{\Delta t}{2}}^{i,j,k} + \left(m_{Dx3}\Big|^{i,j,k}\right)I_{CHx}\Big|_{t-\frac{\Delta t}{2}}^{i,j,k} + \left(m_{Dx4}\Big|^{i,j,k}\right)I_{Dx}\Big|_{t-\Delta t}^{i,j,k}$$

Final Form of the Update Equation for D_y

The update coefficients are computed before the main FDTD loop.

$$m_{Dy0}\Big|^{i,j,k} = \frac{1}{\Delta t} + \frac{\sigma_x^D\Big|^{i,j,k} + \sigma_z^D\Big|^{i,j,k}}{2\varepsilon_0} + \frac{\left(\sigma_x^D\Big|^{i,j,k}\right)\left(\sigma_z^D\Big|^{i,j,k}\right)\Delta t}{4\varepsilon_0^2} \qquad m_{Dy1}\Big|^{i,j,k} = \frac{1}{m_{Dy0}\Big|^{i,j,k}}\left[\frac{1}{\Delta t} - \frac{\sigma_x^D\Big|^{i,j,k} + \sigma_z^D\Big|^{i,j,k}}{2\varepsilon_0} - \frac{\left(\sigma_x^D\Big|^{i,j,k}\right)\left(\sigma_z^D\Big|^{i,j,k}\right)\Delta t}{4\varepsilon_0^2}\right]$$

$$m_{Dy2}\Big|^{i,j,k} = \frac{c_0}{m_{Dy0}\Big|^{i,j,k}} \qquad m_{Dy3}\Big|^{i,j,k} = \frac{1}{m_{Dy0}\Big|^{i,j,k}}\frac{c_0 \Delta t\, \sigma_y^D\Big|^{i,j,k}}{\varepsilon_0} \qquad m_{Dy4}\Big|^{i,j,k} = -\frac{1}{m_{Dy0}\Big|^{i,j,k}}\frac{\Delta t}{\varepsilon_0^2}\left(\sigma_x^D\Big|^{i,j,k}\right)\left(\sigma_z^D\Big|^{i,j,k}\right)$$

The integration terms are computed inside the main FDTD loop, but before the update equation.

$$I_{CHy}\Big|_{t-\frac{\Delta t}{2}}^{i,j,k} = \sum_{T=\Delta t/2}^{t-\frac{\Delta t}{2}} C_y^H\Big|_T^{i,j,k} \qquad I_{Dy}\Big|_t^{i,j,k} = \sum_{T=0}^{t} \tilde{D}_y\Big|_T^{i,j,k} \qquad C_y^H\Big|_{t+\frac{\Delta t}{2}}^{i,j,k} = \frac{H_x\Big|_{t+\frac{\Delta t}{2}}^{i,j,k} - H_x\Big|_{t+\frac{\Delta t}{2}}^{i,j,k-1}}{\Delta z} - \frac{H_z\Big|_{t+\frac{\Delta t}{2}}^{i,j,k} - H_z\Big|_{t+\frac{\Delta t}{2}}^{i-1,j,k}}{\Delta x}$$

The update equation is computed inside the main FDTD loop immediately after the integration terms are updated.

$$\tilde{D}_y\Big|_{t+\Delta t}^{i,j,k} = \left(m_{Dy1}\Big|^{i,j,k}\right)\tilde{D}_y\Big|_t^{i,j,k} + \left(m_{Dy2}\Big|^{i,j,k}\right)C_y^H\Big|_{t+\frac{\Delta t}{2}}^{i,j,k} + \left(m_{Dy3}\Big|^{i,j,k}\right)I_{CHy}\Big|_{t-\frac{\Delta t}{2}}^{i,j,k} + \left(m_{Dy4}\Big|^{i,j,k}\right)I_{Dy}\Big|_{t-\Delta t}^{i,j,k}$$

Final Form of the Update Equation for D_z

The update coefficients are computed before the main FDTD loop.

$$m_{Dz0}\Big|^{i,j,k} = \frac{1}{\Delta t} + \frac{\sigma_x^D\big|^{i,j,k} + \sigma_y^D\big|^{i,j,k}}{2\varepsilon_0} + \frac{\left(\sigma_x^D\big|^{i,j,k}\right)\left(\sigma_y^D\big|^{i,j,k}\right)\Delta t}{4\varepsilon_0^2} \qquad m_{Dz1}\Big|^{i,j,k} = \frac{1}{m_{Dz0}\big|^{i,j,k}}\left[\frac{1}{\Delta t} - \frac{\sigma_x^D\big|^{i,j,k} + \sigma_y^D\big|^{i,j,k}}{2\varepsilon_0} - \frac{\left(\sigma_x^D\big|^{i,j,k}\right)\left(\sigma_y^D\big|^{i,j,k}\right)\Delta t}{4\varepsilon_0^2}\right]$$

$$m_{Dz2}\Big|^{i,j,k} = \frac{c_0}{m_{Dz0}\big|^{i,j,k}} \qquad m_{Dz3}\Big|^{i,j,k} = \frac{1}{m_{Dz0}\big|^{i,j,k}}\frac{c_0 \Delta t\, \sigma_z^D\big|^{i,j,k}}{\varepsilon_0} \qquad m_{Dz4}\Big|^{i,j,k} = -\frac{1}{m_{Dz0}\big|^{i,j,k}}\frac{\Delta t}{\varepsilon_0^2}\left(\sigma_x^D\big|^{i,j,k}\right)\left(\sigma_y^D\big|^{i,j,k}\right)$$

The integration terms are computed inside the main FDTD loop, but before the update equation.

$$I_{CHz}\Big|_{t-\frac{\Delta t}{2}}^{i,j,k} = \sum_{T=\Delta t/2}^{t-\frac{\Delta t}{2}} C_z^H\Big|_T^{i,j,k} \qquad I_{Dz}\Big|_t^{i,j,k} = \sum_{T=0}^{t} \tilde{D}_z\Big|_T^{i,j,k} \qquad C_z^H\Big|_{t+\frac{\Delta t}{2}}^{i,j,k} = \frac{H_y\big|_{t+\frac{\Delta t}{2}}^{i,j,k} - H_y\big|_{t+\frac{\Delta t}{2}}^{i-1,j,k}}{\Delta x} - \frac{H_x\big|_{t+\frac{\Delta t}{2}}^{i,j,k} - H_x\big|_{t+\frac{\Delta t}{2}}^{i,j-1,k}}{\Delta y}$$

The update equation is computed inside the main FDTD loop immediately after the integration terms are updated.

$$\tilde{D}_z\Big|_{t+\Delta t}^{i,j,k} = \left(m_{Dz1}\big|^{i,j,k}\right)\tilde{D}_z\Big|_t^{i,j,k} + \left(m_{Dz2}\big|^{i,j,k}\right)C_z^H\Big|_{t+\frac{\Delta t}{2}}^{i,j,k} + \left(m_{Dz3}\big|^{i,j,k}\right)I_{CHz}\Big|_{t-\frac{\Delta t}{2}}^{i,j,k} + \left(m_{Dz4}\big|^{i,j,k}\right)I_{Dz}\Big|_{t-\Delta t}^{i,j,k}$$

Final Update Equations for E_x, E_y, and E_z

The update coefficients are computed before the main FDTD loop.

$$m_{Ex1}\Big|^{i,j,k} = \frac{1}{\varepsilon_{xx}\big|^{i,j,k}} \qquad m_{Ey1}\Big|^{i,j,k} = \frac{1}{\varepsilon_{yy}\big|^{i,j,k}} \qquad m_{Ez1}\Big|^{i,j,k} = \frac{1}{\varepsilon_{zz}\big|^{i,j,k}}$$

The update equations are computed inside the main FDTD loop.

$$\tilde{E}_x\Big|_{t+\Delta t}^{i,j,k} = \left(m_{Ex1}\big|^{i,j,k}\right)\tilde{D}_x\Big|_{t+\Delta t}^{i,j,k}$$

$$\tilde{E}_y\Big|_{t+\Delta t}^{i,j,k} = \left(m_{Ey1}\big|^{i,j,k}\right)\tilde{D}_y\Big|_{t+\Delta t}^{i,j,k}$$

$$\tilde{E}_z\Big|_{t+\Delta t}^{i,j,k} = \left(m_{Ez1}\big|^{i,j,k}\right)\tilde{D}_z\Big|_{t+\Delta t}^{i,j,k}$$